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Partitions and partially ordered sets

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Abstract. The paper deals with the connection between partitions, non-isomorphic posets and non-isomorphic continuous posets. This connection is studied from the point of view of finding the recurrence formula for the number P_n of non-isomorphic n-element posets. We also determine the number of all non-isomorphic continuous posets for $n \leq 13$.

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1 Introduction

Let us denote by N the set of all positive integers, $N_0 = N \cup \{0\}$. A partition of the number $n \in N$ is a sequence $a = (a_1, \ldots, a_k) \in N^k$, where $1 \leq k \leq n$, such that $a_1 + \ldots + a_k = n$ and $a_1 \geq \ldots \geq a_k$. The terms a_i are called the parts of a. If the partition a has k_i parts equal to i, then we write $a = \langle 1^{k_1}, 2^{k_2}, \ldots \rangle$, where all terms with $k_i = 0$ may be omitted. For instance, $(4,4,2,2,2,1) = \langle 1^1, 2^3, 4^2 \rangle = 4+4+2+2+2+1$ is a partition of the number 15. The set of all partitions of n will be denoted by $\mathbf{P}(n)$. A composition of the number $n \in N$ is a sequence $a = (a_1, \ldots, a_k) \in N^k$, where $1 \leq k \leq n$, such that $a_1 + \ldots + a_k = n$. For example, $(2,1,2,4,4,2) = 2+1+2+4+4+2$ is a composition of 15. If exactly k summands appear in a composition a, we call a a k-composition. We shall denote by $\mathbf{C}(n)$ the set of all compositions of n and by $\mathbf{C}(n,k)$ the set of all k-compositions of n. We recall that there is a bijection between all k-compositions of n and $(k-1)$-subsets of $\{1, 2, \ldots, n-1\}$. Hence there are $\binom{n-1}{k-1}$ k-compositions of n and 2^{n-1} compositions of n. A comprehensive survey of the theory of partitions can be found in the monographs [1] or [6]. Further, our work [5] is useful for comparison of the procedures and methods which are used in this paper.

A partially ordered set $(A, \leq)$ or poset, for short, is a set A together with a binary relation $\leq$ on A, which is satisfying the following three axioms: 1. For all $x \in A, x \leq x$ (reflexivity). 2. If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry). 3. If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity). We use the obvious notation $x < y$ to express $x \leq y$ and $x \neq y$. When there is a possibility of confusion, we write precisely $(A, \leq_A)$. A binary relation $\leq_A$ is called a partial order or an ordering. We say that two posets $(A, \leq_A)$ and $(B, \leq_B)$ are isomorphic if there exists an order-preserving bijection $\varphi : A \to B$ whose inverse is order-preserving bijection as well; that is, for all $x, y \in A$: $x \leq_A y \Leftrightarrow \varphi(x) \leq_B \varphi(y)$. If two posets $(A, \leq_A)$ and $(B, \leq_B)$ are isomorphic, we write $A \cong B$. Next, we define a set

P_1	=	1	(Folklore)	
P_2	=	2	(Folklore)	
P_3	=	5	(Folklore)	
P_4	=	16	(Folklore)	
P_5	=	63	(Folklore)	
P_6	=	318	(Folklore)	
P_7	=	2 045	(1972)	J. Wright
P_8	=	16 999	(1977)	S. K. Das
P_9	=	183 231	(1984)	R. H. Möhring
P_{10}	=	2 567 284	(1990)	J. C. Culberson and G. J. E. Rawlins
P_{11}	=	46 749 427	(1990)	J. C. Culberson and G. J. E. Rawlins
P_{12}	=	1 104 891 746	(1991)	C. Chaunier and N. Lygerös
P_{13}	=	33 823 327 452	(1992)	C. Chaunier and N. Lygerös

Table 1: Values P_n for non-isomorhic n-elements posets, $n \leq 13$

$P(A) := \{(A, \leq_A);\quad \leq_A$ is a partial order on A$\}$. It is well-known that $\cong$ is an equivalence on $P(A)$. The blocks of a partition of the set $\mathbf{P}_n := P(A)/\cong$ are called non-isomorphic posets. In what follows A will denote the set of n elements. We shall denote by P_n the number of all non-isomorphic n-element posets. Non-isomorphic posets can be represented by means of Hasse diagrams. We define: if $x, y \in A$, then we say y covers x if $x < y$ and if no element $z \in A$ satisfies $x < z < y$. The Hasse diagram of a finite poset A is the graph whose edges are the cover relations, and such that if $x < y$, then y is drawn "above" x (i. e. with the higher horizontal coordinate). We remark that the theory of posets is studied e. g. in the monograph [10]. Further, the basic results on posets are presented in the survey [4]. Moreover, the connection between partitions and posets is investigated in [9], but from a different point of view as in this paper. Now we recall the known values of P_n which were introduced in [2] by C. Chaunier and N. Lygerös. We underline that even finding of P_6 was a difficult problem (see e. g. [8]).

The structure of the paper is as follows. First we draw our attention to relations between numbers of non-isomorphic posets and non-isomorphic continuous posets and to their connection with partitions. Our relations will have the form of formulas for P_n. Next we derive a new expression of the power series of the sequence P_n in the form of an infinite product. This is an analogy to Euler's form of the generating function of the sequence of numbers of partitions. Taking into account the main idea of Euler's method which was used in the proof of his well-known pentagonal theorem, we deduce further result. Moreover we determine another relationship between P_n and the number of non-isomorphic continuous posets. Finally we introduce the number of all non-isomorphic continuous posets for $n \leq 13$. Let us remark that in this paper we shall use only standard and classical methods (i. e. elementary combinatorical techniques and the machinery of formal power series).

2 Continuous partially ordered sets

Definition 1. Let $(A, \leq)$ be a partially ordered set, $x, y \in A$. We say that two elements x and y are *comparable* and we write $x \frown y$, if $x \leq y$ or $y \leq x$. Otherwise x and y are called *incomparable.* For $x, y \in A$ we put $x \sim y$ iff there are $k \in N$ and k elements $x_1, \ldots, x_k \in A$ such that $x \frown x_1, \ldots, x_k \frown y$. The poset $(A, \leq)$ is called *continuous,* if for all $x, y \in A : x \sim y$. Otherwise it is called *discontinuous.* Now we define a set $C(A) := \{(A, \leq_A) \in P(A); (A, \leq_A) \text{ is continuous}\}$. Then $\cong$ is an equivalence relation on $C(A)$ and the blocks of $\mathbf{C}_n := C(A)/\cong$ are called *non-isomorphic continuous posets.* We shall denote by C_n the number of all non-isomorphic n-element posets.

Example 1. There are exactly 10 non-isomorphic continuous 4-element posets, i.e. the set $\mathbf{C}_4$ has 10 elements and $C_4 = 10$. All non-isomorphic continuous 4-element posets are shown in Figure 1.

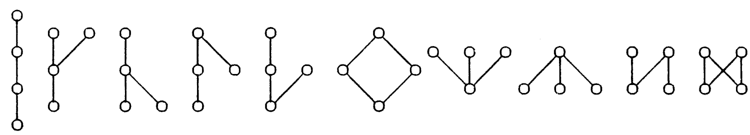

Figure 1

Definition 2. Let $(A, \leq)$ be a poset. For every $x \in A$ we define a set $A_x := \{y \in A; x \sim y\}$. It is evident that for $x, y \in A, x \neq y$ it holds $x \sim y \Leftrightarrow A_x = A_y$ and $\sim$ is an equivalence relation on A. Blocks of the set $A/\sim$ are called *continuous parts* of $(A, \leq)$.

Example 2. Figure 2 shows the Hasse diagram of a poset with 16 elements. This poset is not continuous and has 4 continuous parts. Further, this poset corresponds to a partition $(5, 5, 3, 3) = \langle 3^2, 5^2 \rangle$ of an integer 16.

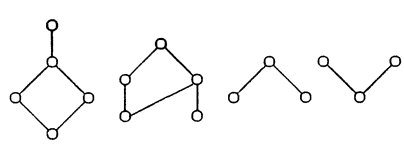

Figure 2

Now we introduce the first assertion, where we show the connection between partitions, non-isomorphic posets and non-isomorphic continuous posets.

Theorem 1. *For each positive integer n we have the formula*

$$P_n = \sum_{\mathbf{P}(n)} \binom{C_1 + k_1 - 1}{k_1} \dots \binom{C_n + k_n - 1}{k_n}, \tag{1}$$

where the sum extends over all partitions $\langle 1^{k_1}, 2^{k_2}, \dots \rangle$ of n.

PROOF: Let $a = \langle 1^{k_1}, 2^{k_2}, \dots \rangle \in \mathbf{P}(n)$ be an arbitrary partition of n and let us consider a part of a which is created by exactly k_i summands i. Let $N(k_i, C_i)$ be the number of all different disarranged k_i-tuples of a non-isomorphic continuous i-element poset. Clearly, $N(k_i, C_i) = \frac{1}{i} C_i(C_i + 1) \dots (C_i + k_i - 1) = \binom{C_i + k_i - 1}{k_i}$, so that $N(k_i, C_i)$ is equal to the number of k_i-element combinations with a repetition of C_i elements. For $k_i = 0$ we put $N(0, C_i) := 1$. Further, by means of the product rule, for each $\langle 1^{k_1}, 2^{k_2} \dots \rangle \in \mathbf{P}(n)$ there are exactly $N(k_1, C_1) \dots N(k_n, C_n)$ elements of the set $\mathbf{P}(n)$, which are composed from k_i i-element continuous parts for $i = 1, \dots, n$. Finally, by the addition rule we have $P_n = \sum_{\mathbf{P}(n)} N(k_1, C_1) \dots N(k_n, C_n)$. (1) is now evident. □

Example 3. We find the number of non-isomorphic posets for $n = 5$ by means of (1). We shall suppose that we know the numbers $C_1 = 1, C_2 = 1, C_3 = 3, C_4 = 10$ and $C_5 = 44$. Clearly,
$\mathbf{P}(5) = \{(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)\} =$
$= \{\langle 5^1 \rangle, \langle 4^1, 1^1 \rangle, \langle 3^1, 2^1 \rangle, \langle 3^1, 1^2 \rangle, \langle 2^2, 1^1 \rangle, \langle 2^1, 1^3 \rangle, \langle 1^5 \rangle\}$. Now we have by the formula (1): $P_5 = \binom{C_5}{1} + \binom{C_4}{1}\binom{C_1}{1} + \binom{C_3}{1}\binom{C_2}{1} + \binom{C_3}{1}\binom{C_1 + 1}{1}$
$+ \binom{C_2 + 1}{2}\binom{C_1}{1} + \binom{C_2}{1}\binom{C_1 + 2}{3} + \binom{C_1 + 4}{5} = 44 + 10 + 3 + 3 + 1 + 1 + 1 = 63.$

3 The recurrence formula for P_n

Let us denote by $N_n(C, k)$ the number of all continuous k-element parts which occur in all elements of $\mathbf{P}_n$. Let x be an arbitrary continuous k-element part from $\mathbf{P}_n$. Then we shall denote by $N_n(x, k)$ the number of all occurrances of x in all elements of $\mathbf{P}_n$. Now we derive a new recurrent formula for P_n which is analogous to that for the number of partitions of an integer n into summands (cf. [5]).

Theorem 2. *For each positive integer n we have the formula*

$$P_n = \frac{1}{n} \sum_{k=0}^{n-1} \alpha(n-k) P_k, \quad \textit{where} \quad \alpha(m) := \sum_{k \mid m} k C_k. \tag{2}$$

PROOF: Clearly, we have the following identity

$$\sum_{k=1}^{n} k \cdot N_n(C, k) = n \cdot P_n. \tag{3}$$

We first determine the value $N_n(C,k)$ for $1 \le k \le n$. Let $n = m_k \cdot k + z_k$, where $z_k < k$ is the remainder after division of n by k and m_k is the partial quotient. Let x be an arbitrary continuous k-element part from $\mathbf{P}_n$ and let $1 \le r \le m_k$. Then we see that the number of all elements of $\mathbf{P}_n$, which contain at least r k-parts x is P_{n-rk}. Hence the number of all k-parts x in $\mathbf{P}_n$ is

$$N_n(x,k) = P_{n-k} + \ldots + P_{n-m_k k}. \tag{4}$$

It is evident that the number $N_n(x,k)$ is the same for every continuous k-part from $\mathbf{P}_n$. Hence by the product rule we have

$$N_n(C,k) = C_k(P_{n-k} + \ldots + P_{n-m_k k}). \tag{5}$$

Further, the relations (3) and (5) together give the following formula for P_n

$$P_n = \frac{1}{n}\sum_{k=1}^{n} kC_k(P_{n-k} + \ldots + P_{n-m_k k}). \tag{6}$$

Now we simplify (6). Let $0 \le s \le n-1$ be a natural number. Let us consider when P_s occurs among the members of the sum $P_{n-k} + \ldots + P_{n-m_k k}$, i.e. when $n - r \cdot k = s$ for some $1 \le r \le m_k$. Clearly, the relation $n - r \cdot k = s$ holds iff k divides $n - s$. Therefore P_s occurs in all the sums $P_{n-k} + \ldots + P_{n-m_k k}$ where k divides $n-s$. Then the number of all occurrances of P_s in the sum $\sum_{k=1}^{n} kC_k(P_{n-k} + \ldots + P_{n-m_k k})$ is exactly $\sum_{k|n-s} kC_k$, since P_s occurs in the sum $\sum_{k=1}^{n} kC_k(P_{n-k} + \ldots + P_{n-m_k k})$ (kC_k)-times for every natural divisor k of the number $n-s$. Thus we have

$$\sum_{k=1}^{n} kC_k(P_{n-k} + \ldots + P_{n-m_k k}) = \sum_{s=0}^{n-1}\left(\sum_{k|n-s} kC_k\right) P_s.$$

This completes the proof of (2). □

Example 4. Now we enumerate the value P_6 by means of our formula (6). We suppose that the values $P_0 = 1, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 16, P_5 = 63$ and the values $C_1 = 1, C_2 = 1, C_3 = 3, C_4 = 10, C_5 = 44, C_6 = 238$ are already known. By formula (2) we have $P_6 = \frac{1}{6}(\alpha(6)P_0+\alpha(5)P_1+\alpha(4)P_2+\alpha(3)P_3+\alpha(2)P_4+\alpha(1)P_5) = \frac{1}{6}((1C_1 + 2C_2 + 3C_3 + 6C_6)P_0 + (1C_1 + 5C_5)P_1 + (1C_1 + 2C_2 + 4C_4)P_2 + (1C_1 + 3C_3)P_3+(1C_1+2C_2)P_4+(1C_1)P_5 = \frac{1}{6}(1440P_0+221P_1+43P_2+10P_3+3P_4+P_5) = \frac{1}{6}(1440 + 221 + 86 + 50 + 48 + 63) = 318$.

4 The formal power series of the sequence P_n

Let $P(x) = \sum_{n=0}^{\infty} P_n x^n$ be the formal power series of the sequence P_n. Using our recurrence (2), we first determine a new form of $P(x)$. We remark that formal power series are studied comprehensively e. g. in [7].

Theorem 3. *The formal power series $P(x)$ of the sequence P_n has the form*

$$P(x) = \prod_{n=1}^{\infty} \left(\frac{1}{1-x^n} \right)^{C_n} . \tag{7}$$

Proof: Let $A(x) = \sum_{n=1}^{\infty} \alpha(n)x^n$ be the formal power series of $\alpha(n)$. Multiplicating formula (2) by nx^n and then evaluating the sum for each positive integer n we obtain the following equation

$$\sum_{n=1}^{\infty} nP_n x^n = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} \alpha(n-k)P_k \right) x^n . \tag{8}$$

It is evident that the left hand side of (8) is equal to $xP'(x)$ and one verifies easily that the right hand side of (8) is equal to the product $A(x)P(x)$. Hence

$$xP'(x) = A(x)P(x). \tag{9}$$

This equation yields $\ln P(x) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n} x^n$, which is nothing else than

$$P(x) = \exp \left(\sum_{n=1}^{\infty} \frac{\alpha(n)}{n} x^n \right) . \tag{10}$$

Furthermore, we arrange the form of the series $\sum_{n=1}^{\infty} \frac{\alpha(n)}{n} x^n$. We get

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n} x^n = \sum_{k=1}^{\infty} kC_k \left(\sum_{n=1}^{\infty} \frac{x^{nk}}{nk} \right) = \sum_{k=1}^{\infty} C_k \ln \frac{1}{1-x^k} .$$

Finally, from this and from (10) we obtain the following relationship

$$P(x) = \exp \left(\sum_{k=1}^{n} C_k \ln \frac{1}{1-x^k} \right) = \prod_{k=1}^{\infty} \left(\frac{1}{1-x^k} \right)^{C_k} ,$$

which is nothing else than (7). This completes the proof. □

Using Theorem 3, we deduce further formula for P_n. In the proof we shall use the main idea of the proof of Euler's pentagonal formula for the number of partitions of an integer n (cf. [1]).

Theorem 4. *For each positive integer n we have the formulas*

$$P_n = -\sum_{k=0}^{n-1} Q_{n-k} P_k \quad \text{and} \quad Q_n = \sum_{S} (-1)^{k_1+\dots+k_n} \binom{C_1}{k_1} \dots \binom{C_n}{k_n} , \tag{11}$$

where the sum extends over the set S of all solutions $[k_1, \dots, k_n] \in \{0, 1, \dots, n\}^n$ of the linear Diophantine equation $1k_1 + 2k_2 + \dots + nk_n = n$.

PROOF: Let $Q(x)=\sum_{n=0}^{\infty} Q_n x^n$ be the formal power series such that $P(x)\cdot Q(x)=1$. By (7) we have $\sum_{n=0}^{\infty} Q_n x^n=\prod_{n=1}^{\infty}(1-x^n)^{C_n}$ and the binomial theorem reads

$$\prod_{n=1}^{\infty}(1-x^n)^{C_n}=\prod_{n=1}^{\infty}\sum_{i=0}^{C_n}(-1)^i\binom{C_n}{i}x^{in}. \tag{12}$$

Hence the general member of the series $\sum_{n=0}^{\infty} Q_n x^n$ has the form

$$(-1)^{j_1+\dots+j_m}\binom{C_{k_1}}{j_1}\dots\binom{C_{k_m}}{j_m}x^{j_1k_1+\dots+j_mk_m}. \tag{13}$$

Let $1\le s\le m$ be an arbitrary positive integer. It is evident that in the form (13) all numbers k_s are different, but the numbers j_s may be the same and further $0\le j_s\le n$ and $1\le k_s\le n$. Now we see that there is a one to one correspondence between the set S of all solutions $[j_1,\dots,j_n]\in\{0,1,\dots,n\}^n$ of the linear Diophantine equation $1j_1+2j_2+\dots+j_mk_m=n$ and all cases when the equation $j_1k_1+\dots+j_mk_m=n$ is satisfied. From this it follows that the coefficient by x^n is

$$Q_n=\sum_S(-1)^{j_1+\dots+j_n}\binom{C_1}{j_1}\dots\binom{C_n}{j_n}. \tag{14}$$

Clearly, in the product $P(x)Q(x)$ the coefficient by x^n is $\sum_{k=0}^{n}P_kQ_{n-k}$. Now we recall that $P(x)Q(x)=1$. Hence in this product the coefficient by x^n is equal to 0 for $n\ge 1$. This implies $\sum_{k=0}^{n}P_kQ_{n-k}=0$ and so we have $P_n=-\sum_{k=0}^{n-1}P_kQ_{n-k}$. This completes the proof. □

Example 5. We again suppose that the numbers C_1,C_2,C_3,C_4 and C_5 are already known. First we find the number Q_5 by means of the formula (11). We shall solve the linear Diophantine equation $1j_1+2j_2+3j_3+4j_4+5j_5=5$ over the set $\{0,1,2,3,4,5\}^5$. There are exactly 6 solutions of this equation and the set S of all such solutions is:
$S=\{[5,0,0,0,0],[0,0,0,0,1],[2,0,1,0,0],[3,1,0,0,0],[0,1,1,0,0],[1,0,0,1,0]\}$.
Now we have $Q_5=(-1)^5\binom{C_1}{5}+(-1)^1\binom{C_5}{1}+(-1)^{2+1}\binom{C_1}{2}\binom{C_3}{1}+$
$+(-1)^{3+1}\binom{C_1}{3}\binom{C_2}{1}+(-1)^{1+1}\binom{C_2}{1}\binom{C_3}{1}+(-1)^{1+1}\binom{C_1}{1}\binom{C_4}{1}=$
$=-C_5+C_2C_3+C_1C_4=-44+3+10=-31$. Analogously we find the values Q_1,Q_2,Q_3 and Q_4. Now by the formula (11) we have $P_5=-(P_0Q_5+P_1Q_4+P_2Q_3+P_3Q_2+P_4Q_1)=31P_0+7P_1+2P_2+P_3+P_4=63$.

Now we introduce two forms of the formal power series $A(x)$ of $\alpha(n)$ and then we mention the connection of such series with the known series from number theory.

Corollary 1. *We have the following relations for the formal power series $A(x)$*

$$A(x)=\sum_{n=1}^{\infty}C_n\cdot\frac{nx^n}{1-x^n}=\sum_{n=1}^{\infty}\frac{x}{n}\cdot C'(x^n), \tag{15}$$

i. e. $A(x)$ is a special case of the Lambert series. (see e. g. [3], page 146).

PROOF: We prove (15) by different ways of the summation of $A(x)$. First we have

$$A(x)=\sum_{n=1}^{\infty}\alpha(n)x^n=\sum_{n=1}^{\infty}nC_n\sum_{k=1}^{\infty}x^{kn}=\sum_{n=1}^{\infty}nC_nx^n\sum_{k=0}^{\infty}x^{kn}=\sum_{n=1}^{\infty}C_n\frac{nx^n}{1-x^n}.$$

Quite similarly,

$$A(x)=\sum_{n=1}^{\infty}\alpha(n)x^n=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}nC_nx^{kn}=\sum_{n=1}^{\infty}\frac{x}{n}\sum_{k=1}^{\infty}(C_nx^{kn})'=\sum_{n=1}^{\infty}\frac{x}{n}C'(x^n).$$

□

Remark. *We note that the important special cases of the Lambert series in the number theory are*

$$\sum_{n=1}^{\infty}\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty}d(n)x^n \quad and \quad \sum_{n=1}^{\infty}\frac{nx^n}{1-x^n}=\sum_{n=1}^{\infty}\sigma(n)x^n,$$

where $d(n)$ is the number and $\sigma(n)$ is the sum of all positive divisors of n, respectively. These series converge for every $|x|<1$. The connection between the arithmetic function $\sigma(n)$ and partitions is studied e. g. in [5].

Taking into account formula (2) and Table 1, we can directly compute the numbers C_n of non-isomorphic continuous posets. We have also determined initial members of the sequence Q_n for $n \le 13$.

Finally we deduce another identity for P_n. In the proof we shall use relation (10) from the proof of Theorem 3.

Theorem 5. *We have the following identity for the number P_n*

$$P_n=\sum_{a_1+\ldots+a_k\in\mathbf{C}(n)}\frac{1}{k!}\frac{\alpha(a_1)}{a_1}\cdots\frac{\alpha(a_k)}{a_k}. \tag{16}$$

PROOF: Let $F(x):=\sum_{n=1}^{\infty}\frac{\alpha(n)}{n}x^n$. Then (10) yields $P(x)=e^{F(x)}$. Let us develop function P into the series of powers of the function F. We have

$$P(x)=\sum_{m=0}^{\infty}\frac{F^m(x)}{m!}=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\sum_{n=1}^{\infty}\frac{\alpha(n)}{n}x^n\right)^m.$$

Hence after involution of the series F we obtain the development of function P into the series of powers of x. Now we determine the coefficient by x^n in the development of function P. Let k be a natural number. Clearly, each member of

C_0	=	1	Q_0	=	1
C_1	=	1	Q_1	=	-1
C_2	=	1	Q_2	=	-1
C_3	=	3	Q_3	=	-2
C_4	=	10	Q_4	=	-7
C_5	=	44	Q_5	=	-31
C_6	=	238	Q_6	=	-184
C_7	=	1 650	Q_7	=	-1 351
C_8	=	14 512	Q_8	=	-12 524
C_9	=	163 341	Q_9	=	-146 468
C_{10}	=	2 360 719	Q_{10}	=	-2 177 570
C_{11}	=	43 944 974	Q_{11}	=	-41 374 407
C_{12}	=	1 055 019 099	Q_{12}	=	-1 008 220 289
C_{13}	=	32 664 484 238	Q_{13}	=	-31 558 946 774

Table 2: Initial values of C_n and Q_n for $n \leq 13$.

the series F^k arises as the product of k factors. The general member of the series F^k has the form

$$\left(\frac{\alpha(a_1)}{a_1}x^{a_1}\right)\cdots\left(\frac{\alpha(a_k)}{a_k}x^{a_k}\right) = \frac{\alpha(a_1)}{a_1}\cdots\frac{\alpha(a_k)}{a_k}x^{a_1+\ldots+a_k}.$$

If $k > m$, then there is no member with x^n or with the lower power of x in the series F^k, so that only a part of the series $\sum_{k=1}^{n}\frac{1}{k!}F^k$ contains members with x^n. Furthermore, in the series F^k the coefficient by x^n is $\frac{\alpha(a_1)}{a_1}\cdots\frac{\alpha(a_k)}{a_k}$ iff $a_1+\ldots+a_k = n$. Clearly, the number of coefficients with this property is equal to card$\mathbf{C}(n,k)$. So in the series F^k the coefficient by x^n is

$$\sum_{a_1+\ldots+a_k\in\mathbf{C}(n,k)}\frac{\alpha(a_1)}{a_1}\cdots\frac{\alpha(a_k)}{a_k}$$

and finally in the series $\sum_{m=0}^{\infty}\frac{F^m}{m!}$ the coefficient by x^n is equal to

$$\sum_{k=1}^{n}\frac{1}{k!}\sum_{a_1+\ldots+a_k\in\mathbf{C}(n,k)}\frac{\alpha(a_1)}{a_1}\cdots\frac{\alpha(a_k)}{a_k} = \sum_{a_1+\ldots+a_k\in\mathbf{C}(n)}\frac{1}{k!}\frac{\alpha(a_1)}{a_1}\cdots\frac{\alpha(a_k)}{a_k}.$$

By comparison of the coefficients by x^n with the series $\sum_{n=0}^{\infty}P_nx^n$ we obtain (16). This completes the proof. □

Example 6. Now we demonstrate the identity (16) for $n = 4$. Since the values C_1, C_2, C_3 and C_4 are already known, we easily compute that $\alpha(1) = 1$, $\alpha(2) = 3$, $\alpha(3) = 10$ and $\alpha(4) = 43$. Further, applying the identity (19) we have

$$P_4 = \frac{1}{1!}\frac{\alpha(4)}{4} + \frac{1}{2!}\left(\frac{\alpha(3)}{3}\frac{\alpha(1)}{1} + \frac{\alpha(1)}{1}\frac{\alpha(3)}{3} + \frac{\alpha(2)}{2}\frac{\alpha(2)}{2}\right) +$$

$$+\frac{1}{3!}\left(\frac{\alpha(2)}{2}\frac{\alpha(1)}{1}\frac{\alpha(1)}{1}+\frac{\alpha(1)}{1}\frac{\alpha(2)}{2}\frac{\alpha(1)}{1}+\frac{\alpha(1)}{1}\frac{\alpha(1)}{1}\frac{\alpha(2)}{2}\right)+$$

$$+\frac{1}{4!}\frac{\alpha(1)}{1}\frac{\alpha(1)}{1}\frac{\alpha(1)}{1}\frac{\alpha(1)}{1}=\frac{1}{1!}\frac{43}{4}+\frac{1}{2!}\frac{107}{12}+\frac{1}{3!}\frac{9}{2}+\frac{1}{4!}\frac{1}{1}=16.$$

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