Hans-Joachim Böckenhauer

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# Two Open Problems in Communication in Edge-Disjoint Paths Modes

*Hans-Joachim Böckenhauer*

**Abstract:** Two open problems in communication in edge-disjoint paths modes are solved. The results achieved are the following:

1. The complete binary tree of height $h$ has the gossip complexity $2 \cdot h$ in the one-way listen-in edge-disjoint paths mode.

2. A class of graphs is constructed such that every graph of $n$ nodes in this class has (maximum) broadcast complexity $\lceil \log_2 n \rceil + 2$. This result shows that the upper bound on the (maximum) broadcast complexity in this communication mode, as it was shown in [8], is tight.

**Key Words:** communication algorithms, parallel computations

**Mathematics Subject Classification:** 90B12, 68Q22

## 1. Introduction and Definitions

The study and the comparison of the computational power of distinct interconnection networks as candidates for the use as parallel architectures for existing parallel computers is an intensively investigated research branch of current theory of parallel computing. One of the fundamental approaches helping to search for the best (most effective) structures of interconnection networks is the study of the communication facilities of networks (i.e., of the complexity (effectivity) of solving fundamental communication tasks of information dissemination).

Some of the basic communication tasks are broadcast, accumulation and gossip (an overview of the study of their complexity according to one-way and two-way communication modes can be found in [3], [4], [5]).

*Broadcast*, *accumulation*, and *gossip* can be described as follows. Assume that each vertex (processor) in a graph (network)$G$ has some piece of information. The *cumulative message* $I(G)$ of $G$ is the set of all pieces of information originally distributed in all vertices of $G$. To solve the *broadcast* [*accumulation*] problem for a given graph $G$ and a vertex $u$ of $G$, we have to find a communication strategy (using the edges of $G$ as communication links) such that all vertices in $G$ learn the piece of information residing in $u$ [that $u$ learns the cumulative message of $G$]. To solve the *gossip* problem for a given graph $G$, a communication strategy such that all vertices in $G$ learn the cumulative message of $G$ must be found. Since the above stated communication problems are solvable only in connected graphs, we note that from now on we use the notion "graph" for connected undirected graphs.

The meaning of a "communication strategy" depends on the communication mode. A communication strategy is realized by a *communication algorithm* consisting of a number of *communication steps (rounds)*. The rules describing what can happen in one communication step (round) are defined exactly by the communication mode. Here, we consider the following two modes:

1. *One-way edge-disjoint paths mode* (1EDP mode)

   One round can be described as a set $P = \{P_1, \ldots, P_k\}$ for some $k \in \mathbb{N}$, where $P_i = x_{i,1}, \ldots, x_{i,\ell_i}$ is a simple path of length $\ell_i - 1$, $i = 1, \ldots, k$. The paths $P_1, \ldots, P_k$ are called the *active paths* of this round. The executed communication of this round consists of the submission of the whole actual knowledge of $x_{i,1}$ to $x_{i,\ell_i}$ via path $P_i$ for any $i = 1, \ldots, k$. The node $x_{i,1}$ is called the *sender* of $P_i$, $x_{i,\ell_i}$ is called the *receiver* of $P_i$, and the nodes $x_{i,j}$ for $2 \leq j \leq \ell_i - 1$ are called *connectors (inner nodes)* of $P_i$. The connectors of $P_i$ do not learn the message submitted from $x_{i,1}$ to $x_{i,\ell_i}$, they are only used to realize the connection from $x_{i,1}$ to $x_{i,\ell_i}$.

   The set of paths $P$ must satisfy the following conditions:

   (1.1) $\forall i, j \in \{1, \ldots, k\}, i \neq j : P_i$ and $P_j$ are edge-disjoint,

   (1.2) $\{x_{i,1} | i = 1, \ldots, k\} \cap \{x_{i,\ell_i} | i = 1, \ldots, k\} = \emptyset$, i. e. no node may simultaneously be sender and receiver in one round,

   (1.3) $|\{x_{i,1} | i = 1, \ldots, k\}| = |\{x_{i,\ell_i} | i = 1, \ldots, k\}| = k$, i. e. no node may be the sender (receiver) for more than one path,

   (1.4) $\{x_{i,1}, x_{i,\ell_i} | i = 1, \ldots, k\} \cap \{x_{r,s_r} | r \in \{1, \ldots, k\}, s_r \in \{2, \ldots, \ell_r - 1\}\} = \emptyset$, i. e. the nodes of the paths of $P$ can be partitioned into three disjoint sets: the set of senders, the set of receivers, and the set of connectors.

2. *One-way listen-in edge-disjoint paths mode* (1LEDP mode)

   As in the previous mode, a round can be described as a set of paths $P = \{P_1, \ldots, P_k\}$ satisfying the conditions (1.1),(1.2),(1.3), and (1.4). The difference is that here the executed communication of this round consists of the submission of the whole actual knowledge of $x_{i,1}$ to all other vertices of $P_i$ for any $i = 1, \ldots, k$. Thus, after the execution of the round all vertices of each path $P_i$ know the message submitted from the sending endpoint $x_{i,1}$ for any $i = 1, \ldots, k$.

The disjoint paths modes were introduced and investigated in [1], [2], [6], [7], [8], [9], [10], [11].

Now, we fix the notation used in this paper. For any graph $G = (V, E)$, $V(G) = V$ denotes the set of vertices of $G$, and $E(G) = E$ denote the set of edges of $G$. In what follows we will denote broadcast, accumulation and gossip as problems $B$, $A$, and $R$. For any given graph $G$ and a vertex $u$ of $G$, let $B^e_u(G)$ $[B^{le}_u(G)]$ denote the number of rounds (complexity) of the optimal broadcast algorithm from $u$ in $G$ in the 1EDP [1LEDP] mode. This means that $B^e_u(G)$ $[B^{le}_u(G)]$ for a graph $G$ and a vertex $u$ in $G$, is the necessary and sufficient number of rounds of the 1EDP [1LEDP] mode to broadcast the piece of information originally residing in vertex $u$ to all other vertices in $G$. Similarly, $A^e_u(G)$ $[A^{le}_u(G)]$ denotes the number of rounds (complexity) of the optimal accumulation algorithm for $G$ and $u$ in the 1EDP [1LEDP] mode.

Finally, for any graph $G$, let $R^e(G), R^{le}(G)$ be the number of rounds (complexity) of the optimal gossip algorithm for $G$ in the 1EDP, 1LEDP mode respectively.

For any graph $G$ we define

$$\begin{aligned}
\mathbf{B^e(G)} &= \max\{B^e_u(G) | u \in V(G)\}, \\
\mathbf{B^e_{min}(G)} &= \min\{B^e_u(G) | u \in V(G)\}, \\
\mathbf{A^e(G)} &= \max\{A^e_u(G) | u \in V(G)\}, \\
\mathbf{A^e_{min}(G)} &= \min\{A^e_u(G) | u \in V(G)\}, \\
\mathbf{B^{le}(G)} &= \max\{B^{le}_u(G) | u \in V(G)\}, \\
\mathbf{B^{le}_{min}(G)} &= \min\{B^{le}_u(G) | u \in V(G)\}, \\
\mathbf{A^{le}(G)} &= \max\{A^{le}_u(G) | u \in V(G)\}, \\
\mathbf{A^{le}_{min}(G)} &= \min\{A^{le}_u(G) | u \in V(G)\}.
\end{aligned}$$

For any $k \geq 2, h \geq 1$ let $CkT_h$ denote the complete $k$-ary tree of height $h$, and let $sCkT_h$ denote the complete $k$-ary tree of height $h$ with one additional node attached to the root.

In this paper we will deal with two open problems from [2], [8] in communication in the edge-disjoint paths modes as defined above.

In Chapter 2 we will show that for any $h \geq 4$ the gossip complexity of the complete binary tree $C2T_h$ of height $h$ is

$$R^{le}(C2T_h) = 2 \cdot h.$$

This was left as an open problem in [2], where it was only shown that

$$2 \cdot h - 1 \leq R^{le}(C2T_h) \leq 2 \cdot h.$$

In [8] it was shown that for any graph $G$ of $n$ nodes

$$\lceil \log_2 n \rceil \leq A^e(G) = B^e(G) \leq \lceil \log_2 n \rceil + 2, \text{ and}$$

$$\lceil \log_2 n \rceil \leq A^e_{\min}(G) = B^e_{\min}(G) \leq \lceil \log_2 n \rceil + 1.$$

It was also shown there that the lower bounds and also the upper bound for $B^e_{\min}$ are sharp, but it was left as an open problem whether there is a graph $G$ with n nodes and $B^e(G) = \lceil \log_2 n \rceil + 2$.

In Chapter 3 we will construct a graph $T_k$ for any $k \geq 2$ with

$$B^e(T_k) = \lceil \log_2(|V(T_k)|) \rceil + 2.$$

## 2. Gossiping in complete binary trees in the 1LEDP mode

In this chapter we will prove that the gossip complexity of the complete binary tree of height $h$ in the $1LEDP$ mode is $2 \cdot h$ for any $h \geq 4$. This result is an improvement of Theorem 6.7 in [2], where it was shown that $2h - 1 \leq R^{le}(C2T_h) \leq 2h$ for any $h \geq 4$.

**Definition 2.1.** *For any $sC2T_h$ let $B^{le}_{+}(sC2T_h)$ [$A^{le}_{+}(sC2T_h)$] be the broadcast [accumulation] complexity from the additional vertex $s$ attached to the root of $C2T_h$.*

**Lemma 2.2.** *For $h \geq 1$*

*(i)* $B^{le}_{\min}(C2T_h) = B^{le}(C2T_h) = h$

*(ii)* $A^{le}_{\min}(C2T_h) = A^{le}(C2T_h) = h$

*(iii)* $B^{le}_{+}(sC2T_h) = h + 1.$

*Proof.* (i), (ii) This is shown in the proof of Theorem 6.3 in [2].

(iii) $B^{le}_{+}(sC2T_h) \leq h + 1$ follows directly from (i).

We will now prove $B^{le}_{+}(sC2T_h) \geq h + 1$ by induction over $h$. It is obvious that $B^{le}_{+}(sC2T_1) \geq 2$ holds. As induction hypothesis, assume that $B^{le}_{+}(sC2T_h) \geq h + 1$. Now let $A$ be a broadcast algorithm for $sC2T_{h+1}$. If the additional node $s$ sends its information $I(s)$ to its neighbour $r$ in the first round of $A$, the algorithm must still broadcast in the $C2T_{h+1}$ rooted at $r$. This takes $h+1$ additional rounds, according to (i). If $s$ sends its information to another vertex in $sC2T_{h+1}$ in the first round, there is still an uninformed $C2T_h$ subtree $T$ rooted at a son of $r$ left. Thus, $A$ must send the information to $T$ and broadcast it there. The complexity of this task is the same as for broadcast from the additional node $s$ in $sC2T_h$, since in every round only one vertex can send from $sC2T_{h+1} \backslash T$ to $T$. Thus, it takes $h + 1$ additional rounds according to the induction hypothesis. □

**Lemma 2.3.** *Let $T$ be a tree and $A$ be any 1LEDP gossip algorithm for $T$. Let $t$ be the smallest number of rounds after which there is at least one vertex in $T$ knowing the cumulative message. Then all vertices knowing the cumulative message after $t$ rounds lay on one path $P$ of $T$, and moreover $P$ is a part of an active path in the $t$-th round.*

*Proof.* This is proved in Lemma 6.5 in [2]. □

**Lemma 2.4.** *For any $h \geq 4$:*

$$2h - 1 \leq R^{le}(C2T_h) \leq 2h.$$

*Proof.* This is proved in Theorem 6.7 in [2]. □

**Theorem 2.5.** *For any $h \geq 4$:*

$$R^{le}(C2T_h) = 2h$$

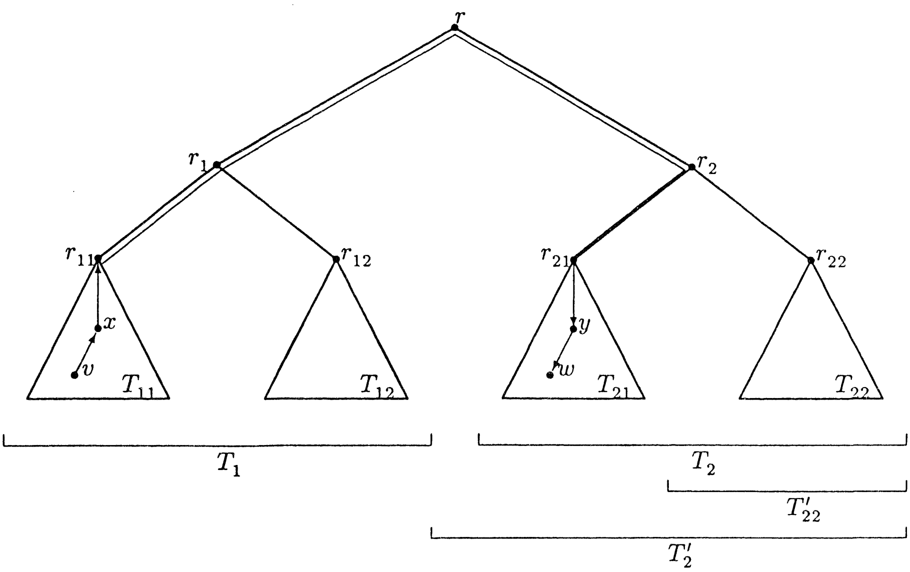


Figure 2.1: The active path $P$ in round $t$

*Proof.* $R^{le}(C2T_h) \leq 2h$ follows directly from Lemma 2.4.

To show that $R^{le}(C2T_h) \geq 2h$ let $T = C2T_h$, let $A$ be a gossip algorithm for $T$ and let $t$ be the minimal number of rounds after which one vertex of $T$ knows the cumulative message $I(T)$. Let $r$ be the root of $T$ and let $r_1, r_2$ be the sons of $r$, let $r_{i1}, r_{i2}$ be the sons of $r_i$ for $i = 1, 2$. For $\alpha \in \{1, 2, 11, 12, 21, 22\}$ let $T_\alpha$ be the subtree of $T$ rooted at $r_\alpha$. Let $T'_2$ be the $sC2T_{h-1}$ subtree of $T$ containing $r$ and $T_2$, let $T'_{22}$ be the $sC2T_{h-2}$ subtree of $T$ containing $r_2$ and $T_{22}$.

If there is a $C2T_{h-1}$ subtree $T' \in \{T_1, T_2\}$ in which no vertex knows $I(T)$ after $t$ rounds, then there is a piece of information $I$, which is unknown for any vertex in $T'$, since the root of $T'$ must know the cumulative message of $T'$. Thus, $A$ must still send $I$ to $T'$ and broadcast it there, i.e. the number of rounds of $A$ is $t + B^{le}_{+}(sC2T_{h-1}) = t + h \geq 2h$, since $t \geq h$ according to Lemma 2.2(ii),(iii).

We now consider the case that there is a vertex $x$ in $T_1$ and a vertex $y$ in $T_2$, such that both $x$ and $y$ know $I(T)$ after $t$ rounds.

Following Lemma 2.3 we know that $x$ and $y$ lie on one path $P$ of $T$ that is active in round $t$.

W.l.o.g. we can assume that $x$ lies in $T_{11}$ [or $x = r_1$] and $y$ lies in $T_{21}$ [or $y = r_2$] and the active path $P$ from a vertex $v$ to a vertex $w$ is directed from $T_{11}$ to $T_{21}$ like shown in Figure 2.1.

Since $x$ can learn only from $v$ in round $t$, $x$ must know $I(T_2)$ after $t - 1$ rounds. Thus, $r$ knows $I(T_2)$ after $t - 1$ rounds because all information from $T_2$ is flowing to $x$ via $r$. This implies that $r$ knows $I(T'_2)$ after $t - 1$ rounds. Thus, $t - 1 \geq A^{le}_{+}$ $+(sC2T_{h-1})$ holds. Since it is possible to construct a broadcast algorithm with the same number of rounds from every accumulation algorithm for a graph $G$ and any vertex $v \in V(G)$ by inverting the sequence of rounds and the direction of the active

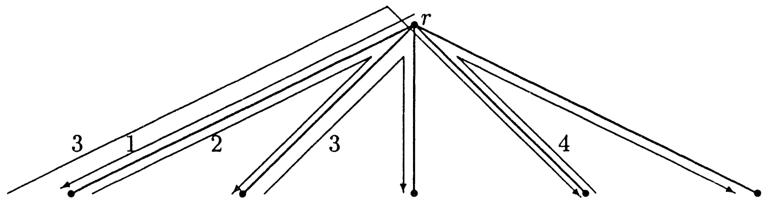


Figure 3.1: Broadcast in $C5T_1$ from the root $r$

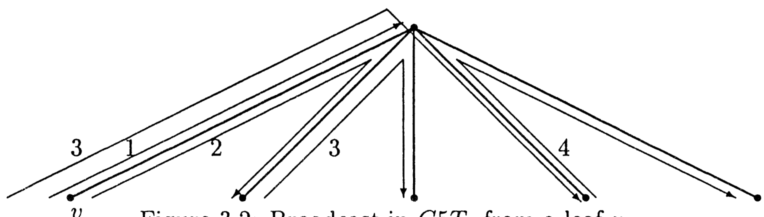


Figure 3.2: Broadcast in $C5T_1$ from a leaf $v$

paths, $A_v^{le}(G) \geq B_v^{le}(G)$ holds for any graph $G$ and any vertex $v \in V(G)$. Following Lemma 2.2 we get $t-1 \geq B_+^{le}(sC2T_{h-1}) = h$. Thus, we know $t \geq h+1$.

Since $t$ is the minimal number of rounds after which a vertex can know $I(T)$, there is no vertex in $T_{22}$ that knows $I(T)$ after $t$ rounds, according to Lemma 2.3.

Thus, after round $t$ the algorithm $A$ has at least to broadcast that information in $T'_{22}$ that was sent from $v$ to $w$ in round $t$. This takes at least $h-1$ rounds, according to Lemma 2.2.

Thus, $A$ needs at least $t+h-1 \geq h+1+h-1 = 2h$ rounds for gossiping in $T$.

□

## 3. Broadcast in the 1EDP mode

In this chapter we will show that the upper bound on the (maximum) broadcast complexity in the $1EDP$ mode from Theorem 2.1 in [8] is tight. In that paper the following inequality was proved for any graph $G$ of $n$ nodes:

$$\lceil \log_2 n \rceil \leq B^e(G) \leq \lceil \log_2 n \rceil + 2$$

We want to show that there is a graph $G$ with $n$ vertices and $B^e(G) = \lceil \log_2 n \rceil + 2$ for some $n \geq n_0$ for any $n_0 \in \mathbb{N}$. First we prove the following lemma:

**Lemma 3.1.** $B^e(C5T_1) = B^e_{\min}(C5T_1) = 4$

*Proof.* $B^e(G) \geq B^e_{\min}(G)$ holds for any graph $G$. Hence, it is sufficient to show that $B^e(C5T_1) \leq 4$ and $B^e_{\min}(C5T_1) \geq 4$.

$B^e(C5T_1) \leq 4$ follows from the algorithms shown in Figure 3.1 for broadcasting from the root of $C5T_1$ and in Figure 3.2 for broadcasting from a leaf of $C5T_1$. In these Figures the arrows denote the communication paths and the numbers denote the round in which the paths are active.

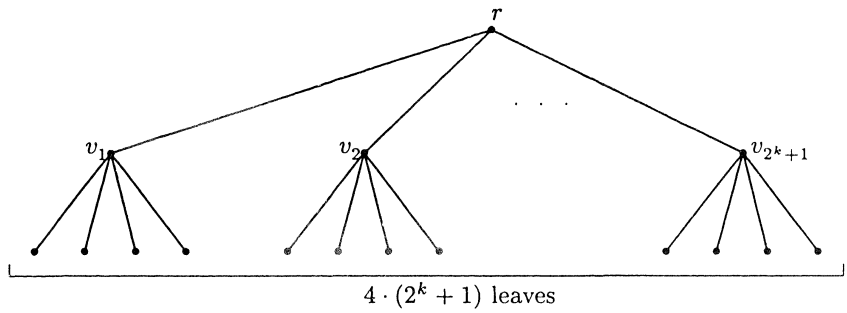


Figure 3.3: The tree $G_k$

We now show that $B^e_{\min}(C5T_1) \geq 4$:

For each communication from one leaf to another leaf the root of $C5T_1$ is needed as an inner vertex, so it cannot send any information in that round. This implies that the number of informed leaves is at most doubled in each round.

Thus, in every algorithm for broadcasting from the root after the first round only one leaf knows the message. This means that after the first three rounds at most 4 leaves are informed. Thus, every communication algorithm needs at least 4 rounds to broadcast from the root of $C5T_1$.

Now we consider broadcasting from a leaf: In that round in which the root learns the information no other communication is possible.Thus, after 3 rounds either the root is uninformed or there are at most 4 leaves informed. This implies that every communication algorithm also needs at least 4 rounds to broadcast from a leaf in the $C5T_1$. □

**Definition 3.2.** *For any $k \in I\!N$ let $G_k$ be the tree shown in Figure 3.3, let $r$ be the root of $G_k$, let $v_1, \ldots, v_{2^k+1}$ be the sons of $r$, and let $T_i$ be the subtree of $G_k$ rooted at $v_i$ for any $i = 1, \ldots, 2^k + 1$.*

**Theorem 3.3.** *For $k \geq 2$ and $n = 5(2^k + 1) + 1$*

$$B^e(G_k) = \lceil \log_2 n \rceil + 2.$$

*Proof.* $B^e(G_k) \leq \lceil \log_2 n \rceil + 2$ follows directly from Theorem 2.1 in [8], since $|V(G_k)| = n$. We show that $B^e_r(G_k) \geq \lceil \log_2 n \rceil + 2$.

We have $\lceil \log_2 n \rceil = k + 3$, since $n = 5(2^k + 1) + 1 = 2^{k+2} + 2^k + 6$ and therefore $2^{k+2} < n \leq 2^{k+3}$. Thus, it suffices to show that $B^e_r(G_k) \geq k + 5$.

We first prove for every $\ell \in I\!N$:

(*) After $\ell + 1$ rounds at most $2^\ell$ subtrees $T_i$ contain an informed vertex.

We prove (*) by induction over $\ell$:

In the first round only one vertex $x$ can be informed by the root. If $x$ sends the message into another subtree in the second round, the root is used as an inner vertex on the active path and cannot send itself. Thus, after $\ell + 1 = 2$ rounds at most $2^\ell = 2$ subtrees contain an informed vertex.

Suppose that the number of subtrees that contain an informed vertex after $\ell + 1$ rounds is at most $2^\ell$.

Since the communication paths are disjoint, in one round from every subtree only one vertex can send into another subtree. If a vertex in one subtree sends to a vertex in another subtree, the root cannot send in this round. Thus, the number of subtrees containing an informed vertex can at most be doubled in each round. This means that after $(\ell + 1) + 1 = \ell + 2$ rounds at most $2 \cdot 2^\ell = 2^{\ell+1}$ subtrees contain an informed vertex.

This completes the proof of $(*)$ and it follows directly from $(*)$ that after $k + 1$ rounds at least one subtree $T \in \{T_1, \ldots, T_{2^{k+1}}\}$ is completely uninformed. Thus, after $k + 1$ rounds the information has to be sent to $T$ and to be distributed there.

The complexity of this task is the same as for broadcasting from a leaf in $C5T_1$, since in every round only one vertex can send from $G_k \setminus T$ to $T$.

From Lemma 3.1 it follows that $B_r^e(G_k) \geq k + 1 + B_{\min}^e(C5T_1) = k + 5$. □

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*Author's address:* Lehrstuhl für Informatik I, RWTH Aachen, Ahornstraße 55, 52056 Aachen, Germany

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