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Sequence transformations and linear recurrences of higher order

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Abstract: Let $k \geq 2$ and $d \geq 1$ be given integers and denote by $\{G_n\}_{n=0}^{\infty}$ a k-order recursive sequence of integers. In the paper some sequence transformations of $\{G_{n+d}/G_n\}_{n=0}^{\infty}$ are investigated.

Key Words: linear recurrence, characteristic polynomial, dominant root, sequence transformation, quicker convergence

Mathematics Subject Classification: AMS Classification Numbers: 11B39, 65B05

1. Introduction

Let $k, A_0, A_1, \ldots, A_{k-1}$ be given integers with $A_{k-1} \neq 0$ and $k \geq 2$. A linear recursive sequence $\{G_n\}_{n=0}^{\infty}$ of order k is defined by the recursion

$$G_{n+1} = A_0 G_n + A_1 G_{n-1} + \cdots + A_{k-1} G_{n-k+1} \qquad (n \geq k-1), \tag{1}$$

where the initial terms $G_0, G_1, \ldots, G_{k-1}$ are fixed integral numbers with $|G_0| + |G_1| + \cdots + |G_{k-1}| \neq 0$. In the special case $k = 2$, $G_0 = 0$, $G_1 = 1$ and $A_0^2 + 4A_1 > 0$ the terms of the sequence (1) will be denoted by U_n, if $A_0 = A_1 = 1$ also holds, then we get the well-known Fibonacci numbers F_n.

The polynomial

$$p(x) = x^k - A_0 x^{k-1} - A_1 x^{k-2} - \cdots - A_{k-2} x - A_{k-1} \tag{2}$$

is said to be the characteristic polynomial of the sequence $\{G_n\}_{n=0}^{\infty}$, the roots of the equation $p(x) = 0$ are denoted by α_i 's $(1 \leq i \leq k)$. In the sequel we suppose that the root α_1 is simple and of the largest absolute value, that is $|\alpha_1| > |\alpha_2| \geq \ldots \geq |\alpha_k| > 0$ and the multiplicity of α_1 is 1. According to the literature, α_1 is the

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dominant root (see, e. g. [7]). Denote by m_i the multiplicity of the distinct α_i 's $(1 \le i \le l,\ \sum_{i=1}^{l} m_i = k)$. Then the Binet-formula for the term G_n is as follows

$$G_n = a\alpha_1^n + p_2(n)\alpha_2^n + p_3(n)\alpha_3^n + \cdots + p_l(n)\alpha_l^n, \tag{3}$$

where the degree of the polynomial p_i $(2 \le i \le l)$ is less than m_i (see, e. g. [7]). The constant a and the polynomials p_i belong to the ring $\mathbb{Q}(\alpha_1, \alpha_2, \dots, \alpha_l)[x]$ and we suppose that the initial terms are chosen such that $a \ne 0$ in (3).

Let $\{x_n\}_{n=0}^{\infty}$ be a convergent sequence of real numbers with $\lim_{n\to\infty} x_n = x$. Consider a sequence transformation T of $\{x_n\}_{n=0}^{\infty}$ into the sequence $\{T_n\}_{n=0}^{\infty}$, which converges to the same limit x. We say that $\{T_n\}_{n=0}^{\infty}$ converges quicker than $\{x_n\}_{n=0}^{\infty}$ if

$$\lim_{n\to\infty} \frac{T_n - x}{x_n - x} = 0,$$

while if this limit is equal to 1, then the two sequences are said to be asymptotically equal.

In this paper we deal with the shifter $(S^{(s)})-$, the multiplier $(M^{(m)})-$ and the Aitken (A) transformations of $\{x_n\}_{n=0}^{\infty}$, which are defined as follows

$$A(x_n) = \frac{x_{n-1}x_{n+1} - x_n^2}{x_{n-1} - 2x_n + x_{n+1}} \quad (n \ge 1), \tag{4}$$

$$S^{(s)}(x_n) = x_{n+s} \quad (1 \le s \text{ fixed integer}), \tag{5}$$

$$M^{(m)}(x_n) = x_{mn} \quad (1 \le m \text{ fixed integer}). \tag{6}$$

Naturally, we suppose that division by zero never occurs in (4).

2. Preliminaries and Results

At first G. M. Phillips [6] proved that if $r_n = \frac{F_{n+1}}{F_n}$ then $A(r_n) = r_{2n}$. This result was generalized by J. H. McCabe and G. M. Phillips [4] for $r_n = \frac{U_{n+1}}{U_n}$, while by M. J. Jamieson [2] for $r_n = \frac{F_{n+d}}{F_n}$ ($d > 1$ integer). J. B. Muskat [5] proved – among others – that $A(r_n) = r_{2n}$ if $r_n = \frac{U_{n+d}}{U_n}$, from which the quicker convergence obviously follows. Z. Zhang [8], [9] and F. Mátyás [3] proved similar results for a generalized class of the linear recurrences of order 2. R. B. Taher and M. Rachidi [7] investigated the Aitken transformation of the sequence $\left\{\frac{G_{n+1}}{G_n}\right\}_{n=0}^{\infty}$ and they stated – without exact proof – that the sequence $\left\{A\left(\frac{G_{n+1}}{G_n}\right)\right\}_{n=1}^{\infty}$ converges quicker than $\left\{\frac{G_{n+1}}{G_n}\right\}_{n=0}^{\infty}$. But, for a correct proof they would have needed stronger conditions in their Proposition 3.1 in [7] (see the right conditions in (7), (8) and the counter-example after the proof of Theorem 2).

The aim of this paper is to investigate the acceleration of the convergence of the sequences obtained from $\left\{\frac{G_{n+d}}{G_n}\right\}_{n=0}^{\infty}$ ($d \geq 1$ fixed integer) by the transformations $S^{(s)}$, $M^{(m)}$ and A. In the following theorems we always suppose that for the distinct roots of (2)

$$|\alpha_1| > |\alpha_2| > |\alpha_3| \geq |\alpha_4| \geq \cdots \geq |\alpha_l| > 0 \tag{7}$$

or

$$\text{if } |\alpha_1| > |\alpha_2| = |\alpha_3| = \cdots = |\alpha_t| \geq |\alpha_{t+1}| \geq \cdots \geq |\alpha_l| > 0 \quad (3 \leq t \leq l), \tag{8}$$

then among the polynomials p_2, p_3, ..., p_t in (3) the polynomial of maximal degree uniquely exists.

Now we formulate our theorems.

Theorem 1. *The sequence $\{S^{(s)}(G_{n+d}/G_n)\}_{n=0}^{\infty}$ does not converge quicker to the same limit α_1^d than the sequence $\{G_{n+d}/G_n\}_{n=0}^{\infty}$ and the two sequences are not asymptotically equal.*

Theorem 2. *The Aitken sequence transformation of $\{G_{n+d}/G_n\}_{n=0}^{\infty}$ converges quicker to the same limit α_1^d than $\{G_{n+d}/G_n\}_{n=0}^{\infty}$.*

Remark. The relations (7) and (8) show that only the existence of the dominant root α_1 likely is not a sufficient condition for Theorem 2.

Theorem 3. *Let $1 \leq m_1 < m_2$ be fixed integers. The sequence*

$$\{M^{(m_2)}(G_{n+d}/G_n)\}_{n=0}^{\infty}$$

converges quicker to the same limit α_1^d than

$$\{M^{(m_1)}(G_{n+d}/G_n\}_{n=0}^{\infty}.$$

3. Proofs

Firstly we mention two lemmas.

Lemma 1. Let α_1 be the dominant root of (2). Then

$$\lim_{n \longrightarrow \infty} \frac{G_{n+d}}{G_n} = \alpha_1^d.$$

Proof. According to (3),

$$\frac{G_{n+d}}{G_n} = \frac{a\alpha_1^{n+d}\left(1 + \frac{1}{a}\sum\limits_{i=2}^{l} p_i(n+d)\left(\frac{\alpha_i}{\alpha_1}\right)^{n+d}\right)}{a\alpha_1^{n}\left(1 + \frac{1}{a}\sum\limits_{i=2}^{l} p_i(n)\left(\frac{\alpha_i}{\alpha_1}\right)^{n}\right)},$$

which implies that $\lim_{n\longrightarrow\infty}\left(\frac{G_{n+d}}{G_n}\right) = \alpha_1^d$ (since $\left|\frac{\alpha_i}{\alpha_1}\right| < 1$ for $2 \le i \le l$).
We mention that $G_n \neq 0$ if $n > n_0$, thus – shifted the indices – we can suppose that $G_n \neq 0$ for all $n \ge 0$.

Lemma 2. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of real numbers and $\lim_{n\longrightarrow\infty} x_n = x$. If $\lim_{n\longrightarrow\infty} \frac{x_{n+1}-x}{x_n-x} = \rho \neq 1$, then $\{A(x_n)\}_{n=1}^{\infty}$ converges quicker to x than $\{x_n\}_{n=0}^{\infty}$.

Proof. This is a result from [1], see Theorem 32, p. 37.

Proof of Theorem 1. By (5) $\{S^{(s)}(G_{n+d}/G_n)\}_{n=0}^{\infty} = \{G_{n+d+s}/G_{n+s}\}_{n=0}^{\infty}$, which – by Lemma 1 – tends to α_1^d as $n \longrightarrow \infty$. Using (3), one can get that

$$C_n^{(s)} := \frac{G_{n+d+s}/G_{n+s} - \alpha_1^d}{G_{n+d}/G_n - \alpha_1^d} = \frac{G_{n+d+s} - G_{n+s}\alpha_1^d}{G_{n+d} - G_n\alpha_1^d} \cdot \frac{G_n}{G_{n+s}}$$

$$= \frac{\sum_{i=2}^{l} \left(p_i(n+d+s)\alpha_i^{n+d+s} - p_i(n+s)\alpha_i^{n+s}\alpha_1^d\right)}{\sum_{i=2}^{l} \left(p_i(n+d)\alpha_i^{n+d} - p_i(n)\alpha_i^n\alpha_1^d\right)} \cdot \frac{G_n}{G_{n+s}}$$

$$= \frac{p_2(n+d+s)-p_2(n+s)\left(\frac{\alpha_1}{\alpha_2}\right)^d + \sum_{i=3}^{l}\left(\frac{\alpha_i}{\alpha_2}\right)^{n+s}\left(p_i(n+d+s)\left(\frac{\alpha_i}{\alpha_2}\right)^d - p_i(n+s)\left(\frac{\alpha_1}{\alpha_2}\right)^d\right)}{p_2(n+d)-p_2(n)\left(\frac{\alpha_1}{\alpha_2}\right)^d + \sum_{i=3}^{l}\left(\frac{\alpha_i}{\alpha_2}\right)^{n}\left(p_i(n+d)\left(\frac{\alpha_i}{\alpha_2}\right)^d - p_i(n)\left(\frac{\alpha_1}{\alpha_2}\right)^d\right)}$$

$$\cdot \frac{\alpha_2^{n+d+s}}{\alpha_2^{n+d}} \cdot \frac{G_n}{G_{n+s}}.$$

If $l = 2$, then $\lim_{n\longrightarrow\infty} C_n^{(s)} = 1 \cdot \alpha_2^s \cdot \frac{1}{\alpha_1^s} = \left(\frac{\alpha_2}{\alpha_1}\right)^s$, which $\neq 0$ and $\neq 1$ since $|\alpha_1| > |\alpha_2| > 0$.
If $l \ge 3$ and the condition (7) holds, then $\lim_{n\longrightarrow\infty} C_n^{(s)} = 1 \cdot \alpha_2^s \cdot \frac{1}{\alpha_1^s} = \left(\frac{\alpha_2}{\alpha_1}\right)^s$, which differs from 0 and 1.
If $l \ge 3$, the condition (8) holds and the polynomial p_j is of the largest degree among $p_2, p_3, \dots, p_t$ $(3 \le t \le l)$, then $\lim_{n\longrightarrow\infty} C_n^{(s)} = \left(\frac{\alpha_j}{\alpha_2}\right)^s \alpha_2^s \cdot \frac{1}{\alpha_1^s} = \left(\frac{\alpha_j}{\alpha_1}\right)^s$, which $\neq 0$ and $\neq 1$. This terminates the proof.

Proof of Theorem 2. By Theorem 1 – in the case $s = 1$ – the limit

$$\lim_{n\longrightarrow\infty} \frac{G_{n+d+1}/G_{n+1} - \alpha_1^d}{G_{n+d}/G_n - \alpha_1^d}$$

exists and differs from 1. Apply Lemma 2 for $\{x_n\}_{n=0}^{\infty} = \{G_{n+d}/G_n\}_{n=0}^{\infty}$ and the desired result immediately follows.

Connected to the Remark we show that if (8) holds and the polynomial p_j of maximal degree does not exist uniquely, then the limit $\lim_{n\longrightarrow\infty} \frac{G_{n+d+1}/G_{n+1}-\alpha_1^d}{G_{n+d}/G_n-\alpha_1^d}$ may not exist, and so Lemma 2 can not be applied. Consider – as a counter–example – the third order linear recursive sequence

$$G^*_{n+1} = 2G^*_n + G^*_{n-1} - 2G^*_{n-2} \quad (n \geq 2),$$

where $G^*_0 = 1$, $G^*_1 = -2$ and $G^*_2 = 4$. The characteristic polynomial is $p^*(x) = x^3 - 2x^2 - x + 2$, the roots of $p^*(x) = 0$ are $\alpha^*_1 = 2$, $\alpha^*_2 = 1$ and $\alpha^*_3 = -1$. The actual form of (3) is as follows

$$G^*_n = 2^n - 2 \cdot 1^n + 2(-1)^n \quad (n \geq 0).$$

Let e. g. $d = 1$, then

$$\frac{G^*_{n+d+1}/G^*_{n+1} - 2^d}{G^*_{n+d}/G^*_n - 2^d} = \begin{cases} \frac{2^{n+2}/(2^{n+1}-4)-2}{(2^{n+1}-4)/2^n-2} \longrightarrow -1, & \text{if } n = 2f \longrightarrow \infty, \\ \frac{(2^{n+2}-4)/2^{n+1}-2}{2^{n+1}/(2^n-4)-2} \longrightarrow -\frac{1}{4}, & \text{if } n = 2f+1 \longrightarrow \infty. \end{cases}$$

This implies that Lemma 2 can not be applied for the sequence $\{G^*_{n+1}/G^*_n\}_{n=0}^{\infty}$. One can verify with e. g. the MAPLE program–package that, unfortunately, the sequence

$$\{A(G^*_{n+1}/G^*_n)\}_{n=0}^{\infty}$$

does not converge to 2, although naturally $\lim_{n \longrightarrow \infty} (G^*_{n+1}/G^*_n) = 2$. This shows that the existence of the dominant root is not always a sufficient condition for the quicker convergence.

Proof of Theorem 3. Let in (6) $x_n = G_{n+d}/G_n$. Then

$$\{M^{(m)}(G_{n+d}/G_n)\}_{n=0}^{\infty} = \{G_{mn+d}/G_{mn}\}_{n=0}^{\infty}.$$

By Lemma 1, the sequences $\{G_{m_1 n+d}/G_{m_1 n}\}_{n=0}^{\infty}$ and $\{G_{m_2 n+d}/G_{m_2 n}\}_{n=0}^{\infty}$ tend to α_1^d as n tends to infinity. Using (3), we get that

$$M_n^{(m_1,m_2)} := \frac{G_{m_2 n+d}/G_{m_2 n} - \alpha_1^d}{G_{m_1 n+d}/G_{m_1 n} - \alpha_1^d} = \frac{G_{m_2 n+d} - \alpha_1^d G_{m_2 n}}{G_{m_1 n+d} - \alpha_1^d G_{m_1 n}} \cdot \frac{G_{m_1 n}}{G_{m_2 n}}$$

$$= \frac{\sum_{i=2}^{l} \left(p_i(m_2 n + d)\alpha_i^{m_2 n+d} - p_i(m_2 n)\alpha_i^{m_2 n}\alpha_1^d\right)}{\sum_{i=2}^{l} \left(p_i(m_1 n + d)\alpha_i^{m_1 n+d} - p_i(m_1 n)\alpha_i^{m_1 n}\alpha_1^d\right)} \cdot \frac{a\alpha_1^{m_1 n} + \sum_{i=2}^{l} p_i(m_1 n)\alpha_i^{m_1 n}}{a\alpha_1^{m_2 n} + \sum_{i=2}^{l} p_i(m_2 n)\alpha_i^{m_2 n}}$$

$$= \left(\frac{\alpha_2}{\alpha_1}\right)^{(m_2-m_1)n} \frac{\sum_{i=2}^{l} \left(\frac{\alpha_i}{\alpha_2}\right)^{m_2 n} \left[p_i(m_2 n + d)\left(\frac{\alpha_i}{\alpha_2}\right)^d - p_i(m_2 n)\left(\frac{\alpha_1}{\alpha_2}\right)^d\right]}{\sum_{i=2}^{l} \left(\frac{\alpha_i}{\alpha_2}\right)^{m_1 n} \left[p_i(m_1 n + d)\left(\frac{\alpha_i}{\alpha_2}\right)^d - p_i(m_1 n)\left(\frac{\alpha_1}{\alpha_2}\right)^d\right]}$$

$$\cdot \frac{1+\frac{1}{a}\sum_{i=2}^{l} p_i(m_1 n)\left(\frac{\alpha_i}{\alpha_1}\right)^{m_1 n}}{1+\frac{1}{a}\sum_{i=2}^{l} p_i(m_2 n)\left(\frac{\alpha_i}{\alpha_1}\right)^{m_2 n}}.$$

Discuss the same cases as we have done in the proof of Theorem 1, then using the inequality $|\frac{\alpha_i}{\alpha_1}| < 1 \quad (2 \le i \le l)$, one can obtain that

$$\lim_{n \longrightarrow \infty} M_n^{(m_1,m_2)} = 0,$$

that is the statement of the theorem has been proved.

Concluding remarks 1. It can be seen that our theorems are valid if the sequence $\{G_n\}_{n=0}^{\infty}$ consists of real or complex element.

2. Numerical examples show that in general $A(G_{n+d}/G_n) \neq M^{(2)}(G_{n+d}/G_n)$. But it would be worth investigating whether the sequences

$$\{A(G_{n+d}/G_n)\}_{n=0}^{\infty} \text{ and } \{M^{(2)}(G_{n+d}/G_n)\}_{n=0}^{\infty}$$

are asymptotically equal or not. Similar questions arise with the secant- the Newton- and the Halley-transformations of the sequence $\{G_{n+d}/G_n\}_{n=0}^{\infty}$, which may be the subject of further investigations.

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