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Optimal Polygonal Interpolation *

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Abstract

The problem of interpolation of function values and mean values with linear spline (polygon) on given knot set is discussed. The solution with minimal norm of function values or values of the derivative is discussed in analytic form (explicit expression for optimal free parameter) and two algorithms for computing such optimal solutions are discussed and demonstrated (with difference equation or pseudoinverse matrix approach).

Key words: Optimal linear splines, interpolants with minimal norm, optimal interpolation of function, mean and derivative values.

1991 Mathematics Subject Classification: 4A15, 65D05

1 Statement of the problem

Let us have given real data

$$\mathbf{x} = \{x_i;\ i = 0(1)n+1,\ h_i = x_{i+1} - x_i\}; \quad \mathbf{g} = \{g_i,\ i = 0(1)n+1\},$$

where $\mathbf{x}$ denotes a simple monotone set of knots on the real axis, $\mathbf{g}$ are function values prescribed in knots and h_i are knotset stepsizes. If we consider the case when these knots are points of interpolation, then there is unique linear spline $s_{11}(x) = s(x)$ on the knotset $\mathbf{x}$ (linear polygon) interpolating given values $\mathbf{g}$. This polygon realizes the unique minimum of the L_2-norm of the first derivative of functions interpolating prescribed values $\mathbf{g}$ (see e.g. [2]).

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In case that the prescribed values g_i, $i = 0(1)n$ are

- function values $g_i = s(t_i)$ in the points $t_i \in (x_i, x_{i+1})$ (points of interpolation between spline knots), (FVI)
- the local mean values $g_i = \frac{1}{h_i} \int_{x_i}^{x_{i+1}} s(x)dx$, (MVI)
- the derivative values $g_i = s'(t_i) = s'(x_i + 0)$, (DVI)

then interpolating linear spline depends on one free parameter (e.g. we can prescribe function value or derivative value in some spline knot). We can then search for such a value of the free parameter, which gives the "best" solution of our problem with respect to some criterion (e.g. some norm of function or derivative values or vectors of their values $s(x_i)$, $s'(t_i)$). Some extremal properties are known for "natural cubic splines" for FVI problem and "natural quadratic splines" for MVI problem in L_2-norm (see [2]).

We will present an explicit analytic solution of several such problems and numerical algorithms for effective computing of the solution for such or similar problems.

2 Function or derivative values interpolation

2.1 Continuity conditions for local parameters

Let us have given the mesh of spline knots x_i and points of interpolation t_i with prescribed function values $g_i = s(t_i)$. We shall denote the local parameters of the polygon with the geometry parameters $d_i = (t_i - x_i)/h_i$, $p_i = 1/d_i$ as

$$s_i = s(x_i), \quad \mathbf{s} = [s_i]_{i=0}^{n+1}, \quad m_i = s'(t_i), \quad \mathbf{m} = [m_i]_{i=0}^{n}. \tag{1}$$

It is easy to show the following relations for one kind of unknown parameters

$$(p_j - 1)s_j + s_{j+1} = p_j g_j\,, \quad (1 - d_j)s_j + d_j s_{j+1} = g_j\,, \tag{2}$$

$$\begin{aligned} h_j(1 - d_j)m_j + d_{j+1}h_{j+1}m_{j+1} &= g_{j+1} - g_j\,, \\ (x_{j+1} - t_j)m_j + (t_{j+1} - x_{j+1})m_{j+1} &= g_{j+1} - g_j \end{aligned} \tag{3}$$

and relation connecting different local parameters $\mathbf{s}, \mathbf{m}$

$$s_{j+1} = s_j + h_j m_j = g_j + h_j(1 - d_j)m_j\,, \quad m_j = (g_j - s_j)/(d_j h_j). \tag{4}$$

We shall start with the discussion of polygons $s_{11}(x) = s(x)$ determined uniquelly by given data $\mathbf{x}$, $\mathbf{g}$ and the function value $s_0 = s(x_0)$ or derivative value $m_0 = s'(t_0)$.

2.2 Solutions depending on free parameter s_0

We can use the mentioned recursive relations (2) between local parameters s_j, g_j of segments of the polygon $s(x)$ written as

$$s_{j+1} = (1 - p_j)s_j + p_j g_j\,, \quad j = 0(1)n. \tag{5}$$

Treating this reccurrence by induction or as the first-order nonhomogeneous difference equation we can obtain for the solution the explicit expression

$$s_j = s_0 c_0^{j-1} + \sum_{i=0}^{j-1} g_i p_i c_{i+1}^{j-1}, \qquad c_i^j = \prod_{k=i}^{j}(1-p_k), \quad c_i^j = 1 \text{ for } i > j. \tag{6}$$

Using relations (4) we obtain for m_j the explicit expression

$$d_j h_j m_j = g_j - s_j = g_j - s_0 c_0^{j-1} - \sum_{i=0}^{j-1} g_i p_i c_{i+1}^{j-1}. \tag{7}$$

From continuous dependence of local parameters s_j, m_j on initial value s_0 described by these expressions (values g_i are given; p_i, h_i, d_i are parameters of the given mesh of knots and points of interpolation) we obtain

$$\frac{ds_j}{ds_0} = c_0^{j-1} = \prod_{i=0}^{j-1}(1-p_i), \qquad d_j h_j \frac{dm_j}{ds_0} = -c_0^{j-1} = -\prod_{i=0}^{j-1}(1-p_i). \tag{8}$$

When we search for such a value of s_0 which gives minimum to the functional

$$F_0(s_0) = \sum_{j=0}^{n+1} s_j^2 = \|\mathbf{s}\|_2^2 = (\mathbf{s}, \mathbf{s})_2\,, \tag{9}$$

then the necessary condition for minimum with respect to s_0,

$$\frac{dF_0}{ds_0} = 2\sum_{j=0}^{n+1} s_j \frac{ds_j}{ds_0} = 0 \tag{10}$$

with (6) and (8) used, gives for such an optimal value s_0^0 the relation

$$\begin{aligned} s_0^0 &= -\Big[\sum_{j=0}^{n+1} c_0^{j-1} \sum_{i=0}^{j-1} g_i p_i c_{i+1}^{j-1}\Big] \Big/ \sum_{j=1}^{n+1} (c_0^{j-1})^2 \\ &= -\Big[\sum_{k=0}^{n} g_k p_k \sum_{i=k}^{n} c_0^i c_{k+1}^i\Big] \Big/ \sum_{j=0}^{n+1} (c_0^{j-1})^2. \end{aligned} \tag{11}$$

When we want to find such a value of s_0 which gives the minimum to the norm of vector $\mathbf{m}$ of first derivatives, then from conditions

$$F_1(s_0) = \sum_{j=0}^{n} m_j^2, \qquad \frac{dF_1}{ds_0} = 2\sum_{i=0}^{n} m_i \frac{dm_i}{ds_0} = 0 \tag{12}$$

and (7), (8) we can explicitly compute such an optimal value as

$$s_0^1 = \Big[\sum_{j=0}^{n} w_j c_0^{j-1}\Big(g_j - \sum_{i=0}^{j-1} c_{i+1}^{j-1} g_i p_i\Big)\Big] \Big/ \sum_{j=0}^{n} w_j (c_0^{j-1})^2 \tag{13}$$

with

$$w_j = 1/(d_j h_j), \quad c_i^j = \prod_{k=i}^{j-1}(1-p_k), \ (c_j^i = 1, \ i > j).$$

When we want to find such a polygon which minimizes (similarly as in case of interpolation in knots) the L_2-norm of the first derivative $s'(x)$

$$||s'(x)||_2^2 = \int_{x_0}^{x_{n+1}} [s'(x)]^2 dx = \sum_{i=0}^{n} h_i m_i^2, \tag{14}$$

then we obtain for optimal parameter value s_0 expression similar to (13), but with $w_j = 1/d_j$.

In the DVI problem we obtain from the first recursion in (4) with known values $g_i = m_i$ and initial value s_0 by induction the explicit expression for components of the vector $\mathbf{s}$:

$$s_j = s_0 + \sum_{i=0}^{j-1} h_i m_i, \quad j = 1(1)n. \tag{15}$$

The optimal value of s_0 which gives minimum to the 2-norm of $\mathbf{s}$ can be then computed analogously with the result

$$(n+2)s_0 = -\sum_{j=0}^{n}(n+1-j)h_j m_j\,. \tag{16}$$

2.3 Solutions depending on parameter m_0

We can use similar approach for obtaining explicit formulas for solution of our problem using relation between local parameters $\mathbf{m}$ of spline given by the formula (3). When we treat this reccurence by induction or as nonhomogeneous linear difference equation with free parameter m_0 we obtain the explicit formula

$$d_j h_j m_j = (d_0 h_0 m_0 - g_0)c_0^{j-1} + g_j - \sum_{i=0}^{j-1} p_i g_i c_{i+1}^{j-1}; \quad j = 0(1)n. \tag{17}$$

The problem to find such an optimal value of m_0 which corresponds to the minimal 2-norm of the vector $\mathbf{m}$ leads to the relations

$$\sum_{j=0}^{n} m_j \frac{dm_j}{dm_0} = 0, \quad \frac{dm_j}{dm_0} = \frac{d_0 h_0}{d_j h_j} c_0^{j-1}$$

which gives for the optimal value m_0^1 the expression

$$d_0 h_0 m_0^1 = g_0 + \Big[\sum_{j=0}^{n} w_j^2 c_0^{j-1}\Big(\sum_{i=0}^{j-1} p_i g_i - g_j\Big)\Big] / \sum_{j=0}^{n} w_j^2 (c_0^{j-1})^2. \tag{18}$$

When we want to use the local parameter s_0 for our problem, then its optimal value can be computed as $s_0^1 = g_0 - h_0 d_0 m_0^1$. When we want to minimize the L_2-norm of the first derivative given as a sum in (14), we obtain similar expression (weighting coefficients in sums are multiplicated by h_j now).

3 Problem of mean value interpolation

In the case of polygonal interpolant the MVI problem with given mean values g_i can be transformed into a special case of the FVI problem with $t_i = (x_i + x_{i+1})/2$ and $g_i = s(t_i)$. We can obtain in a similar way the continuity conditions and explicit solutions, which have now some more simple form.

3.1 Continuity conditions

The local representations of MVI interpolant with different local parameters and the local variable $q = (x - x_i)/h_i$ can be written as

$$s(x) = (1-q)s_i + qs_{i+1} = s_i + h_i m_i q = g_i + h_i m_i (q - \tfrac{1}{2}) = 2g_i q + s_i(1-2q).$$

We can easily obtain the following reccurrences for the local parameters of mean values g_i interpolating polygon $s(x)$:

$$s_j + s_{j+1} = 2g_j\,, \quad j = 0(1)n; \tag{19}$$

$$h_j m_j + h_{j+1} m_{j+1} = 2(g_{j+1} - g_j)\,, \quad j = 0(1)n-1; \tag{20}$$

$$2s_j + h_j m_j = 2g_j\,, \quad j = 0(1)n. \tag{21}$$

From these reccurrences we can obtain by induction the explicit expressions for components of $\mathbf{s}$, $\mathbf{m}$ dependent on initial values s_0, m_0 and data $\mathbf{g}$ as

$$s_{j+1} = (-1)^{j+1} s_0 + 2\sum_{i=0}^{j} (-1)^{j-i} g_i\,, \; j = 0(1)n, \tag{22}$$

$$h_j m_j = 2(-1)^{j+1} s_0 + 2g_j + 4\sum_{i=0}^{j-1} (-1)^{j-i} g_i\,, \; j = 0(1)n, \tag{23}$$

$$s_j = (-1)^{j+1}(g_0 + \frac{1}{2} h_0 m_0) + 2\sum_{i=1}^{j-1} (-1)^{j+i-1} g_i\,, \; j = 1(1)n+1, \tag{24}$$

$$h_j m_j = (-1)^j m_0 h_0 + 2(g_j + (-1)^j g_0) + 4\sum_{i=1}^{j-1} (-1)^{j+i} g_i\,, \; i = 1(1)n. \tag{25}$$

When we search for the initial values of s_0, m_0 giving the minimum of the norm $||\mathbf{s}||_2$, we obtain in a similar way as in the foregoing such a results

$$s_0^0 = \frac{2}{n+2} \sum_{j=0}^{n} (-1)^j [g_j - g_{j-1} + \cdots + (-1)^j g_0]$$

$$= \frac{2}{n+2}\sum_{j=0}^{n}(-1)^{n+j}(j+1)g_{n-j}\,; \tag{26}$$

$$h_0 m_0^0 = \frac{2}{n+2}\Big[\sum_{j=1}^{n}(-1)^j(n+1-j)g_j - ng_0\Big]. \tag{27}$$

The optimal values of initial parameters for minimizing the norm of $\mathbf{m}$ are

$$H_2 s_0^1 = 2\sum_{j=0}^{n}\frac{1}{h_j^2}\Big[g_0 - g_1 + \cdots + (-1)^{j-1}g_{j-1} + \frac{1}{2}(-1)^j g_j\Big], \tag{28}$$

$$h_0 H_2 m_0^1 = 2\sum_{j=1}^{n}\frac{1}{h_j^2}\Big[(-1)^{j+1}g_j - g_0 + 2\sum_{i=1}^{j-1}(-1)^{i+1}g_i\Big], \tag{29}$$

with the notation $H_2 = \sum_{i=0}^{n}(1/h_i^2)$.

4 Difference equations approach

Let us use now the above mentioned difference equations approach to the recursions for parameters s_i, m_i. We can obtain a simple algorithms for computing optimal values of local parameters s_0, m_0 and corresponding sequences of their local parameters in that way. We can learn from the foregoing parts that the vector components of both local parameters $\mathbf{s}$, $\mathbf{m}$ are solutions of nonhomogeneous linear difference equations of the first order. Let us remember some general features of such solutions which can be used in our problems.

4.1 Solution of difference equation considered

Let us have given the vectors of coeficients

$$\mathbf{a} = [a_i], \quad \mathbf{b} = [b_i], \quad \mathbf{c} = [c_i], \quad i = 0(1)n.$$

A general solution of difference equation (see e.g. [4])

$$a_i y_{i+1} = b_i y_i + c_i\,, \quad a(i) \neq 0, \quad i = 0(1)n \tag{30}$$

can be composed from the corresponding solutions of homogeneous and nonhomogeneous equations (HE, NHE). For the particular solutions $\mathbf{u}$ of (HE) corresponding to the initial value $u_0 = 1$ and particular solution $\mathbf{v}$ of (NHE) with initial value $v_0 = 0$ we obtain the explicit formulas

$$u_{j+1} = \prod_{i=0}^{j}\frac{b_i}{a_i}\,, \qquad v_{j+1} = \sum_{i=0}^{j}\frac{c_i}{a_i}\prod_{k=i+1}^{j}\frac{b_k}{a_k}\,. \tag{31}$$

From the algorithmical point of view it is more important that we can easily compute such solutions using directly recursions (30). The solution $\mathbf{y}$ of (NHE) with initial value y_0 we can then write as

$$\mathbf{y} = y_0\mathbf{u} + \mathbf{v}. \tag{32}$$

When we search now for a solution of (NHE) with minimal norm (defined by some proper scalar product) as a function of initial value y_0, then

$$(\mathbf{y},\mathbf{y}) = (y_0\mathbf{u}+\mathbf{v}, y_0\mathbf{u}+\mathbf{v}) \to min \quad \text{is reached for} \quad y_0 = -(\mathbf{v},\mathbf{u})/(\mathbf{u},\mathbf{u}). \tag{33}$$

Starting with this optimal value y_0 we can now compute recursively the optimal sequence $\mathbf{y}$. Let us summarize the description of the algorithm mentioned. The possibility to use various types of the scalar product for different problems will be discussed in some details in the following section.

Algorithm s11opt for computing optimal solution

Input data: coefficients a_i, b_i, c_i of the reccurrence (30),
type of the scalar product corresponding to the problem (norm).

Steps of the algorithm:
1. *Compute the solution* $\mathbf{u}$ *of (HE) with* $u_0 = 1$.
2. *Compute the solution* $\mathbf{v}$ *of (NHE) with* $v_0 = 0$.
3. *Compute the optimal value of* y_0 *defined by (33).*
4. *Compute recursively the optimal solution from* y_0 *and (NHE).*

Such an algorithm can be easily implemented (including the computing of the optimal value of the norm) and the results visualized e.g. in MATLAB.

4.2 Applications to FVI, MVI, DVI problems

We have mentioned basic recursions between given values g_i and local parameters s_i, m_i for the problems of FVI, DVI and MVI with polygons in sections 2.1, 3.1. To find the optimal initial value of s_0 or m_0 which gives minimum to the standard 2-norm of local parameters under consideration, we can use now the *Algorithm s11opt* with proper chosen parameters of the recurrence and we find thus relatively simply the optimal solution of the problem.

We can use this approach also in problems where we want to minimize L_2-norm of $s(x)$—we have to use a proper form of the scalar product in (33) only. Using Simpson's rule of numerical integration we can write (exactly)

$$\begin{aligned}\|s(x)\|_2^2 &= \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} [s(x)]^2 dx = \frac{1}{6}\sum_{i=0}^{n} h_i\Big[s_i^2 + 4\Big(\frac{s_i + s_{i+1}}{2}\Big)^2 + s_{i+1}^2\Big] \\ &= \frac{1}{3}\sum_{i=0}^{n} h_i(s_i^2 + s_i s_{i+1} + s_{i+1}^2) = \frac{1}{3}\mathbf{s}^{\mathbf{T}}\mathbf{R}\mathbf{s}\end{aligned} \tag{34}$$

with the tridiagonal symmetric positive definite matrix $\mathbf{R} = (n+2, n+2)$ with the main diagonal consisting of elements

$$[h_0,\ h_0 + h_1,\ h_1 + h_2, \cdots,\ h_{n-1} + h_n,\ h_n] \tag{35}$$

and subdiagonals

$$\frac{1}{2}\,[h_0, h_1, \ldots, h_n]. \tag{36}$$

When we define now the new scalar product $(\cdot,\cdot)_R$ as $(\mathbf{u},\mathbf{v})_R = (\mathbf{u},\mathbf{R}\mathbf{v})$, we can solve the problem of finding the initial value y_0 minimizing L_2-norm of $s(x)$ by the *Algorithm s1lopt* for diffference equation (5) and scalar product $(\cdot,\cdot)_R$.

Quite similarly we can obtain the optimal value of the parameter m_0 giving minimum to the L_2-norm of $s'(x)$,

$$\|s'(x)\|_2^2 = \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} [s'(x)]^2 dx = \sum_{i=0}^{n} h_i m_i^2. \tag{37}$$

We can see that the last sum in (36) is a quadratic form with diagonal matrix $\mathbf{H} = diag[h_i]$. Thus we can use the foregoing approach with the scalar product defined with the matrix $\mathbf{H}$ and difference equation (19).

Generalizing the examples in the foregoing we can state the results in the following theorem.

Theorem *To find the optimal solution of problems FVI, MVI or DVI, we can use the Algorithm s1lopt with corresponding difference equation for the local parameters chosen and*

- *standard scalar product in case of minimizing 2-norms of vectors* $\mathbf{s}$ *or* $\mathbf{m}$*;*
- *the scalar product* $(\cdot,\cdot)_R$ *with the matrix* $\mathbf{R}$ *given in (35) in case of minimizing norm* $\|s(x)\|_2$*;*
- *the scalar product* $(\cdot,\cdot)_H$, $\mathbf{H} = \mathrm{diag}[h_i]$ *in case of minimizing norm* $\|s'(x)\|_2$.

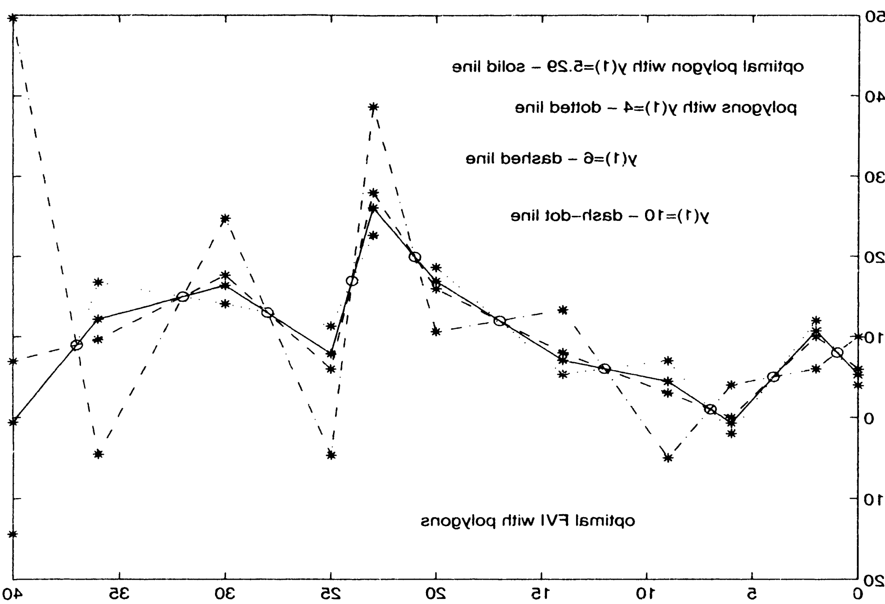

Fig. 1

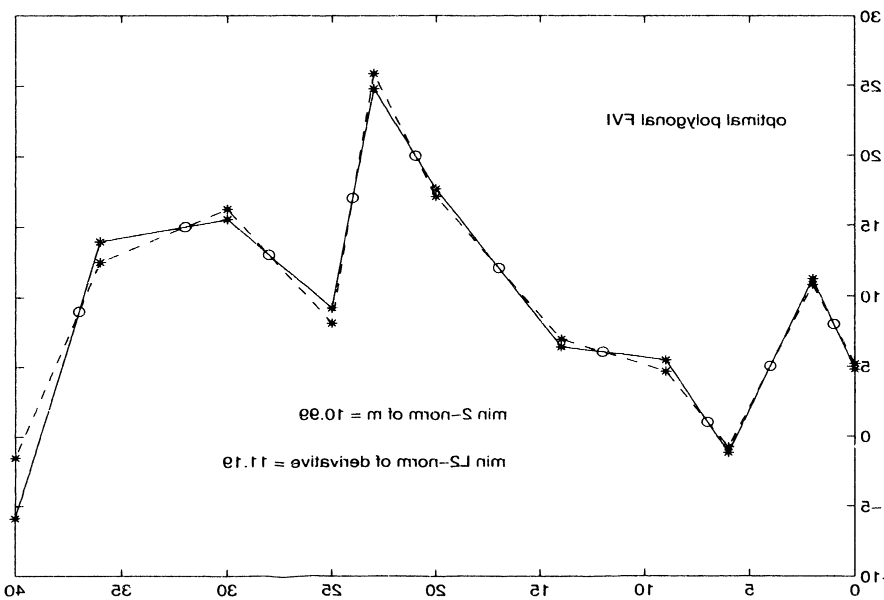

Fig. 2

Examples

Example 1 (FVI problem) Let us take the data

$$\mathbf{x} = [\ 0\ 2\ 6\ 9\ 14\ 20\ 23\ 25\ 30\ 36\ 40\ ],$$
$$\mathbf{t} = [\ 1\ 4\ 7\ 12\ 17\ 21\ 24\ 28\ 32\ 37\ ],$$
$$\mathbf{g} = [\ 8\ 5\ 1\ 6\ 12\ 20\ 17\ 13\ 15\ 9].$$

The interpolatory polygon with minimal 2-norm of $\mathbf{s}$ has initial value $s_0 = 5.29$ and norm equal to 40.76. We can see it plotted with full line on the Fig. 1, where also another solutions with initial values $4, 6, 10$ are plotted. The optimal solution for L_2-norm of $s(x)$ has initial value $s_0 = 5.41$ and the value of its norm is 40.79. The plots of this two solutions are very close together. When we compute the solutions with minimal norms of the derivative, we obtain similar polygons plotted on Fig. 2 with discrete 2-norm of the derivative equal 12.14 and L_2-norm 12.44.

Example 2 (MVI problem) For the MVI problem with data

$$\mathbf{x} = [\ 0\ 2\ 3\ 6\ 7\ 9\ 11\ 15\ ], \qquad \mathbf{g} = [\ 4\ 7\ 11\ 6\ 2\ 8\ 5\ ]$$

we can see on Fig. 3 the optimal interpolating polygons with $s_0 = 3$ and 2-norm of $\mathbf{s}$ equal 20.78 and another one with $s_0 = 2.23$ and L_2-norm of $s(x)$ equal 20.90. On Fig. 4 we se the result of minimizing of the 2-norm of the vector $[\mathbf{s}, \mathbf{m}]$ with norms $\|[\mathbf{s}, \mathbf{m}]\| = 25,55$, $\|\mathbf{s}\| = 20.99$, $\|\mathbf{m}\| = 14.56$.

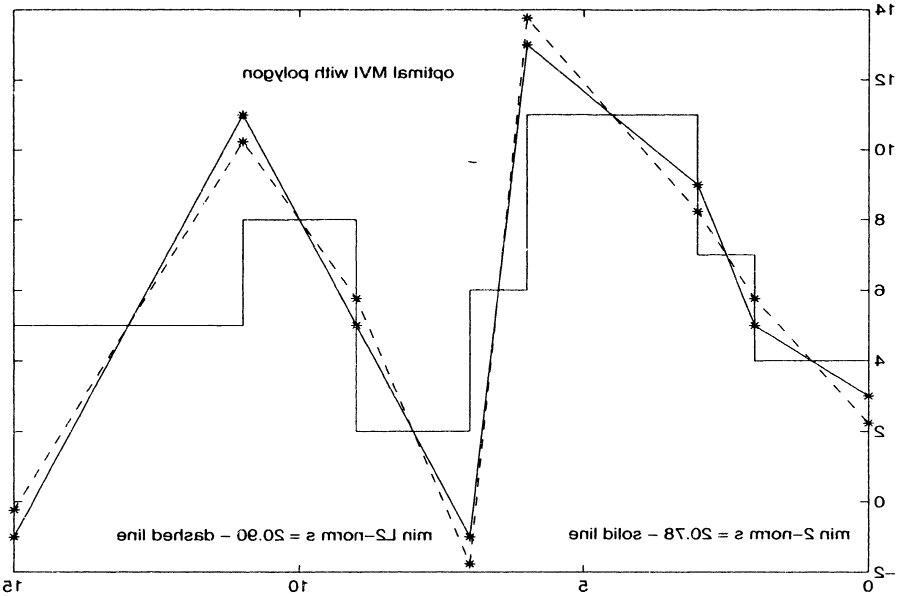

Fig. 3

Example 3 (DVI problem) For the DVI problem with data

$$
\begin{aligned}
\mathbf{x} &= [\ 0\ 2\ 3\ 6\ 7\ 9\ 10\ 13\ 16\ 20\ ], \\
\mathbf{t} &= [\ 1\ 2.2\ 5\ 6.6\ 8\ 9.3\ 11\ 15\ 19\ ], \\
\mathbf{m} &= [\ 0.5\ 1\ 0.2\ -0.4\ -1\ -0.5\ 0\ 1\ 2\ ].
\end{aligned}
$$

we can on Fig. 5 see the optimal interpolatory polygon with 2-norm of the vector $\mathbf{m}$ equal 9.7 and initial value $s_0 = -2.08$ and another solutions with initial values $-4, -3, 0$ with norms equal to $11.5, 10.2, 11.8$.

Example 4 (FVI problems with different knotsets) Let us demonstrate the influence of the knotset $\mathbf{x}$ on the shape of interpolating polygon with identical points of interpolation and values prescribed. For the data

$$
\begin{aligned}
\mathbf{x} &= [\ 0\ 2\ 7\ 13\ 19\ 20.3\ 25.6\ 30.8\ 36.7\ 42.7\ ], \\
\mathbf{x1} &= [\ -1\ 3\ 8\ 13\ 18\ 22\ 25.5\ 30\ 35\ 40\ ], \\
\mathbf{t} &= [\ 0.9\ 5.4\ 10.4\ 15.4\ 20\ 24.5\ 27\ 32\ 38\ ], \\
\mathbf{g} &= [\ 12\ 32\ 66\ 79\ 13.5\ 98\ 32\ 56\ 33\ ].
\end{aligned}
$$

the quite different results of optimal interpolation are plotted on Fig. 6—norms of the vectors of function values are equal to 2747 for the knotset $\mathbf{x}$ and 464.6 for the more centered knotset $\mathbf{x1}$.

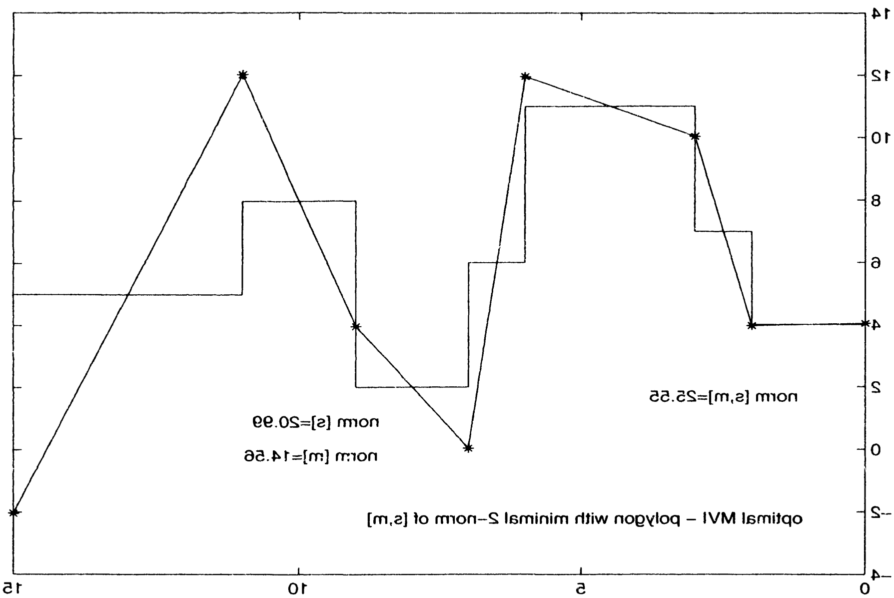

Fig. 4

5 Another numerical approaches

The problems under consideration can be stated as relatively simple optimization problems and some more general optimization techniques known from optimization theory or linear algebra can be used for their numerical solution.

5.1 Constrained optimization techniques

The above mentioned problems of FVI, MVI or DVI with polygon of minimal 2-norm or L_2-norm are typical problems of constrained optimization—here specially of quadratic programming. Let us formulate some of them only in the following (variants of minimized norms are given in brackets):

A) FVI problem—solution with minimal $||\mathbf{s}||_2^2$ $\left(||s(x)||_2^2\right)$:

Find minimum of $\sum_i s_i^2$ $\left(\frac{1}{3}(\mathbf{s}, \mathbf{R}\mathbf{s})\right)$ *under conditions*

$$(p_i - 1)s_i + s_{i+1} = p_i g_i, \; i = 0(1)n. \tag{38}$$

B) MVI problem—solution with minimal $||\mathbf{m}||_2^2$ $\left(||s'(x)||_2^2\right)$:

Find minimum of $\sum_i m_i^2$ $\left((\mathbf{m}, \mathbf{H}\mathbf{m})\right)$ *under conditions*

$$h_i m_i + h_{i+1} m_{i+1} = 2(g_{i+1} - g_i), \; i = 0(1)n - 1. \tag{39}$$

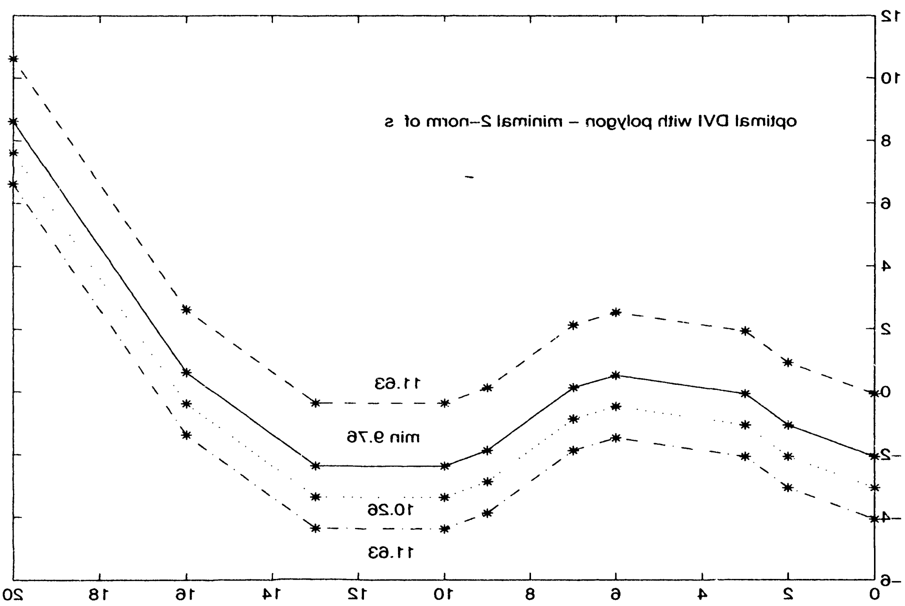

Fig. 5

C) DVI problem w—solution ith minimal $||\mathbf{s}||_2^2$*:*
Find minimum of $\sum_i s_i^2$ *under conditions*

$$-s_i + s_{i+1} = h_i m_i, \ i = 0(1)n. \tag{40}$$

D) Mixed MVI problem—solution with minimal $||[\mathbf{s},\mathbf{m}]||_2^2$*:*
Find minimum of $\sum_i (s_i^2 + m_i^2)$ *under conditions*

$$2s_i + h_i m_i = 2g_i, \quad s_i + s_{i+1} = 2g_i, \ i = 0(1)n. \tag{41}$$

To solve such problems we can use standard algorithms of quadratic programming (see e.g. [3]). The classical technique for constrained optimization with Lagrange's multipliers can be also used (it results in some four block system of linear equations).

5.2 Pseudoinverse matrix approach

When solving the mentioned problems to find interpolating polygon with minimal 2-norm of vectors **s, m** or **[s,m]** we can use also the following statement from linear algebra (see [1], p. 15):

Lemma *Given a matrix* $\mathbf{A} = [m,n]$ *with* $m \leq n$*, vector* $\mathbf{b} = [m,1]$*, then the solution of* $\mathbf{Ax} = \mathbf{b}$ *with minimal 2-norm is given as* $\mathbf{x} = \mathbf{A}^+\mathbf{b}$*, where* $\mathbf{A}^+$ *denotes the pseudoinverse matrix of* $\mathbf{A}$.

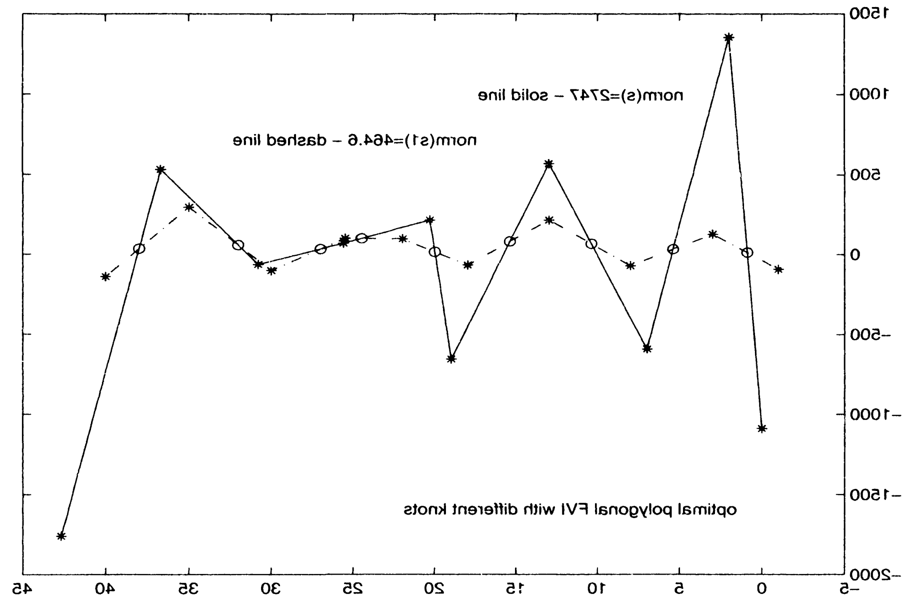

Fig. 6

The continuity conditions for all problems considered above have the form of underdetermined system of linear equations with respect to one or two kinds of local parameters of the polygon we search for. The coefficients of matrices of such systems can be recognized from the recurrences given above. The two-diagonal structure allows us to write down easily the description of the matrix coefficients and of the right-hand side of the system. The optimal solution we then obtain simply using known algorithms for pseudoinverse matrices (e.g. function *pinv* in MATLAB). In all examples mentioned in section 4 we have obtained identical results with *Algorithm s1opt* and with pseudoinverse matrix approach using MATLAB function *pinv*. This approach allows the unexperienced user to find some "good" interpolation polygon without the care of initial values of local parameters.

References

[1] Bjőrck, A.: *Numerical Methods for Least Squares Problems.* SIAM, Philadelphia, 1996, 408 pp.

[2] Kobza, J.: *Splajny.* UP Publ., Olomouc, 1993, 224 pp.

[3] *MATLAB Reference Guide. Optimization Tooolbox.* The MathWorks, 1992.

[4] Elaydi, S. N.: *Introduction to Difference Equations.* Springer-Verlag, New York, 1996.