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Abstract. In this paper we introduce modified Bernstein-Durrmeyer operators and study approximation properties of these operators, including theorems on the degree of approximation.

1. Introduction

Let $C_{[0,1]}$ be the set of all real-valued functions f continuous on $[0,1]$ with the norm

$$\|f\| \equiv \|f(\cdot)\| := \sup_{x\in[0,1]} |f(x)|. \tag{1}$$

In 1912 Bernstein constructed, for any function $f \in C_{[0,1]}$, a sequence of polynomials

$$B_n(f;x) := \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \qquad n \in N := \{1,2,\cdots\}, \tag{2}$$

where

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}, \qquad 0 \le k \le n, \tag{3}$$

$x \in [0,1]$, and proved that $B_n(f) \rightrightarrows f$. These polynomials (2), called Bernstein polynomials, possess many remarkable properties. We present only two of them.

Theorem A. *Let $f \in C_{[0,1]}$ and $B_n(f;\cdot)$ be the Bernstein polynomial for f. Then*

$$\|B_n(f;\cdot) - f(\cdot)\| \le \frac{3}{2}\omega\left(f; n^{-1/2}\right), \qquad n \in N,$$

where $\omega(f;\cdot)$ is the classical modulus of continuity.

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Theorem B. *Let $f \in C^2_{[0,1]}$ and $B_n(f;\cdot)$ be the Bernstein polynomial for f. Then for every $x \in [0,1]$ we have*

$$lim_{n\to\infty} n\{B_n(f;x) - f(x)\} = \frac{1}{2}x(1-x)f''(x).$$

Theorems A and B are classical results of Popoviciu and Voronovskaya [4, Ch. 10].

The Bernstein polynomials and their connections with different branches of analysis, such as convex and numerical analysis have been studied intensively. Basic facts on Bernstein operators, their generalizations and applications, can be found in, e.g., [7-11]. In [6] J.L. Durrmeyer introduced an interesting modification of the Bernstein polynomials defined by

$$M_n(f;x) := (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)\,dt, \tag{4}$$

for $f \in C_{[0,1]}$, where $n \in N$ and $p_{n,k}(\cdot)$ are defined by (3).

Approximation of continuous functions by Bernstein-Durrmeyer operator (defined by (4)), has been investigated by many authors. A careful analysis of such operators, was carried out for the first time by Derriennic in [3]. Subsequently, Ditzian and Ivanov [5] studied their rate of convergence in terms of the so-called Ditzian-Totik modulus of continuity.

Derriennic proved in [3] that:

(a) $M_n f$ is a positive operator,

(b) $M_n(1;x) = 1$,

(c) $lim_{n\to\infty}\|M_n(f;\cdot) - f(\cdot)\| = 0$ for $f \in C_{[0,1]}$.

Let the set of all $f \in C_{[0,1]}$ with derivatives $f', \ldots, f^{(r)}$ belonging also to $C_{[0,1]}$ be denoted by $C^r_{[0,1]}$, $r \in N_0 := \{0,1,2,\ldots\}$ ($C^0_{[0,1]} \equiv C_{[0,1]}$). The norm on $C^r_{[0,1]}$ is given by (1).

In this paper we introduce the following class of operators in $C^r_{[0,1]}$.

Definition. *Fix $r \in N_0$. We define a class of operators $M_{n,r}$ by the formula*

$$M_{n,r}(f;x) := (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\sum_{j=0}^{r}\frac{f^{(j)}(t)}{j!}(x-t)^j\,dt, \qquad x \in [0,1], \tag{5}$$

where $p_{n,k}(\cdot)$ are defined by (3).

Clearly, $M_{n,0}(f;x) = M_n(f;x)$ for $x \in [0,1]$, $n \in N$ and $f \in C_{[0,1]}$.

In this paper we shall study a relation between the rate of approximation by $M_{n,r}$ and the smoothness of the function f.

In the sequel we shall denote the suitable positive constants depending only on a by $K_m(a)$, $m = 1,2,\ldots$.

2. Auxiliary results

In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems.

We may remark here that if $f(x) = x^q$, $x \in [0,1]$, $q \in N_0$, then by Taylor's formula it follows that

$$f(x) = \sum_{j=0}^{q} \frac{f^{(j)}(y)}{j!}(x-y)^j,$$

for every fixed $y \in [0,1]$. This fact and (5) yield

Lemma 1. *Let $f(x) = x^q$, $x \in [0,1]$, $q \in N_0$. Then for every fixed $q \le r \in N_0$ we have*

$$M_{n,r}\left(t^q; x\right) = x^q, \qquad n \in N.$$

In view of [3] it is known that

$$M_n(1;x) = 1, \qquad M_n(x-t;x) = -(1-2x)/(n+2), \tag{6}$$

$$\begin{aligned}(n+q+2)M_n\left((x-t)^{q+1};x\right) &= \\ = x(1-x)\left\{2qM_n\left((x-t)^{q-1};x\right) - M_n'\left((x-t)^q;x\right)\right\} &+ \\ -(1-2x)(q+1)M_n\left((x-t)^q;x\right), &\end{aligned} \tag{7}$$

for $x \in [0,1]$, $n \in N$ and $q \in N$.

Using mathematical induction on $q \in N$ and by (6) and (7), we can prove the following

Lemma 2 ([5], pages 83-84, Lemma 6.3) *For every $q \in N$ we have*

$$M_n\left((x-t)^{2q};x\right) = \sum_{j=0}^{q} a_{j,q,n}(x)\left(\frac{x(1-x)}{n}\right)^{q-j} n^{-2j}, \qquad x \in [0,1], \quad n \in N,$$

where $a_{j,q,n}(x)$ are polynomials in x of fixed degree with coefficients bounded for all n.

Lemma 3. *For every $q \in N$ there exists a positive constant $K_1(q)$ such that*

$$\left\|M_n\left((\cdot - t)^{2q};\cdot\right)\right\| \le K_1(q)n^{-q}, \qquad n \in N.$$

Proof. Applying Lemma 2, we get

$$\left|M_n\left((x-t)^{2q};x\right)\right| \le \sum_{j=0}^{q} |a_{j,q,n}(x)| \left(\frac{x(1-x)}{n}\right)^{q-j} n^{-2j}.$$

By Lemma 1 and by elementary calculations we immediately obtain

$$\left|M_n\left((x-t)^{2q};x\right)\right| \le \sum_{j=0}^{q} K_2(q)4^{j-q}n^{-j-q} \le K_1(q)n^{-q}.$$

Lemma 4. *Fix $r \in N_0$. Then*

$$\|M_{n,r}(f;\cdot)\| \leq \sum_{j=0}^{r} \frac{\|f^{(j)}\|}{j!}, \tag{8}$$

for all $f \in C^r_{[0,1]}$ and $n \in N$.

Proof. If $r=0$, then, by (5) and (1), we have

$$|M_{n,0}(f;x)| \leq \|f\|(n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)dt = \|f\|, \qquad \text{for } x\in[0,1], n\in N.$$

If $r \geq 1$, then, by (1), (3)-(6), we immediately get

$$|M_{n,r}(f;x)| \leq \sum_{j=0}^{r}\frac{1}{j!}M_n\left(|f^{(j)}(t)(x-t)^j|;x\right) \leq$$

$$\leq \sum_{j=0}^{r}\frac{1}{j!}\|f^{(j)}\|M_n\left(|x-t|^j;x\right).$$

Since $|x-t|\leq 1$ for $x,t\in[0,1]$, it follows that $M_n\left(|x-t|^j;x\right)\leq 1$ for $x,t\in[0,1]$. From this we obtain

$$|M_{n,r}(f;x)| \leq \sum_{j=0}^{r}\frac{\|f^{(j)}\|}{j!}, \qquad x\in[0,1], n\in N.$$

This completes the proof of (8).

3. Rate of convergence

In this part we shall state some estimates of the rate of convergence of the operators $M_{n,r}$. We shall use the classical modulus of continuity defined by

$$\omega(f;t) := \sup_{|x-y|\leq t,\ x,y\in[0,1]} |f(x)-f(y)|, \qquad t\geq 0. \tag{9}$$

The methods used to prove Lemma 4 and the Theorems are similar to those used in construction of modified Bernstein polynomials [2, 8, 11].

Theorem 1. *Fix $r \in N_0$. Then there exists a positive constant $K_3(r)$ such that*

$$\|M_{n,r}(f;\cdot)-f(\cdot)\| \leq K_3(r)n^{-r/2}\omega\left(f^{(r)};n^{-1/2}\right) \tag{10}$$

for every $f\in C^r_{[0,1]}$ and $n\in N$.

Proof. For $r=0$ the result is well-known (see, for example, [4, Ch. 10, Theorem 8.2]).

Let $f\in C^r_{[0,1]}$ with $r\geq 1$ and let $t\in[0,1]$ be a fixed point. We apply the following modified Taylor's formula

$$f(x) = \sum_{j=0}^{r}\frac{f^{(j)}(t)}{j!}(x-t)^j +$$

$$+\frac{(x-t)^r}{(r-1)!}\int_0^1 (1-u)^{r-1}\left\{f^{(r)}(t+u(x-t))-f^{(r)}(t)\right\}du, \qquad x\in[0,1].$$

From this we derive the following equality from (5) and (6):

$$f(x)=(n+1)\sum_{k=0}^{n}\left(p_{n,k}(x)\int_0^1 p_{n,k}(t)f(x)dt\right)=M_{n,r}(f(t);x) \tag{11}$$

$$+(n+1)\sum_{k=0}^{n}p_{n,k}(x)\cdot$$

$$\cdot\int_0^1 p_{n,k}(t)\left\{\frac{(x-t)^r}{(r-1)!}\int_0^1(1-u)^{r-1}\left\{f^{(r)}(t+u(x-t))-f^{(r)}(t)\right\}du\right\}dt,$$

for $x\in[0,1]$ and $n\in N$. Applying (9) and the inequality $\omega(g;\lambda t)\le(1+\lambda)\omega(g;t)$ for $g\in C_{[0,1]}$ and $\lambda,t\in[0,1]$, we get

$$|f^{(r)}(t+u(x-t))-f^{(r)}(t)|\le\omega\left(f^{(r)};u|x-t|\right)\le$$

$$\le\omega\left(f^{(r)};|x-t|\right)\le\omega\left(f^{(r)};n^{-1/2}\right)\left(1+n^{1/2}|x-t|\right)$$

for $0\le u,x\le 1$ and $n\in N$. This inequality and (11) and (4) imply that

$$|f(x)-M_{n,r}(f(t);x)|\le \tag{12}$$

$$\le\omega\left(f^{(r)};n^{-1/2}\right)(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n,k}(t)\frac{|x-t|^r}{r!}\left(1+n^{1/2}|x-t|\right)dt=$$

$$=\omega\left(f^{(r)};n^{-1/2}\right)\frac{1}{r!}\left\{M_n(|x-t|^r;x)+n^{1/2}M_n\left(|x-t|^{r+1};x\right)\right\}$$

for all $x\in[0,1]$ and $n\in N$. Further, by Hölder inequality and Lemma 3 and (6), we have

$$M_n(|x-t|^q;x)\le\left(M_n(1;x)M_n\left((x-t)^{2q};x\right)\right)^{1/2}= \tag{13}$$

$$=\left(M_n\left((x-t)^{2q};x\right)\right)^{1/2}\le\left(\frac{K_1(q)}{n^q}\right)^{1/2}, \qquad x\in[0,1],\ n,q\in N.$$

Collecting (13) and (12) we obtain

$$|M_{n,r}(f;x)-f(x)|\le K_3(r)n^{-r/2}\omega\left(f^{(r)};n^{-1/2}\right),$$

for $x\in[0,1]$ and $n\in N$. This completes the proof of (10).

From Theorem 1 we derive the following

Corollary. *If* $f\in C^r_{[0,1]}$, $r\in N_0$, *and* $f^{(r)}\in Lip\alpha$ *with* $0<\alpha\le 1$, *i.e.* $\omega\left(f^{(r)};t\right)=O(t^\alpha)$ *for* $t>0$, *then*

$$\|M_{n,r}(f;\cdot)-f(\cdot)\|=O(n^{-(r+\alpha)/2}), \qquad n\in N.$$

Now we shall give the Voronovskaya type theorem.

Theorem 2. *Suppose that $f \in C_{[0,1]}^{r+2}$ with a fixed $r \in N_0$. Then for every $x \in [0,1]$ we have*

$$M_{n,r}(f;x) - f(x) = \frac{(-1)^r f^{(r+1)}(x) M_n\left((t-x)^{r+1};x\right)}{(r+1)!} + \tag{14}$$
$$+\frac{(-1)^r (r+1) f^{(r+2)}(x) M_n\left((t-x)^{r+2};x\right)}{(r+2)!} + o_x\left(\frac{1}{n^{1+r/2}}\right)$$

as $n \to \infty$.

Proof. The assertion (14) for the Bernstein-Durrmeyer operators $M_{n,0}(f)$ and $f \in C_{[0,1]}^2$ is given in [3].

Fix $r \in N$ and $x \in [0,1]$. If $f \in C_{[0,1]}^{r+2}$, then $f^{(j)} \in C_{[0,1]}^{r+2-j}$, $0 \le j \le r$. Hence, for every $f^{(j)}$ we can write Taylor's formula:

$$f^{(j)}(t) = \sum_{i=0}^{r+2-j} \frac{f^{(j+i)}(x)}{i!}(t-x)^i + \varphi_j(t;x)(t-x)^{r+2-j}, \qquad 0 \le j \le r, \tag{15}$$

for $t \in [0,1]$, where $\varphi_j(t) \equiv \varphi_j(t;x)$ is a function such that $\varphi_j(t)t^{r+2-j}$ belongs to $C_{[0,1]}^{r+2-j}$ and $\lim_{t\to x}\varphi_j(t) = 0$. From this we get

$$M_{n,r}(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\sum_{j=0}^{r}\frac{(x-t)^j}{j!}\sum_{i=0}^{r+2-j}\frac{f^{(j+i)}(x)}{i!}(t-x)^i\,dt+ \tag{16}$$
$$+(n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\sum_{j=0}^{r}\frac{(x-t)^j}{j!}\varphi_j(t;x)(t-x)^{r+2-j}\,dt :=$$
$$:= A_{n,r}(x) + B_{n,r}(x), \quad n \in N.$$

We may observe that

$$A_{n,r}(x) =$$
$$= (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\sum_{j=0}^{r}\frac{(x-t)^j}{j!}\sum_{l=j}^{r+2}\frac{f^{(l)}(x)}{(l-j)!}(t-x)^{l-j}\,dt =$$
$$= (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\sum_{j=0}^{r}\frac{(-1)^j}{j!}\sum_{l=j}^{r+2}\frac{f^{(l)}(x)}{(l-j)!}(t-x)^{l}\,dt =$$
$$= (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\sum_{j=0}^{r}\frac{(-1)^j}{j!}\left\{\sum_{l=j}^{r}\frac{f^{(l)}(x)}{(l-j)!}(t-x)^l + \right.$$
$$\left. + \frac{f^{(r+1)}(x)}{(r+1-j)!}(t-x)^{r+1} + \frac{f^{(r+2)}(x)}{(r+2-j)!}(t-x)^{r+2}\right\}dt =$$
$$= (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\sum_{l=0}^{r}\frac{f^{(l)}(x)}{l!}(t-x)^l\sum_{j=0}^{l}\binom{l}{j}(-1)^j dt+$$
$$+\frac{f^{(r+1)}(x)}{(r+1)!}(n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)(t-x)^{r+1}\sum_{j=0}^{r}\binom{r+1}{j}(-1)^j dt+$$

$$+\frac{f^{(r+2)}(x)}{(r+2)!}(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n,k}(t)\,(t-x)^{r+2}\sum_{j=0}^{r}\binom{r+2}{j}(-1)^j dt=$$

$$=f(x)+(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n,k}(t)\sum_{l=1}^{r}\frac{f^{(l)}(x)}{l!}(t-x)^l\sum_{j=0}^{l}\binom{l}{j}(-1)^j dt+$$

$$+\frac{f^{(r+1)}(x)}{(r+1)!}(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n,k}(t)\,(t-x)^{r+1}\sum_{j=0}^{r}\binom{r+1}{j}(-1)^j dt+$$

$$+\frac{f^{(r+2)}(x)}{(r+2)!}(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n,k}(t)\,(t-x)^{r+2}\sum_{j=0}^{r}\binom{r+2}{j}(-1)^j dt$$

for $n \in N$. Using the equalities

$$\sum_{j=0}^{r}\binom{r}{j}(-1)^j=0, \qquad r\in N,$$

$$\sum_{j=0}^{r}\binom{r+1}{j}(-1)^j=(-1)^r,$$

$$\sum_{j=0}^{r}\binom{r+2}{j}(-1)^j=(r+1)(-1)^r$$

and (3), (4) we deduce that

$$A_{n,r}(x)=f(x)+\frac{(-1)^r f^{(r+1)}(x)\,M_n\left((t-x)^{r+1};x\right)}{(r+1)!}+ \tag{17}$$

$$+\frac{(-1)^r(r+1)f^{(r+2)}(x)\,M_n\left((t-x)^{r+2};x\right)}{(r+2)!}, \qquad n\in N.$$

Observe that

$$B_{n,r}(x)=$$

$$(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n,k}(t)\,(t-x)^{r+2}\,\Phi_r(t;x)\,dt=M_n\left((t-x)^{r+2}\Phi_r(t);x\right),$$

for $n \in N$, where

$$\Phi_r(t)\equiv\Phi_r(t;x):=\sum_{j=0}^{r}\frac{(-1)^j}{j!}\varphi_j(t;x), \qquad t\in[0,1],$$

and Φ_r is a function belonging to $C_{[0,1]}$ and $\lim_{t\to x}\Phi_r(t)=\Phi_r(x)=0$. By the Hölder inequality and by Lemma 3, we get

$$|B_{n,r}(x)|\le\left(M_n\left(\Phi_r^2(t);x\right)\right)^{1/2}\left(M_n\left((t-x)^{2r+4};x\right)\right)^{1/2}\le$$

$$\le\left(\frac{K_1(q)}{n^{r+2}}\right)^{1/2}\left(M_n\left(\Phi_r^2(t);x\right)\right)^{1/2}, \qquad n\in N.$$

Since $\Phi_r^2\in C_{[0,1]}$, we have by statement (c)

$$\lim_{n\to\infty}M_n\left(\Phi_r^2(t);x\right)=\Phi_r^2(x)=0.$$

From the above we deduce that

$$B_{n,r}(x) = o_x\left(\frac{1}{n^{1+r/2}}\right), \qquad \text{as } n \to \infty. \tag{18}$$

Collecting (16), (17) and (18) we obtain (14).

Theorem 1 and Theorem 2 in our paper show that the operators $M_{n,r}$, $n \in N$, give better degree of approximation of functions $f \in C^r_{[0,1]}$ than B_n.

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