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Some Properties of Solutions of a Class of Nonlinear Difference Equations

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Abstract

The existence of nonoscillatory bounded solutions of the second order nonlinear difference equation

$$\Delta(r(n)\Delta y(n)) + f(n, y(n), \Delta y(n)) = 0$$

is investigated.

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In this paper we investigate some properties of solutions of the nonlinear difference equation

$$(E) \qquad \Delta(r(n)\Delta y(n)) + f(n, y(n), \Delta y(n)) = 0,$$

where $r : N_0 \to R_+$, $f : N_0 \times R^2 \to R$. Here $N_0 := \{n_0, n_0 + 1, \dots\}$, n_0 is a given nonnegative integer, R_+ denotes the set of positive reals. For a function $y : N_0 \to R$ the difference operator Δ is defined by

$$\Delta y(n) = y(n+1) - y(n), \qquad \Delta^2 y(n) = \Delta(\Delta y(n)).$$

Throughout this paper we assume that the function $f(n,u,v)$ is continuous in the region $D:\ n\in N_0,\ u_0\le u\le u_1,\ 0\le v\le v_1$ for each fixed n, and we denote

$$\alpha(n) = \sum_{k=n_0}^{n-1} \frac{1}{r(k)}.$$

By a solution of Eq. (E) we mean a real sequence defined for $n\ge n_1$ (for some $n_1\in N_0$), which satisfies Eq. (E).

A nontrivial solution y of Eq. (E) is called oscillatory if for any $k\in N_0$ there exists $n\ge k$ such that $y(n)y(n+1)\le 0$. Otherwise it is called nonoscillatory.

In Th. 1 we consider assumptions under which a nonoscillatory bounded solution of Eq. (E) exists. In Th. 2 we give conditions under which Eq. (E) has a solution lying between solutions of other given equations. Similar problems for differential equations have been investigated by S. Belohorec [1]. Some asymptotic properties of the difference equation

$$\Delta(r(n)\Delta x(n)) + f(n,x(n),\Delta x(n)) = h(n)$$

were given by A. Drozdowicz [2].

Theorem 1 *Let the function f be nonnegative and nondecreasing with respect to the last two arguments on the region D. If for some constants c_0 and c_1 $(u_0 < c_0 \le u_1,\ 0 < c_1 \le v_1)$*

$$\sum_{i=n_0}^{\infty} \alpha(i+1) f\left(i, c_0, \frac{c_1}{r(i)}\right) < \infty$$

then for every m, $u_0\le m < c_0$ there exists $n_1(m)\in N_0$ such that for all $n_2\ge n_1$, $n_2\in N_0$, there exists a solution y of Eq. (E) defined for $n\ge n_1$, such that $y(n_2)=m$ and y increases to a constant $c\le c_0$.

Conversely, if Eq. (E) has such a solution, then for arbitrary numbers c_2, c_3 such that $u_0\le c_2 < c$, $0\le c_3\le \lim_{n\to\infty} r(n)\Delta y(n)$ we have

$$\sum_{i=n_0}^{\infty} \alpha(i+1) f\left(i, c_2, \frac{c_3}{r(i)}\right) < \infty.$$

Proof I. Let c_0 and c_1 be such constants and let m be an arbitrary number satisfying $u_0\le m < c_0$. Then there exists $n_1(m)\ge n_0$ such that for every $n_2\ge n_1$, $n_2\in N_0$, we have

$$\begin{aligned} &\sum_{i=n_2}^{\infty} \alpha(i+1) f\left(i, c_0, \frac{c_1}{r(i)}\right) \le c_0 - m, \\ (1)\qquad &\sum_{i=n_2}^{\infty} f\left(i, c_0, \frac{c_1}{r(i)}\right) \le c_1. \end{aligned}$$

Consider the equation

$$(2) \qquad y(n) = m + \sum_{i=n_2}^{n-1} (\alpha(i+1) - \alpha(n_2))\, f(i, y(i), \Delta y(i)) + (\alpha(n) - \alpha(n_2)) \sum_{i=n}^{\infty} f(i, y(i), \Delta y(i)).$$

We prove that Eq. (2) has a solution y passing through the point (n_2, m) and increasing to some constant $c \leq c_0$. This solution is also a solution of Eq. (E), because

$$\begin{aligned}
\Delta y(n) &= m + \sum_{i=n_2}^{n} (\alpha(i+1) - \alpha(n_2))\, f(i, y(i), \Delta y(i)) \\
&\quad + (\alpha(n+1) - \alpha(n_2)) \sum_{i=n+1}^{\infty} f(i, y(i), \Delta y(i)) \\
&\quad - m - \sum_{i=n_2}^{n-1} (\alpha(i+1) - \alpha(n_2))\, f(i, y(i), \Delta y(i)) \\
&\quad - (\alpha(n) - \alpha(n_2)) \sum_{i=n}^{\infty} f(i, y(i), \Delta y(i)) \\
&= (\alpha(n+1) - \alpha(n_2))\, f(n, y(n), \Delta y(n)) \\
&\quad + (\alpha(n) + \frac{1}{r(n)} - \alpha(n_2)) \sum_{i=n+1}^{\infty} f(i, y(i), \Delta y(i)) \\
&\quad - (\alpha(n) - \alpha(n_2))(f(n, y(n), \Delta y(n)) + \sum_{i=n+1}^{\infty} f(i, y(i), \Delta y(i)) \\
&= (\alpha(n+1) - \alpha(n_2)) f(n, y(n), \Delta y(n)) \\
&\quad + \frac{1}{r(n)} \sum_{i=n+1}^{\infty} f(i, y(i), \Delta y(i)) - (\alpha(n) - \alpha(n_2)) f(n, y(n), \Delta y(n)) \\
&= \frac{1}{r(n)} f(n, y(n), \Delta y(n)) + \frac{1}{r(n)} \sum_{i=n+1}^{\infty} f(i, y(i), \Delta y(i)) \\
&= \frac{1}{r(n)} \sum_{i=n}^{\infty} f(i, y(i), \Delta y(i)).
\end{aligned}$$

Therefore

$$(3) \qquad r(n)\Delta y(n) = \sum_{i=n}^{\infty} f(i, y(i), \Delta y(i))$$

and

$$\Delta(r(n)\Delta y(n)) =$$

$$= \sum_{i=n+1}^{\infty} f(i, y(i), \Delta y(i)) - \sum_{i=n}^{\infty} f(i, y(i), \Delta y(i)) = -f(n, y(n), \Delta y(n)).$$

The existence of a solution of Eq. (2) will be prove by the method of succesive approximations. If we put $y_1(n) = m$ and

$$y_{k+1}(n) = m + \sum_{i=n_2}^{n-1} (\alpha(i+1) - \alpha(n_2)) f(i, y_k(i), \Delta y_k(i)) \tag{4}$$

$$+ (\alpha(n) - \alpha(n_2)) \sum_{i=n}^{\infty} f(i, y_k(i), \Delta y_k(i))$$

for $k = 1, 2, \ldots$, then for $n \geq n_2$ and for every k the following inequalities hold

$$m \leq y_k(n) \leq c_0, \qquad 0 \leq \Delta y_k(n) \leq \frac{c_1}{r(n)}, \tag{5}$$

which can be proved by induction in next way. For example we prove that $m \leq y_k(n) \leq c_0$. We can see that

$$\alpha(i) - \alpha(j) = \sum_{k=n_1}^{i-1} \frac{1}{r(k)} - \sum_{k=n_1}^{j-1} \frac{1}{r(k)} \geq 0 \quad \text{for } i \geq j.$$

Therefore, by this and (4)

$$y_{k+1}(n) \geq m.$$

Moreover, by (5) and (4)

$$\begin{aligned} y_{k+1}(n) &\leq m + \sum_{i=n_2}^{n-1} \alpha(i+1) f(i, y_k(i), \Delta y_k(i)) + \alpha(n) \sum_{i=n}^{\infty} f(i, y_k(i), \Delta y_k(i)) \\ &\leq m + \sum_{i=n_2}^{n-1} \alpha(i+1) f\left(i, c_0, \frac{c_1}{r(i)}\right) + \sum_{i=n}^{\infty} \alpha(i+1) f\left(i, c_0, \frac{c_1}{r(i)}\right) \\ &= m + \sum_{i=n_2}^{\infty} \alpha(i+1) f\left(i, c_0, \frac{c_1}{r(i)}\right) \leq m + c_0 - m = c_0. \end{aligned}$$

Similarly, it may be proved by induction that for every $n \geq n_2$ the sequences $\{y_k(n)\}$, $\{\Delta y_k(n)\}$ are nondecreasing. Thus there exists $y(n)$ such that for every $n \geq n_2$ we have

$$\lim_{k\to\infty} y_k(n) = y(n), \qquad \lim_{k\to\infty} \Delta y_k(n) = \Delta y(n).$$

Therefore, using the Levi's Theorem [3, 4] we get by (4) that y is a solution of Eq. (E) for $n \geq n_2$ and this solution has the required properties.

Nontrivial modification is that we used Levi's Theorem for sequences, which are also measurable functions. Then in the thesis of Levi's Theorem the symbl $\int$ reduces to $\sum$.

II. Let y be a solution of Eq. (E) considered in the first part. Then there exists a number $n_2 \geq n_0$, $n_2 \in N_0$ such that for $n \geq n_2$ we have

$$c_2 \leq y(n) \leq c \quad \text{and} \quad 0 \leq \lim_{n\to\infty} r(n)\Delta y(n) \leq r(n)\Delta y(n).$$

Now, from Eq. (2) and the last inequalities we get

$$c \geq y(n) \geq m + \sum_{i=n_2}^{n-1} (\alpha(i+1) - \alpha(n_2)) f(i, y(i), \Delta y(i))$$

$$\geq \sum_{i=n_2}^{n-1} \alpha(i+1) f\left(i, c_2, \frac{c_3}{r(i)}\right) - \alpha(n_2) \sum_{i=n_2}^{n-1} f(i, y(i), \Delta y(i))$$

for all $n \geq n_2$. From this, by (3)

$$\sum_{i=n_0}^{\infty} \alpha(i+1) f\left(i, c_2, \frac{c_3}{r(i)}\right) < \infty$$

which proves the theorem. □

Remark 1 A similar property—asymptotically constant solutions—for the difference equation

$$\Delta(r_{n-1}\Delta u_{n-1}) + f(n, u_n) = 0$$

can be found in [6], (Th. 1).

In the following theorem we omit the assumption of monotonicity of the function f.

Theorem 2 *Assume that $F_1(n,u,v)$ and $F_2(n,u,v)$ are continuous and non-decreasing (in u and v) functions, for every fixed $n \geq n_0$. Moreover, let*

$$0 \leq F_1(n,u,v) \leq f(n,u,v) \leq F_2(n,u,v) \tag{6}$$

for every point of D.

Denote

$$\Delta(r(n)\Delta w(n)) + F_1(n, w(n), \Delta w(n)) = 0, \tag{$E1$}$$

$$\Delta(r(n)\Delta z(n)) + F_2(n, z(n), \Delta z(n)) = 0, \tag{$E2$}$$

If there exist constants c_0, c_1, $(u_0 < c_0 \leq u_1,\ 0 < c_1 \leq v_1)$ such that

$$\sum_{i=n_0}^{\infty} \alpha(i+1) F_2\left(i, c_0, \frac{c_1}{r(i)}\right) < \infty \tag{7}$$

then for every $m, u_0 \leq m < c_0$, there exists $n_1(m) \in N_0$ such that for all $n_2 \geq n_1, n_2 \in N_0$, a solution of Eq. (E) defined for $n \geq n_2$ passes through the (n_2, m). By Th. 1 this solution lies between the solutions $w(n)$ and $z(n)$ of Eq. (E1) and Eq. (E2), passing through this point.

Conversely, if Eq. (E) has such a solution, then

$$\sum_{i=n_0}^{\infty} \alpha(i+1) F_1(i, c_2, \frac{c_3}{r(i)}) < \infty,$$

where c_2 and c_3 are such as in Th. 1.

Proof Let $w(n)$ and $z(n)$ be solutions of Eq. (E1) and Eq. (E2) such that $w(n_2) = m$, $z(n_2) = m$ whose existence is proved in Th. 1. We will prove that there exists solution y of Eq. (E) such that

$$w(n) \leq y(n) \leq z(n) \quad \text{for } n \geq n_2 \tag{8}$$

and $y(n_2) = m$.

Let l_∞ be Banach space of bounded sequences with "sup" norm. Let $T = \{x \in l_\infty : w(n) \leq x(n) \leq z(n), \text{ for } n \geq n_2\}$. We define operator A in the following way

$$Ax(n) = m + \sum_{i=n_2}^{n-1} (\alpha(i+1) - \alpha(n_2)) f(i, x(i), \Delta x(i))$$

$$+ (\alpha(n) - \alpha(n_2)) \sum_{i=n}^{\infty} f(i, x(i), \Delta x(i)).$$

Set T and operator A have following properties:

1. Set T is nonempty, convex and compact subset of l_∞. To prove that T is compact subset of l_∞ it is easy to construct a finite ε-net for the set T.

2. Assume that $x(n) \in T$. From definition of A, (6) and (8) we have

$$Ax(n) \leq m + \sum_{i=n_2}^{n-1} (\alpha(i+1) - \alpha(n_2)) F_2(i, x(i), \Delta x(i))$$

$$+ (\alpha(n) - \alpha(n_2)) \sum_{i=n}^{\infty} F_2(i, x(i), \Delta x(i)) = z(n)$$

and $Ax(n) \geq w(n)$ for $n \geq n_2$.

Therefore $Ax(n) \in T$ eg. operator A maps T into T.

3. Operator A is continuous on T.

Let $x_k, x \in T$, $\lim_{k\to\infty} x_k = x$. From (7), Lebegue Theorem [5] and continuity of f we have

$$\lim_{k\to\infty} Ax_k(n) = \lim_{k\to\infty} (m + \sum_{i=n_2}^{n-1} (\alpha(i+1) - \alpha(n_2)) \times f(i, x_k(i), \Delta x_k(i)) + (\alpha(n) - \alpha(n_2)) \sum_{i=n}^{\infty} f(i, x_k(i), \Delta x_k(i)))$$

$$= m + \sum_{i=n_2}^{n-1} (\alpha(i+1) - \alpha(n_2)) f(i, x(i), \Delta x(i)) + (\alpha(n) - \alpha(n_2)) \sum_{i=n}^{\infty} f(i, x(i), \Delta x(i)) = Ax(n).$$

Then by Schauder's Theorem operator A has a fixed point on T. Let y be this point. Then $Ay(n) = y(n)$ i.e. Eq. (E) has a solution y such that $y(n_2) = m$ and $w(n) \le y(n) \le z(n)$ for $n \ge n_2$. This proves the first part of our theorem. The proof of the second part is similar to that of Th. 1. □

The most essential modification in our proof is the use of Banach space instead of Frechet space dealt by Belohorec [1].

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