Acta Mathematica et informatica Universitatis Ostraviensis 4(1996) 105-120

State hypergroups of automata

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Abstract. A functorial passage from the category of automata without outputs and their homomorphisms into the category of preorder hypergroups and strong homomorphisms based on the concept of inertial relation extended into apreording of a statě set is ušed for the describing of some basic properties of automata. Further, the relational hypergroup product is treated in connection with products of automata.

1991 Mathematics Subject Classification: 20N20, 68Q70

In the páper [16] there are described constructions of some hyperstructures on sets of words formed from the given input alphabets and on the statě sets of corresponding automata. Using the hyperoperation *x* -f *y* = *{x,y}* on the set *A** of words over a given alphabet *A* and the binary operation of concatenation, the authors of [16] háve introduced a notion of a hyper-ringoid, and the equivalence on *A** determined by the equality of lengths of words has served them for a construction of the so called strengthen join hypergroup. Further in [16] there are defined various hyperoperations on the statě set of a deterministic and non-deterministic acceptor. One of them is defined in such a way that the result of the hyperproduct applied on a pair of stat.es is the union of blocks of the corresponding states within the grade equi valence. Here, by a grade of a statě *s* is called the set grád *s* of all the words transfering the statě 5 into the set of hnal states and two states *s*, *$2* are said to be grade equivalent if grád *S* = grád *$2-* Such a defined hyperoperation on the statě set of an automaton creates a join hypergroup. Relationships between properties of automata and their corresponding hyperstructures are not treated in this páper [16]. Other current papers devoted to the mentioned topič in some other directions are [15], [17].

In this contribution we use a similar idea as it is described above (originally ušed in [16] independent ly within [7]) and to any automaton without output (in the sense e.g. [1], [2], [8], [9], [19]) we assign a commutative hypergroup (a quasi-ordering hypergroup — [4], [6], determined by the Warner quasi-ordering — [23]) in such a way that we get a functorial passage from the category of automata with the samé input alphabet into the category of commutative hypergroups and their strong homomorphisms. Some basic properties of automata will be described in terms of hypergroups. Finally a relational product of quasi-order hypergroups corresponding to the heterogenous product of automata will be introduced.

Recall first, that a hypergroup (in the sense of Marty) i. e. a pair $(H, \circ)$, where $H \neq \emptyset$ and $\circ : H \times H \to \exp' H$ $(= \exp H \setminus \{\emptyset\})$ is an associative hyperoperation ($(a \circ b) \circ c = a \circ (b \circ c)$, where $A \circ B = \bigcup_{(a,b) \in A \times B} a \circ b$ for all nonempty subsets A, B of H) satisfying the reproduction axiom $a \circ H = H = H \circ a$ for any $a \in H$, is said to be a quasi-ordering hypergroup if for any a, $b \in H$ we have

$$a \in a^2 = a^3 , \quad a \circ b = a^2 \cup b^2 .$$

If moreover $a^2 = b^2$ implies $a = b$ for any pair a, $b \in H$ then $(H, \circ)$ is called an ordering hypergroup ([6]). It is evident that a (quasi-)ordering hypergroup is commutative and extensive (which means $\{a, b\} \subseteq a \circ b$ for any pair $a, b \in H$).

Let $(H, \circ)$ be a hypergroup, K be a nonempty subset of H which is multiplicatively closed, i. e. $K \circ K \subseteq K$. If $(K, \circ)$ satisfies the reproduction axiom then it is called a subhypergroup of the hypergroup $(H, \circ)$ - cf. [4], Definition 7, p. 8.

It is to be noted that a quasi-ordering hypergroup can be defined in this space-saving form:

A quasi-ordering hypergroup is a hypergroupoid $(H, \circ)$ (i. e. $H \neq \emptyset, \circ : H \times H \to \exp' H$) such that $a \in a^3 \subseteq a^2$, $a \circ b = a^2 \cup b^2$ for any $a, b \in H$.

It is easy to see that $(H, \circ)$ satisfies the reproduction axiom. Indeed, $a \circ H \subseteq H$ for any $a \in H$ and for any $x \in H$ we have $x \in a \circ x \subseteq a \circ H$, thus $H \subseteq a \circ H$ for every $a \in H$. Hence $(H, \circ)$ is a commutative quasi-hypergroup. We show that the hyperoperation $\circ$ is also associative. Define a binary relation $r \subseteq H \times H$ by $x \, r \, y$ if and only if $y \in x^2$, i. e. $x^2 = r(x)$ for any $x \in H$. Since the hypergroupoid $(H, \circ)$ is extensive, the relation r is reflexive and for $x \, r \, y$, $y \, r \, z$ we have $y \in x^2$, $z \in y^2$ which implies $z \in x^4 = x^2$ thus $x \, r \, z$, consequently $r^2 = r$. Now, suppose $a, b, c \in H$ are arbitrary elements. Then

$$a \circ (b \circ c) = a \circ (r(b) \cup r(c)) = a \circ r(\{b, c\}) = \bigcup_{x \in r(\{b,c\})} a \circ x =$$
$$= r(a) \cup \bigcup_{x \in r(\{b,c\})} r(x) = r(a) \cup r^2(\{b, c\}) = r(\{a, b, c\}) .$$

Similarly $(a \circ b) \circ c = c \circ (b \circ a) = r(\{a, b, c\})$, hence $a \circ (b \circ c) = (a \circ b) \circ c$ and we have $(H, \circ)$ is an extensive commutative hypergroup.

As a motivating example for our consideration take the Peano algebra $(\mathbb{N}, \sigma)$ of all positive integers with the successor function $\sigma : \mathbb{N} \to \mathbb{N}$ and define a hyperoperation $\circ : \mathbb{N} \times \mathbb{N} \to \exp' \mathbb{N}$ in this way: For $m, n \in \mathbb{N}$ denote $a = \min\{m, n\}$ and

$$m \circ n = \{x \in \mathbb{N}, \, \exists \, k \in \mathbb{N} \quad \text{or } k = 0 : \sigma^k(a) = x\} = \{x \in \mathbb{N}; a \leq x\} .$$

Then $(\mathbb{N}, \circ)$ is an ordering hypergroup. On the other hand, if we define a binary relation $r \subseteq \mathbb{N} \times \mathbb{N}$ by the rule $(m, n) \in r$ iff $n \circ m \circ n = n^2$, then r is an ordering on the set $\mathbb{N}$ identical with the usual one $\leq$.

The above mentioned simple relationship between quasi-ordering hypergroups and quasi-ordered sets can be also described as there follows:

For any quasi-ordering hypergroup $(H, \circ)$, the pair (H, r), where $a\,r\,b$ iff $a \circ b \circ a = a^2$, is a quasi-ordered set and if for any quasi-ordered set (X, r) we put $x \circ y = r(x) \cup r(y)$ $(= \{t; x\,r\,t\} \cup \{s; y\,r\,s\})$ then $(X, \circ)$ is a quasi-ordering hypergroup.

Now, let A^* be the free monoid of words over an arbitrary (nonempty) alphabet A; suppose e stands for the empty word.

In accordance with [2], [3], [8], [9] and other publications, by an automaton we mean a triad $\mathbb{A} = (S, A, \delta)$, where S, A are arbitrary sets $(A \neq \emptyset)$, which are called — in the given order — a set of states (or a state set), a set of input symbols (or an input alphabet) and $\delta : S \times A^* \to S$ is a mapping called a transition function, which satisfies these two conditions: $\delta(s, e) = s$ for any state $s \in S$, (the identity axiom),

$\delta(s, ab) = \delta(\delta(s, a), b)$ for any state $s \in S$ and any pair of words $a, b \in A^*$ (the homomorphism axiom). It is to be noted that the transition function $\delta : S \times A^* \to S$ is a usual extension of the next-state function $\delta_{\mathbb{A}}^{(0)} : S \times A \to S$.

An automaton $\mathbb{A}_1 = (S_1, A, \delta_1)$ is said to be a subautomaton of an automaton $\mathbb{A} = (S, A, \delta)$ if $S_1 \subseteq S$, $\delta(s, a) \in S_1$ for any state $s \in S_1$ and any word $a \in A^*$ and further $\delta_1 = \delta \uparrow S \times A^*$, i. e. δ_1 is a restriction of the transition function δ on $S_1 \times A^*$.

For $\emptyset \neq T \subseteq S$ we denote $\delta(T, A^*) = \{\delta(t, a); t \in T, a \in A^*\}$. (In [1], [2] this operator is denoted only by δ, i. e. $\delta(T)$ stands for the above set in the mentioned papers). If T is a singleton, e. g. $T = \{t\}$, we write $\delta(t, A^*)$ instead of $\delta(\{t\}, A^*)$.

Let $\mathbb{A} = (S, A, \delta)$ be an automaton. We define a binary hyperoperation on the state set S of $\mathbb{A}$ by

$$s \circ t = \delta(s, A^*) \cup \delta(t, A^*)$$

for any pair of states s, $t \in S$. Since A^* contains the empty word e, we have s, $t \in s \circ t$, thus $s \circ t \neq \emptyset$ and commutativity of this hyperoperation is also evident. Further, $s^2 = \delta(s, A^*)$ thus $s \circ t = s^2 \cup t^2$ and

$$\begin{aligned} s^3 = s \circ s^2 = s \circ \delta(s, A^*) &= \textstyle\bigcup_{t \in \delta(s, A^*)} s \circ t = \delta(s, A^*) \cup \bigcup_{t \in \delta(s, A^*)} \delta(t, A^*) = \\ &= s^2 \cup \delta(\delta(s, A^*), A^*) = s^2 \cup \{\delta(\delta(s, a), b); a \in A^*, b \in A^*\} = \\ &= s^2 \cup \{\delta(s, ab); ab \in A^* A^* = A^*\} = s^2 \cup \{\delta(s, c); c \in A^*\} = s^2 \cup \delta(s, A^*) = \\ &= s^2 \cup s^2 = s^2\,, \end{aligned}$$

i. e. $s \in s^2 = s^3$ for any state $s \in S$. Consequently, the hyperoperation $\circ$ is associative and we get that $(S, \circ)$ is a quasi-ordering hypergroup. The same also follows from the fact, that $s \circ t = \{u \in S; (s, u) \in r \text{ or } (t, u) \in r\} = r(s) \cup r(t)$, where r is the transitive cover of the inertial relation ν on the state set of $\mathbb{A}((s, t) \in \nu$ if $\delta(s, a) = t$ for some input symbol $a \in A)$ which has been used and studied by M. W. Warner in [23]. The just defined quasi-ordering hypergroup $(s, \circ)$ will be called the state hypergroup of the automaton $\mathbb{A} = (S, A, \delta)$ and denoted also $\mathbb{H}(\mathbb{A})$.

Recall that a nonempty subautomaton $\mathbb{B} = (T, A, \delta)$ (automata with empty state sets will not be considered) of an automaton $\mathbb{A} = (S, A, \delta)$ is said to be separated if $\delta(S \setminus T, A^*) \cap T = \emptyset$. An automaton is called connected if it does not possess any separated proper subautomaton.

Definition. We say that a commutative hypergroup $(H, *)$ is an inner disjoint product of its subhypergroups $(H_1, *)$, $(H_2, *)$, if $H = H_1 * H_2$, $H_1 \cap H_2 = \emptyset$.

Definition. A commutative hypergroup $(H, *)$ is said to be inner irreducible if for any pair H_1, H_2 of its subhypergroups such that $H_1 * H_2 = H$ we have $H_1 \cap H_2 \neq \emptyset$, i.e. if $(H, *)$ is not any inner disjoint product of some pair of its subhypergroups.

Theorem 1. *An automaton $\mathbb{A} = (S, A, \delta)$ is connected if and only of its state hypergroup $(S, \circ)$ is inner irreducible.*

PROOF: Suppose the automaton $\mathbb{A}$ is connected and $(H_1, \circ)$, $(H_2, \circ)$ are subhypergroups of the state hypergroup $(S, \circ)$ such that $H_1 \circ H_2 = S$. For any state $s \in H_1$ and any word $a \in A^*$ we have

$$\delta(s, a) \in \delta(s, A^*) = s \circ s \in H_1 ,$$

thus (H_1, A, δ_1) with $\delta_1 = \delta \upharpoonright H_1 \times A^*$ is a subautomaton of the connected automaton $\mathbb{A}$. Then $\delta(S \setminus H_1, A^*) \cap H_1 \neq \emptyset$ hence there exists a pair of states $(t, s) \in H_1 \times (S \setminus H_1)$ and a word $a \in A^*$ such that

$$t = \delta(s, a) \subseteq \delta(s, A^*) = s \circ s .$$

From the equality $H_1 \circ H_2 = S$ there follows the existence of a pair $(u, v) \in H_1 \times H_2$ such that

$$s \in u \circ v = \delta(u, A^*) \cup \delta(v, A^*) .$$

Since $u \in H_1$, $\delta(u, A^*) = u \circ u \in H_1$, $s \in S \setminus H_1$ and $\delta(v, A^*) = v \circ v \subseteq H_2$, we have

$$t \in s \circ s \subseteq \textstyle\bigcup_{x \in v\circ v,\ y \in v \circ v} x \circ y = (v \circ v) \circ (v \circ v) = v^3 \circ v =$$
$$= v^2 \circ v = v^3 = v^2 \in H_2 ,$$

thus $t \in H_1 \cap H_2$, hence $H_1 \cap H_2 \neq \emptyset$ which means that the state hypergroup $(S, \circ)$ of the automaton $\mathbb{A}$ is inner irreducible.

Now suppose $(S, \circ)$ is inner irreducible and simultaneously the automaton $\mathbb{A}$ is disconnected. Then

$$\delta(S \setminus S_1, A^*) \cap S_1 = \emptyset$$

for some subautomaton (S_1, A, δ_1) of the automaton $\mathbb{A}$. Denoting $H_1 = S_1$, $H_2 = S \setminus S_1$ then from the equality $\delta_1(S_1, A^*) = S_1$ there follows $(H_1, \circ)$ is a subhypergroup of the hypergroup $(S, \circ)$. Let $s, t \in H_2$. Since $\delta(H_2, A^*) \cap H_1 = \emptyset$, we have $\delta(H_2, A^*) = H_2$, thus

$$s \circ t = \delta(s, A^*) \cup \delta(t, A^*) \subseteq H_2 ,$$

hence $(H_2, \circ)$ is also a subhypergroup of the hypergroup $(S, \circ)$. Evidently $H_1 \circ H_2 \subseteq S$. On the other hand if $s \in S$ then either $s \in S_1 = H_1$ or $s \in S \setminus S_1 = H_2$. Then

$$s \in \delta(s, A^*) \cup \delta(t, A^*) = s \circ t = t \circ s ,$$

where t is an arbitrary element of H_2 if $s \in H_1$ and it is an arbitrary element of H_1 if $s \in H_2$. Hence $S \subseteq H_1 \circ H_2$, therefore $S = H_1 \circ H_2$. Moreover

$$H_1 \cap H_2 = S_1 \cap (S \setminus S_1) = \emptyset ,$$

which contradicts the assumption of the irreducibility of the hypergroup $(S, \circ)$, consequently the automaton $\mathbb{A}$ is connected. □

Recall that the automaton (S, A, δ) is said to be strongly connected if for any pair of its states $s, t \in S$ there exists a word $a \in A^*$ such that $\delta(s, a) = t$.

By [20] a hypergroup $(H, *)$ is said to be cyclic if for some $h \in H$ we have $H = \bigcup_{k \in \mathbb{N}} h^k$ and it is called single-power cyclic (more exactly n-single-power cyclic) if there exist $h \in H$, $n \in \mathbb{N}$ such that $H = h^n$. In this case the element h is called n-generating.

Theorem 2. *An automaton $\mathbb{A} = (S, A, \delta)$ is strongly connected if and only if its state hypergroup is 2-single-power cyclic and each its state $s \in S$ is a 2-generating element of this hypergroup.*

PROOF: Suppose the automaton $\mathbb{A}$ is strongly connected and $s \in S$ is its arbitrary state. Then $s^2 = s \circ s \subseteq S$. For a state $t \in S$ and a suitable word $a \in A^*$ we have

$$t = \delta(s, a) \subseteq \delta(s, A^*) = s \circ s ,$$

thus $S \subseteq s^2$, hence we have the equality $s^2 = S$ holds.

Suppose s, $t \in S$. By the assumption that the state hypergroup $(S, \circ)$ is 2-single-power cyclic and s is a 2-generating element of that, we have

$$S = s^2 = s \circ s = \delta(s, A^*) ,$$

thus $t = \delta(s, a)$ for some word $a \in A^*$, consequently the automaton $\mathbb{A}$ is strongly connected. □

An automaton $\mathbb{A} = (S, A, \delta)$ is said to be retrievable if for any state $s \in S$ and any word $a \in A^*$ there exists a word $b \in A^*$ such that $\delta(s, a\,b) = s$.

Properties of automata are studied in [1] with the use of the concept of source which is based on that of a predecessor. A state s is a predecessor of a state t if t can be reached from s (by a finite input sequence, including the empty word e), i. e. if $t = \delta(s, a)$ for some word $a \in A^*$. If $\mathbb{A}_1 = (S_1, A, \delta)$ is a subautomaton of an automaton $\mathbb{A} = (S, A, \delta)$, then the $\mathbb{A}_1$-source of $T \subseteq S_1$ is the set $\sigma_{\mathbb{A}_1}(T)$ of predecessors of T which are states of $\mathbb{A}_1$, i. e.

$$\sigma_{\mathbb{A}_1}(T) = \{t \in S_1;\ \exists\, a \in A^* :\ \delta(t, a) \in T\}\,.$$

Properties of the set-mapping $\sigma_{\mathbb{A}}$ are treated in detail in [1], §3 and §4. In §6 of the mentioned paper there is characterized separatedness of a subautomaton of an automaton $\mathbb{A} = (S, A, \delta)$ by the condition $\sigma_{\mathbb{A}}(S_1) = S_1$. The proof is easy and similarly there can be obtained that an automaton is retrievable if and only if every its subautomaton is separated. The consequence is:

Lemma 1. ([1], Theorem 8(i)). *An automaton $\mathbb{A}$ is retrievable if and only if $\sigma_{\mathbb{A}}(T) = T$ for the state set T of any subautomaton of $\mathbb{A}$.*

From the definition of a source there follows with respect to lemma 1 that an automaton $\mathbb{A}$ is retrievable if and only if it is a union of its strongly connected subautomata, thus $\mathbb{A} = \big(\bigcup_{i\in I} S_i, A, \delta\big)\ \big(= \bigcup_{i\in I} \mathbb{A}_i\big)$, where $\mathbb{A}_i = (S_i, A, \delta_i)$, $\delta_i = \delta \uparrow (S_i \times A^*)$, $i \in I$.

Theorem 3. *An automaton $\mathbb{A} = (S, A, \delta)$ is retrievable if and only if any inner irreducible subhypergroup of the state hypergroup $(S, \circ)$ is 2-single-power cyclic.*

PROOF: Since the automaton $\mathbb{A}$ is retrievable if and only if it is a union of its strongly connected subautomata $\mathbb{A}_i = (S_i, A, \delta_i)$, $i \in I$, we have $S_i \cap S_j = \emptyset$ for $i, j \in I$, $i \neq j$ – in the opposite case we would have $S_i = S_j$. Further it is clear that $(H, \circ)$ is a subhypergroup of the state hypergroup $(S, \circ)$ of the automaton $\mathbb{A}$ if and only if there exists a nonempty set $J \subseteq I$ such that $H = \bigcup_{i\in J} S_i$ and a subhypergroup $(H, \circ)$ of the hypergroup $(S, \circ)$ is inner irreducible if and only if $H = S_j$ for exactly one index $j \in I$. Indeed, if $H = \bigcup_{k\in J} S_k$, where $J \subseteq I$ is a subset containing at least two indices then for any $i \in J$ we have

$$S_i \circ \Big(\bigcup_{k\in J} S_k\Big) = H, \quad S_i \cap \Big(\bigcup_{k\in J,\, k\neq i} S_k\Big) = \emptyset\,.$$

This implies – with respect to Theorem 2 – the assertion. □

An automaton $\mathbb{A} = (S, A, \delta)$ is said to be reflexive if for any state $s \in S$ there exists a word $a \in A^*$, $a \neq e$ such that $\delta(s, a) = s$.

We give some sufficient (but not necessary) conditions for the reflexivity of automata:

Theorem 4. *Let $(S, \circ)$ be the state hypergroup of an automaton $\mathbb{A} = (S, A, \delta)$. Each of the following equivalent conditions implies the reflexivity of $\mathbb{A}$:*

1° *For any $s \in S$ there exists $t \in s^2, t \neq s$ such that $s \in t^2$.*

2° *For any $s \in S$ there exists $t \in S, t \neq s$ such that $s \circ t \circ s = s^2$, $t \circ s \circ t = t^2$.*

3° *For any $s \in S$ the equation $x^2 = s^2$ has a solution different from the state s.*

Proof: Suppose the condition 1° holds, i. e. for any $s \in S$ there is $t \in s^2 = \{\delta(s, a); a \in A^*\}$ such that $t \neq s \in t^2 = \{\delta(t, a); a \in A^*\}$, i. e. for suitable words a_1, $a_2 \in A^*$, $a_1 \neq e \neq a_2$ we have $\delta(s, a_1) = t$, $\delta(t, a_2) = s$. Put $a = a_1 a_2$. Then $a \neq e$ and $\delta(s, a) = \delta(s, a_1 a_2) = \delta(\delta(s, a_1), a_2) = s$, thus the automaton $\mathbb{A}$ is reflexive. Equivalence of conditions $1^\circ, 2^\circ, 3^\circ$ is evident. □

It is easy that the assignement $\mathbb{A} \mapsto (S, \circ)$ of a hypergroup to an automaton is functorial, more precisely if of $\mathbb{A}_1 = (S_1, A, \delta_1)$, $\mathbb{A}_2 = (S_2, A, \delta_2)$ are automata with the same input alphabet, $(S_1, \circ_1)$, $(S_2, \circ_2)$ are their state hypergroups and $f : \mathbb{A}_1 \to \mathbb{A}_2$ is a homomorphism, which means

$$f(\delta_1(s, a)) = \delta_2(f(s), a)$$

for any pair $(s, a) \in S \times A$, then $f : (S_1, \circ_1) \to (S_2, \circ_2)$ is a strong homomorphism or a good homomorphism of corresponding hypergroups, i. e.

$$f(s \circ_1 t) = f(s) \circ_2 f(t)$$

for any pair of states s, $t \in S_1$.

In what follows, denote by $\mathbb{H}(\mathbb{A})$ the state hypergroup $(S, \circ)$ of an automaton $\mathbb{A} = (S, A, \delta)$. A quite natural task is to explaine a behaviour of state hypergroups with respect to products of automata. In the Algebraic Theory of Automata the most used construction of products defines the product-automaton on the cartesian product of the state sets of the given automata. In the direct product, one can fix the input alphabet or take the cartesian product of the input alphabets as the new input alphabet. In [3] and [9] these two kinds of products are distinguished as homogeneous and heterogeneous product. We adopt notation of the paper [9] and recall exact definitions of the mentioned types of products. It is to be noted that in [9] the author has introduced and studied another concept of an automaton product – called a cartesian composition of automata – which possess very nice properties from the point of view of structural questions of the Algebraic Automata Theory.

Let $\mathbb{A}_1 = (S_1, A, \delta_1)$, $\mathbb{A}_2 = (S_2, A, \delta_2)$ be two automata with the same input alphabet A. Then the homogeneous (direct) product $\mathbb{A}_1 \times \mathbb{A}_2$ is the automaton $(S_1 \times S_2, A, \delta)$, with $\delta((s, t), a) = (\delta_1(s, a), \delta_2(t, a))$ for all $s \in S_1$, $t \in S_2$, $a \in A^*$.

Let $\mathbb{A}_1 = (S_1, A_1, \delta_1)$, $\mathbb{A}_2 = (S_2, A_2, \delta_2)$ be automata. Then the heterogeneous (direct) product $\mathbb{A}_1 \otimes \mathbb{A}_2$ is the automaton $(S_1 \times S_2, A_1 \times A_2, \delta)$, where $\delta : (S_1 \times S_2) \times (A_1 \times A_2)^* \to S_1 \times S_2$ is defined in this way:

If $(s_1, s_2) \in S_1 \times S_2$, $a = (a_1^{(1)}, a_2^{(1)})(a_1^{(2)}, a_2^{(2)}) \dots (a_1^{(k)}, a_2^{(k)}) \in (A_1 \times A_2)^*$ then

$$\delta\Big((s_1, s_2), a\Big) = \Big(\delta_1(s_1, a_1), \delta_2(s_2, a_2)\Big), \text{where}$$
$$a_1 = a_1^{(1)} a_1^{(2)} \dots a_1^{(k)} \in A_1^*,\ a_2 = a_2^{(1)} a_2^{(2)} \dots a_2^{(k)} \in A_2^*.$$

It is easy to verify that the function δ satisfies the identity axiom and the homomorphism axiom. It is an extension of the next-state function

$$\delta^{(0)}_{\mathbb{A}_1 \otimes \mathbb{A}_2} : (S_1 \times S_2) \times (A_1 \times A_2) \to S_1 \times S_2$$

defined by $\delta\Big((s,t),(a,b)\Big) = \Big(\delta_1(s,a), \delta_2(t,b)\Big)$ for all $s \in S_1$, $t \in S_2$, $a \in A_1$, $b \in A_2$.

Both products are products in the sense used in category theory (cf. [3]) however – as the following example shows – they are not compatible with the direct product of hypergroups in the sense that, in general, none of hypergroups $\mathbb{H}(\mathbb{A}_1 \times \mathbb{A}_2)$, $\mathbb{H}(\mathbb{A}_1 \otimes \mathbb{A}_2)$ is isomorphic to the hypergroup $\mathbb{H}(\mathbb{A}_1) \times \mathbb{H}(\mathbb{A}_2)$, where for (s_1, s_2), $(t_1, t_2) \in \mathbb{H}(\mathbb{A}_1) \times \mathbb{H}(\mathbb{A}_2)$ we have

$$(s_1, s_2) * (t_1, t_2) = (s_1 \circ_1 t_1) \times (s_2 \circ_2 t_2).$$

After some calculation, one can see that the hyperoperation $*$ determines a hypergroup structure on the cartesian product $S_1 \times S_2$ of state sets of automata $\mathbb{A}_1$, $\mathbb{A}_2$; see e.g. [18], Theorem 40.

Example. Consider automata $\mathbb{A}_i = (S_i, A, \delta)$, $i = 1, 2$, where $S_1 = \{s_1, s_2, s_3\}$, $S_2 = \{t_1, t_2\}$, $A = \{a, b\}$ and

$$\delta_1(s_1, a) = \delta_1(s_2, a) = \delta_1(s_3, a) = s_3\,,\ \delta_1(s_1, b) = \delta_1(s_3, b) = s_2\,,$$
$$\delta_1(s_2, b) = s_3\,,\ \delta_2(t_1, a) = t_2\,,\ \delta_2(t_2, a) = t_1\,,\ \delta_2(t_1, b) = \delta_2(t_2, b) = t_2\,.$$

Let ν be the inertial relation of the automaton $\mathbb{A}_1 \times \mathbb{A}_2$ (i.e. $(s_i, t_j)\nu(s_k, t_m)$ if and only if either $(s_k, t_m) = (\delta_1(s_i, a), \delta_2(t_j, a))$ or $(s_k, t_m) = (\delta_1(s_i, b), \delta_2(t_j, b))$ and μ be the inertial relation of the heterogeneous direct product $\mathbb{A}_1 \otimes \mathbb{A}_2$. By $\widetilde{\nu}$, $\widetilde{\mu}$ will be denoted the reflexive and transitive covers of relations ν, μ respectively. We describe these relations by their zero-one incident tables (or matrices). Values in parentheses correspond to relations $\widetilde{\nu}, \widetilde{\mu}$:

$\nu(\widetilde{\nu})$	(s_1,t_1)	(s_1,t_2)	(s_2,t_1)	(s_2,t_2)	(s_3,t_1)	(s_3,t_2)
(s_1,t_1)	0(1)	0(0)	0(0)	1(1)	0(1)	1(1)
(s_1,t_2)	0(0)	0(1)	0(0)	1(1)	1(1)	0(1)
(s_2,t_1)	0(0)	0(0)	0(1)	0(1)	0(1)	1(1)
(s_2,t_2)	0(0)	0(0)	0(0)	0(1)	1(1)	1(1)
(s_3,t_1)	0(0)	0(0)	0(0)	1(1)	0(1)	1(1)
(s_3,t_2)	0(0)	0(0)	0(0)	1(1)	1(1)	0(1)

$\mu(\widetilde{\mu})$	(s_1,t_1)	(s_1,t_2)	(s_2,t_1)	(s_2,t_2)	(s_3,t_1)	(s_3,t_2)
(s_1,t_1)	0(1)	0(0)	0(1)	1(1)	0(1)	1(1)
(s_1,t_2)	0(0)	0(1)	0(1)	1(1)	1(1)	1(1)
(s_2,t_1)	0(0)	0(0)	0(1)	0(1)	0(1)	1(1)
(s_2,t_2)	0(0)	0(0)	0(1)	0(1)	1(1)	1(1)
(s_3,t_1)	0(0)	0(0)	0(1)	0(1)	1(1)	1(1)
(s_3,t_2)	0(0)	0(0)	0(1)	1(1)	1(1)	0(1)

Then for (s_i,t_j), $(s_k,t_m) \in \mathbb{H}(\mathbb{A}_1 \times \mathbb{A}_2)$ we have $(s_i,t_j) \circ (s_k,t_m) = \widetilde{\nu}(s_i,t_j) \cup \widetilde{\nu}(s_k,t_m)$ and for these pairs considered as elements of the hypergroup $\mathbb{H}(\mathbb{A}_1 \otimes \mathbb{A}_2)$ we have similarly $(s_i,t_j) \circ (s_k,t_m) = \widetilde{\mu}(s_i,t_j) \cup \widetilde{\mu}(s_k,t_m)$.

Hypergroups $\mathbb{H}(\mathbb{A}_1 \times \mathbb{A}_2)$, $\mathbb{H}(\mathbb{A}_1 \otimes \mathbb{A}_2)$ cannot be isomorphic because of for any pairs $(s,t) \in \mathbb{H}(\mathbb{A}_1 \times \mathbb{A}_2)$ we have $3 \leq |(s,t)^2| \leq 4$, especially

$$(s_2,t_2)^2 = \{(s_2,t_2),(s_3,t_1),(s_3,t_2)\}$$

and for any pair (s,t) considered as an element of the hypergroup $\mathbb{H}(\mathbb{A}_1 \otimes \mathbb{A}_2)$ we have $4 \leq |(s,t)^2| \leq 5$, especially

$$(s_1,t_1)^2 = \{(s_1,t_1),(s_2,t_2),(s_3,t_1),(s_3,t_2),(s_2,t_1)\}\,.$$

None of the above mentioned hypergroups is isomorphic to the direct product $\mathbb{H}(\mathbb{A}_1) \times \mathbb{H}(\mathbb{A}_2)$ because of e.g. for $(s_1,t_2) \in \mathbb{H}(\mathbb{A}_1) \times \mathbb{H}(\mathbb{A}_2)$ we have

$$(s_1,t_1)^2 = s_1^2 \times t_1^2 = S_1 \times S_2$$

which is a six-element set.

Lemma 2. *Let $\{(H_k,*_k); k = 1,2,\dots,n\}$ be a system of quasi-order hypergroups, $H = H_1 \times \dots \times H_n$ and for $(x_1,\dots,x_n) \in H$, $(y_1,\dots,y_n) \in H$ let*

$$(x_1,\dots,x_n) * (y_1,\dots,y_n) = (x_1^2 \times \dots \times x_n^2) \cup (y_1^2 \times \dots \times y_n^2)\,.$$

Then $(H,*)$ is a quasi-order hypergroup.

Proof: It is easy to see that $* : H \times H \to \exp'(H)$ is a commutative hyperoperation on the set H. If for $(x_1,\dots,x_n) \in H$, $(y_1,\dots,y_n) \in H$ we put

$$(y_1,\dots,y_n) r_H (x_1,\dots,x_n)$$

whenever $(x_1,\dots,x_n) \in y_1^2 \times \dots \times y_n^2$, i.e. $x_k \in y_k^2 = y_k *_k y_k$ for any k, we have (H,r_H) is the direct product of quasi-ordered sets $(H_1,r_1),\dots,(H_n,r_n)$, where $(y_k,x_k) \in r_k$ means $x_k \in y_k^2$. Indeed, $x_k \in y_k^2$ implies $x_k^2 \subseteq y_k^4$ and

$$y_k *_k x_k *_k y_k = \bigcup_{t \in y_k^2} (t^2 \cup x_k^2) = \bigcup_{t \in y_k^2} t^2 \cup x_k^2 = (y_k^2 *_k y_k^2) \cup$$
$$\cup x_k^2 = y_k^4 \cup x_k^2 = y_k^4 = y_k^2\,.$$

Since

$$(x_1, \ldots, x_n) * (y_1, \ldots, y_n) = \prod_{i=1}^n x_i^2 \cup \prod_{i=1}^n y_i^2 = \prod_{i=1}^n r_i(x_i) \cup \prod_{i=1}^n r_i(y_i) = \\ = r_H((x_1, \ldots, x_n)) \cup r_H((y_1, \ldots, y_n)),$$

we get the assertion. □

Definition. Let $\{(H_1, *_1), \ldots, (H_n, *_n)\}$ be a system of quasi-order hypergroups. The hypergroup $(H, *)$ constructed in the previous Lemma 2 is called a relational product of hypergroups $(H_1, *_1), \ldots, (H_n, *_n)$ and it will be denoted by

$$(H, *) = (H_1, *_1) \times_{\text{rel}} \cdots \times_{\text{rel}} (H_n, *_n) = \prod_{i=1}^{n}{}_{\text{rel}}(H_i, *_i).$$

Theorem 5. *Let $\mathbb{A}_i = (S_i, A_i, \delta_i)$ be an automaton, $\mathbb{H}(\mathbb{A}_i)$ be the state hypergroup of the automaton $\mathbb{A}_i$, $i = 1, 2, \ldots, n$. Then we have*

$$\mathbb{H}(\mathbb{A}_1 \otimes \ldots \otimes \mathbb{A}_n) = \mathbb{H}(\mathbb{A}_1) \times_{rel} \cdots \times_{rel} \mathbb{H}(\mathbb{A}_n).$$

PROOF: Consider a finite system of automata $\mathbb{A}_i = (S_i, A_i, \delta_i)$, $i = 1, 2, \ldots, n$ and denote

$$\mathbb{A} = (S, A, \delta) = \mathbb{A}_1 \otimes \ldots \otimes \mathbb{A}_n,$$

i.e. $S = \prod_{i=1}^{n} S_i$, $A = \prod_{i=1}^{n} S_i$ and the next-state function

$$\delta : S \times A^* \to S$$

which is defined by

$$\delta((s_1, \ldots, s_n), u) = (\delta_1(s_1, u_1), \ldots, \delta_n(s_n, u_n)), \tag{1}$$

where $(s_1, \ldots, s_n) \in S$ and $u = (a_1^{(1)}, \ldots, a_n^{(1)})\,(a_1^{(2)}, \ldots, a_n^{(2)}) \ldots (a_1^{(k)}, \ldots, a_n^{(k)}) \in (A_1 \times \ldots \times A_n)^* = A^*$, $u_i = a_i^{(1)} a_i^{(2)} \ldots a_i^{(k)}$ for $i = 1, 2, \ldots, n$. Since the carrier sets of hypergroups $\mathbb{H}(\mathbb{A}_1 \otimes \ldots \otimes \mathbb{A}_n)$, $\prod_{i=1}^{n}{}_{\text{rel}}\mathbb{H}(\mathbb{A}_i)$ are the same, namely S, we show that the hyperoperation $\circ$ of the first state hypergroup coincides with the hyperoperation $*$ of the second one.

Suppose $(s_1, \ldots, s_n) \in S$, $(t_1, \ldots, t_n) \in S$ are arbitrary n-tuples of states and

$$(q_1, \ldots, q_n) \in (s_1, \ldots, s_n) \circ (t_1, \ldots, t_n) = \delta((s_1, \ldots, s_n), A^*) \cup \delta((t_1, \ldots, t_n), A^*).$$

Then for some word $u = (a_1^{(1)}, \ldots, a_n^{(1)})\,(a_1^{(2)}, \ldots, a_n^{(2)}) \ldots (a_1^{(k)}, \ldots, a_n^{(k)}) \in A^*$ we have either $(q_1, \ldots, q_n) = \delta((s_1, \ldots, s_n), u)$, i.e. $q_i = \delta_i(s_i, a_i^{(1)} \ldots a_i^{(k)})$, $a_i \in A_i$, $i = 1, 2, \ldots, n$ or

$$(q_1,\dots,q_n)=\delta((t_1,\dots,t_n),u)\,,$$

i.e. $q_i=\delta_i(t_i,a_i^{(1)}\dots a_i^{(k)})$ for $i=1,2,\dots,n$. Then

$$\begin{aligned}(q_1,\dots,q_n)&\in\textstyle\prod_{i=1}^n\{\delta_i(s_i,u_i);\ \ u_i\in A_i^*\}\cup\prod_{i=1}^n\{\delta_i(t_i,v_i);\ v_i\in A_i^*\}=\\&=(s_1^2\times\dots\times s_n^2)\cup(t_1^1\times\dots\times t_n^2)=(s_1,\dots,s_n)*(t_1,\dots,t_n)\,,\end{aligned}$$

thus we have

$$(s_1,\dots,s_n)\circ(t_1,\dots,t_n)\subseteq(s_1,\dots,s_n)*(t_1,\dots,t_n)\,. \tag{2}$$

Now suppose

$$(q_1,\dots,q_n)\in(s_1,\dots,s_n)*(t_1,\dots,t_n)=(s_1^2\times\dots\times s_n^2)\cup(t_1^2\times\dots\times t_n^2)\,.$$

Then either $q_i\in s_i^2=s_i\circ_i s_i=\delta_i(s_i,A_i^*)$ for $i=1,2,\dots,n$
or $q_i\in t_i^2=\delta_i(t_i,A_i^*)$, $i=1,2,\dots,n$, hence for suitable words $u_1=a_1^{(1)}\dots a_1^{(k_1)}\in A_1^*,\dots,u_n=a_n^{(1)}\dots a_n^{(k_n)}\in A_n^*$ we have $q_i=\delta_i(s_i,u_i)$, $i=1,2,\dots,n$ in the first case and $q_i=\delta_i(t_i,u_i)$,
$i=1,2,\dots,n$ in the second case. Denote $k=\max\{k_1,\dots,k_n\}$ and $\bar u_i=a_i^{(1)}\dots\dots a_i^{(k_i)}\dots a_i^{(k)}$, where $a_i^{(k_i+1)}=\dots=a_i^{(k)}=e_i$ (the empty word from A_i^*) if $k_i<k$ and $\bar u_i=a_i^{(1)}\dots a_i^{(k_i)}$ if $k_i=k$, $i=1,2,\dots,n$. Then

$$\delta_i(s_i,u_i)=\delta_i(s_i,\bar u_i),\ \ \delta_i(t_i,u_i)=\delta_i(t_i,\bar u_i)$$

for any $i=1,2,\dots,n$ and the length of all words $\bar u_1,\dots,\bar u_n$ is the same $|\bar u_i|=k$ for $i=1,2,\dots,n$. With respect to (1) we have either

$$\begin{aligned}(q_1,\dots,q_n)&=(\delta_1(s_1,\bar u_1),\dots,\delta_n(s_n,\bar u_n))=(\delta_1(s_1,a_1^{(1)}\dots a_1^{(k)}),\dots\\&\dots,\delta_n(s_n,a_n^{(1)}\dots a_n^{(k)}))=\delta((s_1,\dots,s_n),(a_1^{(1)},\dots,a_n^{(1)})\,(a_1^{(2)},\dots,a_n^{(2)})\dots\\&\qquad\dots(a_1^{(k)},\dots,a_n^{(k)}))\quad\text{or}\\(q_1,\dots,q_n)&=(\delta_1(t_1,\bar u_1),\dots,\delta_n(t_n,\bar u_n))=\delta((t_1,\dots,t_n),(a_1^{(1)},\dots,a_n^{(1)})\dots\\&\qquad\dots(a_1^{(k)},\dots,a_n^{(k)}))\,,\end{aligned}$$

where $(a_1^{(1)},\dots,a_n^{(1)})\ (a_1^{(2)},\dots,a_n^{(2)})\dots(a_1^{(k)},\dots,a_n^{(k)})\in\Big(\prod_{i=1}^n A_i\Big)^*=A^*$, consequently

$$\begin{aligned}(q_1,\dots,q_n)&\in\{\delta((s_1,\dots,s_n),u);\ u\in A^*\}\cup\{\delta((t_1,\dots,t_n),v);\ v\in A^*\}=\\&=\delta((s_1,\dots,s_n),A^*)\cup\delta((t_1,\dots,t_n),A^*)=(s_1,\dots,s_n)\circ(t_1,\dots,t_n)\,,\end{aligned}$$

therefore the inclusion

$$(s_1,\dots,s_n)*(t_1,\dots,t_n)\subseteq(s_1,\dots,s_n)\circ(t_1,\dots,t_n)$$

also holds and with respect to the opposite inclusion (2) we get the equality $\mathbb{H}(\mathbb{A}) = \prod_{i=1}^{n}{}_{\mathrm{rel}}\mathbb{H}(\mathbb{A}_i)$. □

A significant class of commutative hypergroups form join spaces – [4] (Def. 156), [18], [22], which generalize some important classical and modern geometrical structures. Recall that a join space is a commutative hypergroup $(J,\cdot)$ satisfying the so called transposition law, i. e. for any quadrupple $a,b,c,d \in J$ such that $a/b \cap c/d \neq \emptyset$ we have $a \cdot d \cap b \cdot c \neq \emptyset$, where

$$a/b = \{x \in J;\ a \in b \cdot x\} \neq \emptyset$$

for any pair $a,b \in J$.

Theorem 3 from [6] yields necessary and sufficient conditions under which order hypergroups are join spaces. This can be easily generalized for the case of quasi-ordering hypergroups as there follows:

Proposition 1. *Let (S,r) be a quasi-ordered set, $*_r : S \times S \to \exp' S$ be a hyperoperation determined by r, i. e. $(S,*_r)$ is a quasi-ordering hypergroup in which $a *_r b = r(a) \cup r(b)$. Then the following conditions are equivalent:*

1° *The hypergroup $(S,*_r)$ is a join space.*

2° *If $a,b \in S$ is an arbitrary pair such that xra, xrb for some $x \in S$ then there exists $y \in S$ such that ary, bry.*

Now we easily prove

Theorem 6. *Let $(S,\circ)$ be a state hypergroup of an automaton $\mathbb{A} = (S,A,\delta)$. Then the following conditions are equivalent:*

1° *The hypergroup $(S,\circ)$ is a join space.*

2° *For any pair of states $(s,t) \in S \times S$ such that $s \circ t \subseteq u^2$ for a suitable state $u \in S$, there exists a state $v \in S$ with the property $v^2 \subseteq s^2 \cap t^2$.*

3° *For any pair of states $(s,t) \in S \times S$ such that there exists a pair of words $(a,b) \in A^* \times A^*$ and a state $u \in S$ with $\delta(u,a) = s$, $\delta(u,b) = t$, we have $\delta(s,c) = \delta(t,d)$ for some pair $(c,d) \in A^* \times A^*$.*

PROOF: We prove implications 1° ⇒ 2° ⇒ 3° ⇒ 1°.

1° ⇒ 2°: Let r be a quasi-order on S determining the hyperoperation $\circ$. Suppose $(s,t) \in S \times S$ is a pair of states such that $s \circ t \subseteq u^2$ for some $u \in S$. Then $r(s) \cup r(t) \subseteq r(u)$, which implies $s \in r(u)$, $t \in r(u)$, i. e. urs and urt. By condition 2° of Proposition 1 there exists $v \in S$ such that srv, trv, i. e. $v \in r(s)$, $v \in r(t)$, consequently

$$r(v) \subseteq r^2(s) \cap r^2(t) \subseteq r(s) \cap r(t)\,,$$

i.e. $v^2 \subseteq s^2 \cap t^2$.

$2^\circ \Rightarrow 3^\circ$: Since $(S, \circ)$ is a state hypergroup of the automaton $\mathbb{A} = (S, A, \delta)$, the above used quasi-order $r \subseteq S \times S$ satisfies

$$r(s) = \delta(s, A^*) = \{\delta(s, a); a \in A^*\}$$

for any state $s \in S$. Suppose $(s, t) \in S \times S$ is a pair of states such that $\delta(u, a) = s$, $\delta(u, b) = t$ for some pair $(a, b) \in A^* \times A^*$ and some state $u \in S$. Then urs, urt thus $s, t \in r(u) = u^2$, consequently

$$s \circ t = r(s) \cup r(t) \subseteq r^2(u) \cup r^2(u) = r^2(u) \subseteq r(u) = u^2 .$$

By 2° there exists $v \in S$ such that

$$v \in r(v) = v^2 \subseteq s^2 \cap t^2 = r(s) \cap r(t) ,$$

which implies srv, trv again. Since r is a transitive cover of the inertial relation $\nu \subseteq S \times S$ (defined by $s\nu t \equiv \delta(s, x) = t$ for some $x \in A$), we have

$$\delta(s, c) = v = \delta(t, d)$$

for a suitable pair of words $(c, d) \in A^* \times A^*$. Thus 3° holds.

$3^\circ \Rightarrow 1^\circ$: It is evident that condition 3° is equivalent to condition 2° of Proposition 1 if srt means $\delta(s, a) = t$ for some $a \in A^*$. Then by Proposition 1 we have the quasi-order hypergroup $(S, \circ)$ is a join space. □

Theorem 40 from [18] says that the direct product of two join spaces is a join space. Similarly as in this mentioned case it is not difficult to show that also the relational product of quasi-order join spaces is a join space.

Recall that if (H_1, r_1), (H_2, r_2) are sets with binary relations we denote – in accordance with [5], Definition 5C.2 – by $(H_1, r_1) \times_{\text{rel}} (H_2, r_2)$ their relational (cartesian) product, thus if (x_1, x_2), $(y_1, y_2) \in (H, r) = (H_1, r_1) \times_{\text{rel}} (H_2, r_2)$, where $H = H_1 \times H_2$, then $(x_1, x_2)\, r\, (y_1, y_2)$ if and only if $x_1 r_1 y_1$, $x_2 r_2 y_2$.

Lemma 3. *Let $\{(H_i, *_i); i = 1, 2, \dots, n\}$ be a family of quasi-order hypergroups which are join spaces. Then their relational product*

$$(H_1, *_1) \times_{rel} \dots \times_{rel} (H_n, *_n)$$

is also a join space.

PROOF: Denote $(H, *) = (H_1, *_1) \times_{\text{rel}} \dots \times_{\text{rel}} (H_n, *_n)$. By Lemma 2 we have $(H, *)$ is a quasi-ordering hypergroup. Let $r \subseteq H \times H$ be the corresponding quasi-ordering. Since for any n-tuple $(x_1, \dots, x_n) \in H$ holds

$$(x_1, \dots, x_n) * (x_1, \dots, x_n) = r(x_1, \dots, x_n) = x_1^2 \times \dots \times x_n^2 = r_1(x_1) \times \dots \times r_n(x_n)$$

(where $r^\wedge$ is the quasi-order determining the hyperoperation */., k = 1,2,.. . ,n), we háve (íf, r) = (#1,^) x_{fel} ... x_{rel} *{H$_n$,r$_n$),* i.e. (x_x,..., ar_n)r(j/i,..., y_n) if and only if *x^kUk* for each fc G {1, 2,..., n}.

Suppose (xi,..., *x_n),* (yi,..., y_n) G iř are such n-tnples that *[u*,. .., u_n)r(xi,... . . ., *x_n),* (tii j • • • j *u_n)r(yi,..., y_n)* for a suitable n-tuple (t/i,..., *u_n)* £ i/. Then *UkTkXk, UkVkVk* for fc = 1, 2,..., n and since each of hypergronps (#&, */-) is a join space, by Proposition 1 for any fc G {1,2, ...,n} there exists t^ G #/c such that *XkrkVk, Vk,$r_k$$v_k$.* Then **(a?i,...,** *$x_n$)r($v_u$* ..., v„), (j/i,..., 2/$_n$)r(vi,..., *$v_n$),* hence by Proposition 1 again, we háve the product hypergroup *(H,* *) is a join space. D

With respect to Lemma 3, we get the following result as an immediate corollary of Theorem 5:

Proposition 2. LeíjA,-; *i* — 1,2,...,n} *be a finite family of automata such that their statě hypergroups* H(A;)$_;$ *i* — 1, 2,. .., *n are join spaces. Then the statě hypergroup*

H(Ai (g) ...<g> A_n)

Ž5 a/.so *a join space.*

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