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Inner Product in l-Groups

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Abstract: We investigate a new conception of an inner product on lattice ordered groups. The inner product is motivated with a scalar product of vectors in vector spaces. Basic and characteristic properties of the inner product are described.

Key Words: inner product, lattice ordered groups

Mathematics Subject Classification: 06F15

A scalar product of vectors has a basic part in the theory of vector spaces. Vector spaces together with lattice ordered groups (briefly l-groups) form vector lattices (see [1]). Let us investigate in this paper so called an inner product on l-groups that is motivated with the scalar product of vectors without using of the structure of vector lattices.

1. Motivation. The formula $(\mathbf{u},\mathbf{v}) = \|\mathbf{u}/2+\mathbf{v}/2\|^2 - \|\mathbf{u}/2-\mathbf{v}/2\|^2$ holds for the scalar product of vectors $\mathbf{u},\mathbf{v}$ from a real vector space. If we rewrite the right side of this formula for vector lattices (which are abelian l-groups) similarly such that we substitute $\|\mathbf{u}\|^2$ (i.e., the square of the lenght of the vector $\mathbf{u}$) with $|\mathbf{u}| = \mathbf{u} \vee -\mathbf{u}$ (i.e., the absolute value of $\mathbf{u}$) then we obtain $|\mathbf{u}/2+\mathbf{v}/2| - |\mathbf{u}/2-\mathbf{v}/2| = [(\mathbf{u}/2+\mathbf{v}/2) \vee -(\mathbf{u}/2+\mathbf{v}/2)] - [(\mathbf{u}/2-\mathbf{v}/2) \vee -(\mathbf{u}/2-\mathbf{v}/2)] = \{[(\mathbf{u}/2+\mathbf{v}/2) \vee (-\mathbf{u}/2-\mathbf{v}/2)] + (\mathbf{v}/2 - \mathbf{u}/2)\} \wedge \{[(\mathbf{u}/2+\mathbf{v}/2) \vee (-\mathbf{u}/2-\mathbf{v}/2)] + (\mathbf{u}/2-\mathbf{v}/2)\} = (\mathbf{v} \vee -\mathbf{u}) \wedge (\mathbf{u} \vee -\mathbf{v})$.

Now we can define an inner product of l-groups.

2. Definition. Let $(G, +, \vee, \wedge)$ be an l-group and $x, y \in G$. Then *an inner product $x.y$ of elements x,y* is $x.y = (x \vee -y) \wedge (y \vee -x)$.

We want to analyze the inner product without the assumption of commutatitivity of an l-group G.

3. Remarks. 1. We have $x.x \geq 0, x.y = y.x$ and $x.x = 0 \Leftrightarrow x = 0$, for $x, y \in G$.

2. K. L. M. Swamy [4] and T. Kovář [3] investigated so called autometrics in a commutative l-group G. The standard autometric has the form $\rho(x, y) = |x - y|$, for $x, y \in G$ and it is in connection with the inner product such that $\rho(x, y) = (x - y).(x - y)$.

3. No unit element e exists in an l-group G with respect to inner product. Namely, if $x = x.e = (x \vee -e) \wedge (-x \vee e) = (x \wedge -x) \vee (-e \wedge -x) \vee (x \wedge e) \vee (-e \wedge \wedge e)$ then $-|e| \leq x$, for any $x \in G$. That is a contradiction with the fact that the smallest element does not exists in G.

4. **Proposition.** *If G is an l-group and $x, y, z \in G$ then it holds:*
a) $x.y \geq x \wedge y, |x|.|y| = |x| \wedge |y|$,
b) $|x| = x.x, -|x| = x.(-x), x.|x| = x$,
c) $x^+.x^- = 0, 0.x = x.0 = 0$,
d) $-z + x.y + z = (-z + x + z).(-z + y + z)$.

Proof. a),b) follow directly from the definition 2.
c) We have $x^+.x^- = (x^+ \vee -x^-) \wedge (x^- \vee -x^+) = x^- \vee -x^+ = -(x^+ \wedge -x^-) = = 0, 0.x = (0 \vee -x) \wedge (0 \vee x) = x^+ \wedge -x^- = 0$ and $x.0 = 0$ similarly.
d) $-z + x.y + z = -z + [(x \vee -y) \wedge (-x \vee y)] + z = [(-z + x + z) \vee (-z - y + + z)] \wedge [(-z + y + z) \vee (-z - x + z)] = (-z + x + z).(-z + y + z)$.

5. **Lemma.** *Let G be an l-group and $x, y \in G$. Then*

$$x + y = (x \vee y) + (x \wedge y) \Leftrightarrow (-x + y)^+ = (y - x)^+.$$

Proof. We have $(x \vee y) + (x \wedge y) = [2x \vee (y + x)] \wedge [(x + y) \vee 2y]$ and thus $x + y = = (x \vee y) + (x \wedge y) \Leftrightarrow 0 = -x + \{[(2x \vee (y + x)] \wedge [(x + y) \vee 2y]\} - y = [(x - y) \vee (-x + + y + x - y)] \wedge [0 \vee (-x + y)] = \{[0 \vee (-x + y)] + (x - y)\} \wedge [0 \vee (-x + y)] = [0 \vee (-x + + y)] + [0 \wedge (x - y)] = (-x + y)^+ + (x - y)^- \Leftrightarrow (-x + y)^+ = -(x - y)^- = (y - x)^+$.

6. **Proposition.** *If G is an l-group, $x, y \in G$ and $(x + y)^- = (y + x)^-$ then $|x - y| = = |x| \vee |y| - x.y$ holds.*

Proof. The proposition N,[2],p.113 implies that $|x| \vee |y| - x.y = (x \vee -y) \vee (-x \vee \vee y) - [(x \vee -y) \wedge (-x \vee y)] = |(x \vee -y) - (-x \vee y)| = |(x \vee -y) + (x \wedge -y)| = |x - y|$. Namely, $(-x - y)^+ = (-y - x)^+$ and we have $x - y = (x \vee -y) + (x \wedge -y)$, see Lemma 5.

7. **Proposition.** *If G is an l-group and $x, y \in G$ then $|x.y| = |x| \wedge |y| = |x|.|y|$ holds.*

Proof. We have $|x.y| = [(x \vee -y) \wedge (-x \vee y)] \vee -[(x \vee -y) \wedge (-x \vee y)] = [(x \vee -y) \wedge \wedge (-x \vee y)] \vee [(-x \wedge y) \vee (x \wedge -y)] = (x \wedge -x) \vee (x \wedge y) \vee (-y \wedge -x) \vee (-y \wedge y) \vee (-x \wedge \wedge y) \vee (x \wedge -y) = -|y| \vee -|x| \vee [x \wedge (y \vee -y)] \vee [-x \wedge (y \vee -y)] = -(|x| \wedge |y|) \vee (x \wedge |y|) \vee \vee (-x \vee |y|) = -(|x| \wedge |y|) \vee [(x \vee -x) \wedge |y|] = -(|x| \wedge |y|) \vee (|x| \wedge |y|) = |x| \wedge |y| = |x|.|y|$, see 4.a.

8. **Corollary.** *If G is an l-group and $x, y \in G$ then $-(x.y) = x.(-y) = (-x).y$ hold.*

Proof. First, $(-x).y = (-x \vee -y) \wedge (x \vee y) = x.(-y)$ and further $(-x.y) = (-x \vee \vee -y) \wedge (x \vee y) = (x \wedge -x) \vee (y \wedge -y) \vee (-x \wedge y) \vee (x \wedge -y) = -|x| \vee -|y| \vee -[(x \vee \vee -y) \wedge (y \vee -x)] = -(|x| \wedge |y| \wedge x.y) = -(x.y)$ hold.

Recall, that elements x,y of an l-group G are *orthogonal* when $|x| \wedge |y| = 0$. Let us denote $x\delta y$.

9. Corollary. *Let G be an l-group and $x, y \in G$. Then it holds:*
1. *Elements x,y are orthogonal if and only if $x.y = 0$.*
2. *If G is commutative then $x.y = 0 \Leftrightarrow |x| + |y| = |x + y| = |x - y|$.*

Proof. The part 1. follows immediately from the proposition 7.

2.$\Rightarrow$: Propositions 6. and 8. imply $|x + y| = |x - (-y)| = |x| \vee |-y| - x.(-y) = |x| \vee |y| + x.y = |x| \vee |y|$ and also $|x - y| = |x| \vee |y| - x.y = |x| \vee |y| = |x| + |y|$.

$\Leftarrow$: Similarly, we have $|x| \vee |y| - x.y = |x - y| = |x + y| = |x| \vee |y| + x.y$ and thus $2(x.y) = 0$ and $x.y = 0$, because G is a torsion free group.

10. Proposition. *Let G be an l-group and $x, y \in G$. Then for the following propositions*

(i) $x.y \leq 0$,
(ii) $x \wedge y \leq 0 \leq x \vee y$,
(iii) $x \wedge y \leq x + y \leq x \vee y$,
(iv) $|x - y| \geq |x + y|$,

it holds (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv). If moreover, G is commutative then also (iv) $\Rightarrow$ (i) holds.

Proof. (i) $\Leftrightarrow$ (ii) : We have $0 \geq x.y = (x \vee -y) \wedge (-x \vee y) = (x \wedge -x) \vee (x \wedge y) \vee (-y \wedge -x) \vee (-y \wedge y) = -|x| \vee -|y| \vee (x \wedge y) \vee -(x \vee y) \Leftrightarrow x \wedge y \leq 0 \leq x \vee y$.

(ii) $\Leftrightarrow$ (iii) : In an l-group G it holds $x \vee y = x - (x \wedge y) + y$ and further $x \vee y = y \vee x = y - (x \wedge y) + x, x \wedge y = y - (x \vee y) + x$. These facts follow $x \vee y \geq 0 \geq x \wedge y \Leftrightarrow y - (x \wedge y) + x \geq 0 \geq y - (x \vee y) + x \Leftrightarrow -(x \wedge y) \geq -y - x \geq -(x \vee y) \Leftrightarrow x \vee y \geq x + y \geq x \wedge y$.

(iii) $\Rightarrow$ (iv) : The previous facts imply $x \wedge y \leq 0 \leq x \vee y$. Therefore we have $x \wedge y \leq (x+y)^+ \leq x \vee y$ and $x \wedge y \leq (x+y)^- \leq x \vee y$, i.e., $-(x \vee y) \leq -(x+y)^- \leq -(x \wedge y)$. Therefore we obtain $|x + y| = (x + y)^+ - (x + y)^- \leq (x \vee y) - (x \wedge y) = |x - y|$, see [2] ,p.113,N.

(iv) $\Rightarrow$ (i) : If G is commutative then the propositions 6. and 8. imply $|x - y| = |x| \vee |y| - x.y$ and $|x + y| = |x| \vee |y| + x.y$. If (iv) is true then $-x.y \geq x.y$, i.e., $2(x.y) \leq 0$ and $x.y \leq 0$, because G is torsion free.

11. Proposition. *If G is an l-group and $x, y \in G$ then $|x| \wedge |y| = (x \wedge y) \vee -(x.y) \vee -(x \vee y)$ holds.*

Proof. We have $|x| \wedge |y| = (x \vee -x) \wedge (y \vee -y) = (x \wedge y) \vee [(x \wedge -y) \vee (-x \wedge y)] \vee (-x \wedge -y) = (x \wedge y) \vee -(x.y) \vee -(x \vee y)$.

12. Definition. *A .-ideal I in an l-group G* is a subgroup in G fulfilling the condition: $x \in I, g \in G \Rightarrow g.x \in I$.

13. Proposition. *Let G be an l-group. Then I is a .-ideal in G if and only if I is a convex l-subgroup in G.*

Proof. $\Rightarrow$: If $x \in I, g \in G, 0 \leq |g| \leq |x|$ then $|g| = |g| \wedge |x| = |g|.|x| \in I$ (see 7.), because $|x| = x.x \in I$. We have $g = g.|g| \in I$ and together I is a convex subgroup in G. For $x, y \in I$ it holds $0 \leq |x \wedge y| \leq |x| \wedge |y| = |x|.|y| \in I$. This fact implies $x \wedge y \in I$ and I is also an l-subgroup in G.

$\Leftarrow$: With respect to 7. it holds $0 \leq |g.x| = |g| \wedge |x| \leq |x|$ and thus $g.x \in I$.

14. Proposition. *If G is a commutative l-group and $x, y, z \in G^+$. Then it holds:*

a) $(x+y) \wedge z \leq (x \wedge z) + (y \wedge z)$,

b) $(x+y) \wedge z = (x \wedge z) + (y \wedge z) \Leftrightarrow (x+y-z) \wedge x \wedge y \wedge z \leq 0$,

c) $x \wedge y \wedge z = 0 \Rightarrow (x+y).z = x.z + y.z$,

d) $x \wedge y = 0 \Rightarrow (kx).y = k(x.y) = x.(ky)$, *for any integer number k.*

Proof. a) We have $(x+y) \wedge z \leq (x+y) \wedge [(x \wedge y \wedge z) + z] = (x+y) \wedge (x+z) \wedge (y+z) \wedge 2z = (x \wedge z) + (y \wedge z)$.

b) From the part a) of the proof it follows: $(x+y) \wedge z = (x \wedge z) + (y \wedge z) \Rightarrow (x+y) \wedge z = (x+y) \wedge [(x \wedge y \wedge z) + z] \Rightarrow (x+y-z) \wedge 0 = (x+y-z) \wedge (x \wedge y \wedge z)$. On the contrary $0 \wedge (x+y-z) = 0 \wedge (x+y-z) \wedge (x \wedge y \wedge z) = (x+y-z) \wedge (x \wedge y \wedge z) \Rightarrow (x+y) \wedge z = (x+y) \wedge [(x \wedge y \wedge z) + z] = (x \wedge z) + (y \wedge z)$.

c) The previous part of the proof and 7. implies $x.z + y.z = (x \wedge z) + (y \wedge z) = (x+y) \wedge z = (x+y).z$.

d) The proposition follows for $k > 0$ from b; for k=0 follows from 4.c and for $k < 0$ we have $(kx).y = (-|k|x).y = -[(|k|x).y] = -|k|(x.y) = k(x.y)$, see 8.

15. Corollary. *If G is an l-group then $(|x| + |y|).|z| \leq |x|.|z| + |y|.|z|$ holds for $x, y, z \in G$.*

Proof. Proof follows from 7. and 14.a.

16. Theorem. *1. If G is an l-group then for $x, y, z \in G^+ \cup G^-$ it holds:*

a) $x.y = sgn x.\, sgn y(|x| \wedge |y|)$, *where* $sgn x = 1$ *for* $0 \neq x \in G^+$, $sgn x = -1$ *for* $0 \neq x \in G^-$ *and* $sgn 0 = 0$.

b) $x.(y.z) = (x.y).z$,

c) $|(x+y).z| \leq |x.z| + |y.z| + |x.z|$.

2. If G is a representable l-group then parts b and c hold for $x, y, z \in G$.

3. If G is a commutative l-group then $|(x+y).z| \leq |x.z| + |y.z|$ holds for $x, y, z \in G$.

Proof. 1a: We have $x.y = (x \vee -y) \wedge (-x \vee y) = x \wedge y = |x| \wedge |y|$ for $x, y \in G^+$, $x.y = (x \vee -y) \wedge (-x \vee y) = -y \wedge -x = |x| \wedge |y|$ for $x, y \in G^-$ and $x.y = (x \vee -y) \wedge (-x \vee y) = -x \vee y = -(x \wedge -y) = -(|x| \wedge |y|)$ for $x \in G^+, y \in G^-$.

1b: We have $x.(y.z) = x.[\operatorname{sgn} y.\operatorname{sgn} z)(|y| \wedge |z|)] = [\operatorname{sgn} x.(\operatorname{sgn} y.\operatorname{sgn} z)].[|x| \wedge (|y| \wedge |z|)] = [(\operatorname{sgn} x.\operatorname{sgn} y).\operatorname{sgn} z].[(|x| \wedge |y|) \wedge |z|] = [(\operatorname{sgn} x.\operatorname{sgn} y).(|x| \wedge |y|)].z = (x.y).z$.

1c: Propositions 7.,14.a and [2] ,p.112,I imply $|(x+y).z| = |x+y| \wedge |z| \leq (|x| + |y| + |x|) \wedge |z| \leq (|x| \wedge |z|) + (|y| \wedge |z|) + (|x| \wedge |z|) = |x.z| + |y.z| + |x.z|$.

2. Let us recall that a representable group G is l-isomorphic with an l-subgroup of a direct product of linearly ordered groups $G_i (i \in I)$. Then for every $i \in I$ it holds $|x.y|_i = |(x.y)_i| = |x_i.y_i| = |x_i|.|y_i| = |x|_i.|y|_i = (|x|.|y|)_i$, see 7. The parts b and c we can prove similarly as in the part 1.

3. A commutative l-group is a representable l-group and $|x+y| \leq |x| + |y|$ holds. These facts imply $|(x+y).z| = |x+y|.|z| = |x+y| \wedge |z| \leq (|x| + |y|) \wedge |z| \leq |x| \wedge |z| + |y| \wedge |z| = |x|.|z| + |y|.|z| = |x.z| + |y.z|$, see 7. and 14.a.

17. Remark. The inequality 16.c does not hold without absolute values. For example, for $x \geq 0, x \neq 0, y = z = -x$ it is $0 = (x+y).z$ and $x.z + y.z + x.z = x.(-x) + (-x).(-x) + x.(-x) = -x$.

18. Corollary. *If G is an l-group and $x, z, u, v \in G$ then it holds:* $x\delta y \Rightarrow x.u\delta y.v$, $x\delta y.v$.

Proof. We have $x\delta y \Rightarrow |x.u| \wedge |y.v| = (|x| \wedge |u|) \wedge (|y| \wedge |v|) = |u| \wedge (|x| \wedge |y|) \wedge |v| = 0$. Therefore $x.u\delta y.v$ holds and the second formula follows similarly.

19. Proposition. *If G is an l-group and $x, y \in G$ then it holds:*

a) $x^+.y^+ = (x \wedge y)^+, x^-.y^- = (-x \wedge -y)^+, x^+.y^- = (-x \vee y)^-$, $x^-.y^+ = (x \vee -y)^-$,

b) $(x.y)^+ = (x^+.y^+) \vee (x^-.y^-) = (x^+.y^+) + (x^-.y^-)$, $(x.y)^- = (x^+.y^-) \wedge (x^-.y^+) = (x^+.y^-) + (x^-.y^+)$,

c) $x.y = x^+.y^+ + x^-.y^- + x^+.y^- + x^-.y^+$.

Proof. a) Theorem 16.1 implies $x^+.y^+ = x^+ \wedge y^+ = (x \wedge y)^+, x^-.y^- = |x^-| \wedge |y^-| = -(x \wedge 0) \wedge -(y \wedge 0) = (-x \wedge -y)^+, x^+.y^- = -[x^+ \wedge (-y^-)] = -[(x \vee 0) \wedge (-y \vee 0)] = -[(x \wedge -y) \vee 0] = (-x \vee y)^-, x^-.y^+ = -[-(x \wedge 0) \wedge (y \vee 0)] = -[(-x \wedge y) \vee 0] = (x \vee -y)^-$.

b) First it holds $(x.y)^+ = [(x \vee -y) \wedge (-x \vee y)] \vee 0 = [(x \vee -y) \vee 0] \wedge [(-x \vee y) \vee 0] = [(x \vee 0) \vee (-y \vee 0)] \wedge [(-x \vee 0) \vee (y \vee 0)] = (x^+ \vee -y^-) \wedge (-x^- \vee y^+) = (x^+ \wedge -x^-) \vee (x^+ \wedge y^+) \vee (-y^- \wedge -x^-) \vee (-y^- \wedge y^+) = 0 \vee (x^+ \wedge y^+) \vee (-x^- \wedge -y^-) = 0 \vee (x \wedge y) \vee (-x \wedge -y) = (x \wedge y)^+ \vee (-x \wedge -y)^+ = x^+.y^+ \vee x^-.y^-$ and also $(x.y)^- = [(x \vee -y) \wedge (-x \vee y)] \wedge 0 = (x \vee -y)^- \wedge (-x \vee y)^- = x^-.y^+ \wedge x^+.y^-$.

Further, it holds $x^+.y^+ + x^-.y^- = (x \wedge y) \vee 0 + (-x \wedge -y) \vee 0 = [(x \wedge y) + (-x \wedge -y)] \vee (-x \wedge -y) \vee (x \wedge y) \vee 0 = [0 \wedge (y - x) \wedge (x - y)] \vee [(-x \vee x) \wedge (x \vee -y) \wedge (-x \vee y) \wedge (y \vee -y)] \vee 0 = |x| \wedge |y| \wedge (x.y)^+ = (x.y)^+$, see 7.and also $x^+.y^- + x^-.y^+ = (-x \vee y) \wedge 0 + (x \vee -y) \wedge 0 = [(-x \vee y) + (x \vee -y)] \wedge [(-x \vee y) \wedge (x \vee -y)] \wedge 0 = [0 \vee (y + x) \vee (-x - y)] \wedge (x.y)^- = (x.y)^-$.

c) Finally, it holds $x.y = (x.y)^+ + (x.y)^- = x^+.y^+ + x^-.y^- + x^+.y^- + x^-.y^+$.

20. Corollary. *Let G be an l-group and $x, y \in G$. Then $x^+.y^+, x^-.y^-$ are orthogonal elements and also $x^+.y^-, x^-.y^+$ are orthogonal elements.*

Proof. Proof follows from 18.

21. Proposition. *If G is an l-group and $x, y, z \in G^+ \cup G^-$ then it holds:*

a) $x \geq 0 \Rightarrow x.(y \vee z) = x.y \vee x.z, x.(y \wedge z) = x.y \wedge x.z$,
$x \leq 0 \Rightarrow x.(y \wedge z) = x.y \vee x.z, x.(y \vee z) = x.y \wedge x.z$,

b) $x \geq 0, y \wedge z = 0 \Rightarrow x.(y + z) = x.y + x.z$,
$x \geq 0, y \vee z = 0 \Rightarrow x.(y + z) = x.y + x.z$,
$x \leq 0, y \wedge z = 0 \Rightarrow x.(y + z) = x.y + x.z$,
$x \leq 0, y \vee z = 0 \Rightarrow x.(y + z) = x.y + x.z$.

Proof. Let us discuss all cases with using of 16.1 and [2],p.102,c:

a) First,if $x \geq 0$ then it holds:

(i) for $y, z \geq 0$ it is $x.(y \vee z) = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = x.y \vee x.z, x.(y \wedge z) = x \wedge (y \wedge z) = (x \wedge y) \wedge (x \wedge z) = x.y \wedge x.z$,

(ii) for $y \geq 0, z \leq 0$ it is $x.(y \vee z) = x \wedge (y \vee z) = x \wedge y = (x \wedge y) \vee -(x \wedge -z) = x.y \vee x.z, x.(y \wedge z) = -(x \wedge -z) = (x \wedge y) \wedge -(x \wedge -z) = x.y \wedge x.z$,

(iii) for $y \leq 0, z \geq 0$ it is $x.(y \vee z) = x \wedge z = -(x \wedge -y) \vee (x \wedge z) = x.y \vee x.z, x.(y \wedge z) = -(x \wedge -y) = -(x \wedge -y) \wedge (x \wedge z) = x.y \wedge x.z$,

(iv) for $y \leq 0, z \leq 0$ it is $x.(y \vee z) = -[x \wedge -(y \vee z)] = -x \vee (y \vee z) = (-x \vee \vee y) \vee (-x \vee z) = -(x \wedge -y) \vee -(x \wedge -z) = x.y \vee x.z, x.(y \wedge z) = -[x \wedge -(y \wedge z)] = = -x \vee (y \wedge z) = (-x \vee y) \wedge (-x \vee z) = -(x \wedge -y) \wedge -(x \wedge -z) = x.y \wedge x.z$.

If $x \leq 0$ then $x.(y \wedge z) = -[(-x).(y \wedge z)] = -[(-x).y \wedge (-x).z] = x.y \vee x.z$ and $x.(y \vee z) = -[(-x).(y \vee z)] = -[(-x).y \vee (-x).z] = x.y \wedge x.z$ hold, see 8.

b) If $x \geq 0, y \wedge z = 0$ then $x.y \wedge x.z = x \wedge y \wedge z = 0$ holds and thus we have $x.(y + z) = x.(y \vee z) = x.y \vee x.z = x.y + x.z$.

If $x \geq 0, y \vee z = 0$ then $x.y \vee x.z = -(x \wedge -y) \vee -(x \wedge -z) = -x \vee y \vee z = 0$ and thus we have $x.(y + z) = x.(y \wedge z) = x.y \wedge x.z = x.y + x.z$.

If $x \leq 0, y \wedge z = 0$ then $x.y \vee x.z = -(-x \wedge y) \vee -(-x \wedge z) = x \vee -y \vee -z = x \vee \vee -(y \wedge z) = x \vee 0 = 0$ and thus we have $x.(y + x) = x.(y \vee z) = x.y \wedge x.z = x.y + x.z$.

If $x \leq 0, y \vee z = 0$ then $x.y \wedge x.z = -x \wedge -y \wedge -z = -x \wedge -(y \vee z) = -x \wedge 0 = 0$ and thus we have $x.(y + z) = x.(y \wedge z) = x.y \vee x.z = x.y + x.z$.

22. Theorem. *If G is an l-group and $x, y, z \in G$ then $x.(y.z) = (x.y).z$ holds.*

Proof. Propositions 16, 19, 20 and 21 imply
$x.(y.z) = x^+.(y.z)^+ + x^-.(y.z)^- + x^+.(y.z)^- + x^-.(y.z)^+ = x^+.(y^+.z^+ + + y^-.z^-) + x^-.(y^+.z^- + y^-.z^+) + x^+.(y^+.z^- + y^-.z^+) + x^-.(y^+.z^+ + y^-.z^-) = = x^+.(y^+.z^+) + x^+.(y^-.z^-) + x^-.(y^+.z^-) + x^-.(y^-.z^+) + x^+.(y^+.z^-) + x^+.(y^-.z^+) + + x^-.(y^+.z^+) + x^-.(y^-.z^-) = (x^+.y^+).z^+ + (x^+.y^-).z^- + (x^-.y^+).z^- + (x^-.y^-).z^+ + + (x^+.y^+).z^- + (x^+.y^-).z^+ + (x^-.y^+).z^+ + (x^-.y^-).z^- + = (x^+.y^+).z^+ + + (x^-.y^-).z^+ + (x^+.y^+).z^- + (x^-.y^-).z^- + (x^+.y^-).z^+ + (x^-.y^+).z^+ + (x^+.y^-).z^- + + (x^-.y^+).z^- = (x^+.y^+ + x^-.y^-).z^+ + (x^+.y^+ + x^-.y^-).z^- + (x^+.y^- + x^-.y^+).z^+ + + (x^+.y^- + x^-.y^+).z^- = (x.y)^+.z^+ + (x.y)^+.z^- + (x.y)^-.z^+ + (x.y)^-.z^- = (x.y).z$.

Let us remark that Proposition 7 implies $|x| \leq |y| \Leftrightarrow |x.y| = |x|$. Now, we shall investigate a similar relation introduced in the following definition.

23. Definition. Let G be an l-group and $x, y \in G$. Then let us define *a relation* [*on* G such that $x[y \Leftrightarrow x.y = x$.

24. Proposition. *If $(G, \leq)$ is a commutative l-group then* [*is an antisymmetric and transitive relation on G with following properties:*

a) The restriction $[/G^+$ of the relation [on G^+ is the lattice order on G^+,

b) xnon $\| 0, y \leq 0, x[y \Rightarrow x = 0$, for $x, y \in G$,

c) $0[x$, for $x \in G$,

d) [is reflexive exactly on G^+.

Proof. First, we shall prove that [is antisymmetric and transitive:

$x[y, y[x \Rightarrow x = x.y = y.x = y$ and $x[y, y[z \Rightarrow x = x.y, y = y.z \Rightarrow x = x.y = = x.(y.z) = (x.y).z = x.z \Rightarrow x[z$, for $x, y, z \in G$.

Further, we have:

a) $x[y \Leftrightarrow x = x.y = x \wedge y \Leftrightarrow x \leq y$, for $x, y \in G^+$,

b) if $x \geq 0, y \leq 0$ then $x[y \Leftrightarrow 0 \leq x = x.y = -(x \wedge -y) = -x \vee y \leq 0 \Leftrightarrow x = 0$ and if $x \leq 0, y \leq 0$ then $x[y \Leftrightarrow 0 \geq x = x.y = -x \wedge -y = -(x \vee y) \geq 0 \Leftrightarrow x = 0$ hold (see Theorem 16),

c) $0 = 0.x \Leftrightarrow 0[x$, for $x \in G$,

d) $x[x \Leftrightarrow x = x.x = |x| \Leftrightarrow x \geq 0$, for $x \in G$.

References

[1] Birkhoff G., *Lattice theory*, Amer. Math. Soc. Providence, R.I., 1973.

[2] Fuchs L., *Partially ordered algebraic systems*, Moscow, Mir, 1965.

[3] Kovář T., *On normal autometrics in commutative lattices ordered groups*, Discuss. Math. (to apper).

[4] Swamy K. L. M., *A general theory of autometrized algebras*, Math. Ann. **157** (1964), 65–74.

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