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Steady Plane Flow of Second-Grade Fluid in Exterior Domains *

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Abstract

We study the steady motion of second-grade fluid in exterior domains with a small but non-zero velocity prescribed at infinity. We split the problem into the Oseen problem and transport equation and look for a fixed point in Sobolev spaces. We proof the existence of strong solutions.

Key words: Second-grade fluid, steady flow, exterior domain, transport equation, Oseen problem.

1991 Mathematics Subject Classification: 76D99, 35Q35

1 Introduction

Let us study the plane flow of second-grade fluid past an obstacle. The second-grade fluid is characterized by the constitutive law (see e.g. [TrNo])

$$\tau = 2\mu\mathbf{D} + 2\alpha_1\mathbf{A_1} + 4\alpha_2\mathbf{D}^2\,, \tag{1.1}$$

where μ is viscosity, α_1 and α_2 are normal stress moduli, $\mathbf{D}$ is the symmetric part of the gradient of velocity and

$$\mathbf{A_1} = \frac{d}{dt}\mathbf{D} + (\nabla\mathbf{v})^T\mathbf{D} + \mathbf{D}\nabla\mathbf{v}\,. \tag{1.2}$$

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Here $\frac{d}{dt}$ denotes the material time derivative and $\mathbf{v}$ the velocity field.

Steady motion of the incompressible second-grade fluid is governed by the balance of momentum

$$\varrho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \nabla \cdot \tau + \varrho \mathbf{f} \tag{1.3}$$

and the balance of mass

$$\nabla \cdot \mathbf{v} = 0\,, \tag{1.4}$$

where ϱ denotes the (constant) density, p the pressure and $\mathbf{f}$ the external forces. This model of non-Newtonian fluid was studied for different types of domains by several authors, see e.g. [DuFo], [DuRa], [GaSe], [NoSeVi], [PiSeVi].

We put the origine of the coordinate system into the obstacle i.e. into the compact body $\mathcal{O}$ with smooth boundary. Inserting (1.1) and (1.2) into (1.3) and using the condition of thermodynamical stability $\alpha_1 + \alpha_2 = 0$ (see [DuFo]) we obtain following system of equations which describes the steady motion of second-grade fluid in the exterior domain $\Omega = \mathbb{R}^2 \setminus \mathcal{O}$

$$\begin{gathered} -\mu\Delta \mathbf{v} - \alpha_1 \mathbf{v} \cdot \nabla \Delta \mathbf{v} + \nabla p = -\varrho \mathbf{v} \cdot \nabla \mathbf{v} + \varrho \mathbf{f} + \\ + \alpha_1 \nabla \cdot [(\nabla \mathbf{v})^T (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)] \\ \nabla \cdot \mathbf{v} = 0 \\ \mathbf{v} = \mathbf{0} \qquad \text{at } \partial\Omega = \partial\mathcal{O} \\ \mathbf{v} \to \mathbf{v}_\infty \qquad \text{as } |\mathbf{x}| \to \infty\,. \end{gathered} \tag{1.5}$$

We shall assume throughout the paper that the prescribed constant velocity at infinity $\mathbf{v}_\infty \neq \mathbf{0}$. We can rotate the coordinate system in such a way that $\mathbf{v}_\infty = \beta(1,0)$. We shall search the solution $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{u} + \mathbf{v}_\infty$; we have for $\mathbf{u}$:

$$\begin{gathered} -\mu\Delta \mathbf{u} - \alpha_1 \mathbf{u} \cdot \nabla \Delta \mathbf{u} - \alpha_1 \beta \Delta \tfrac{\partial \mathbf{u}}{\partial x_1} + \varrho\beta \tfrac{\partial \mathbf{u}}{\partial x_1} + \nabla p = \\ = -\varrho \mathbf{u} \cdot \nabla \mathbf{u} + \varrho \mathbf{f} + \alpha_1 \nabla \cdot [(\nabla \mathbf{u})^T (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)] \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u} = -\mathbf{v}_\infty = -(\beta, 0) \qquad \text{at } \partial\Omega \\ \mathbf{u} \to \mathbf{0} \qquad \text{as } |\mathbf{x}| \to \infty\,. \end{gathered} \tag{1.6}$$

Using the decomposition procedure proposed by Mogilevskij and Solonnikov (see [MoSo]) we consider formally the mapping

$$\mathcal{M} : \mathbf{g} \mapsto (\mathbf{u}, s) \mapsto \mathbf{z}\,,$$

where

$$\begin{gathered} -\Delta \mathbf{u} + \varrho \tfrac{\beta}{\mu} \tfrac{\partial \mathbf{u}}{\partial x_1} + \nabla s = \mathbf{g} \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u} = -(\beta, 0) \qquad \text{at } \partial\Omega \\ \mathbf{u} \to \mathbf{0} \qquad \text{as } |\mathbf{x}| \to \infty\,, \end{gathered} \tag{1.7}$$

it means that the pair $(\mathbf{u}, s)$ satifies the Oseen problem with the right hand side $\mathbf{g}$, and

$$\begin{gathered}\mu\mathbf{z} + \alpha_1(\mathbf{u} + \mathbf{v}_\infty) \cdot \nabla\mathbf{z} = -\varrho\mathbf{u} \cdot \nabla\mathbf{u} + \varrho\mathbf{f} + \\ + \alpha_1\nabla \cdot \left[(\nabla\mathbf{u})^T(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) + \frac{\varrho\beta}{\mu}\frac{\partial\mathbf{u}}{\partial x_1} \otimes \mathbf{u} - s(\nabla\mathbf{u})^T\right] + \alpha_1\frac{\varrho\beta^2}{\mu}\frac{\partial^2\mathbf{u}}{\partial x_1^2}\,,\end{gathered} \tag{1.8}$$

it means that $\mathbf{z}$ satisfies transport equation with the right hand side depending on $(\mathbf{u}, s)$. (In fact, each component of $\mathbf{z}$ satisfies scalar transport equation.) Clearly, if we find a fixed point of the mapping $\mathcal{M}$ in an appropriate space then the corresponding pair $(\mathbf{u}, p)$ with $p = \mu s + \alpha_1(\mathbf{u} + \mathbf{v}_\infty) \cdot \nabla s$ solves the original problem (1.6).

2 Notation, Basic Theorems

We denote by $L^q(\Omega)$ the usual Lebesgue space equipped by the norm

$$||u||_q = \left(\int_\Omega |u|^q d\mathbf{x}\right)^{\frac{1}{q}}.$$

The Sobolev space $W^{k,q}(\Omega)$ contains all measurable functions such that the norm

$$||u||_{k,q} = \left(\sum_{|\alpha|\leq k} ||D^\alpha u||_q^q\right)^{\frac{1}{q}}$$

is finite. If $k = 0$ then $W^{0,q}(\Omega) = L^q(\Omega)$. Here $\alpha = (\alpha_1, \alpha_2)$ is a multiindex and $D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$. By $D^k u$ we understand the vector which consists of all derivatives of k-th order of u. If $k = 1$ we shall write only Du. We do not distinguish between $W^{k,q}(\Omega)$ and $(W^{k,q}(\Omega))^n$ but in order to avoid misunderstanding, all the vector- and tensor-valued functions are printed boldfaced.

We shall mention some well-known theorems which will be often used in the next section:

Theorem 2.1 *Let $\Omega \subseteq \mathbb{R}^n$. Let $u \in L^p(\Omega) \bigcap L^q(\Omega)$, $1 \leq p < q \leq \infty$. Then $u \in L^r(\Omega)$ $\forall r \in (p, q)$ and*

$$||u||_r \leq ||u||_p^\alpha ||u||_q^{1-\alpha}$$

with $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$.

Proof It is an easy consequence of the Hölder inequality. □

Theorem 2.2

1. *Let $\Omega \subset \mathbb{R}^2$ be exterior domain with lipschitzian boundary. Let for some $p \in [1,2)$ $Du \in L^p(\Omega)$. Then there exists a unique constant u_0 such that*
$$\|u - u_0\|_s \leq C\|Du\|_p\,, \quad s = \frac{2p}{2-p}\,,$$
where the constant C does not depend on u.

2. *Let $u \in W^{1,p}(\Omega)$, $p > 2$. Then $u \in C(\overline{\Omega})$ and there exists $C > 0$, independent of u, such that*
$$\|u\|_\infty \leq C\|u\|_{1,p}\,.$$

Proof see [Ga1]. □

Remark 2.3 Let all the assumptions of the first part of Theorem 2.2 be satisfied. If moreover $u \in L^q(\Omega)$ for some $q \in [1,\infty)$ then evidently $u_0 = 0$.

The decomposition (1.7)–(1.8) allows us to study separately the Oseen problem (1.7) and the transport equation (1.8). Let us recall some known facts about these two equations.

Theorem 2.4 *Let $1 < q < \frac{6}{5}$ and $\beta \in (0,\beta_0)$. Let $\mathbf{g} \in W^{k,q}(\Omega)$, $k = 0,1,\cdots$ and $\Omega \subset \mathbb{R}^2$ be exterior domain. Then there exists unique solution to the Oseen problem (1.7). Moreover denoting by $\mathbf{u}$ this solution, we have:*

$$\begin{aligned}\langle\mathbf{u}\rangle_{\beta,q} &\equiv \beta(\|u_2\|_{\frac{2q}{2-q}} + \|Du_2\|_q) + \beta^{\frac{2}{3}}\|\mathbf{u}\|_{\frac{3q}{3-2q}} + \beta^{\frac{1}{3}}\|D\mathbf{u}\|_{\frac{3q}{3-q}} \leq \\ &\leq C(\Omega,q,\beta_0)(\|\mathbf{g}\|_q + \beta^{2(1-\frac{1}{q})}|\ln\beta|^{-1}\|\mathbf{v}_\infty\|_{2-\frac{1}{q},q,\partial\Omega}) \qquad (2.1)\\ [\mathbf{u}]_k &\equiv \beta^{2(1-\frac{1}{q})}(\|D^2\mathbf{u}\|_{k,q} + \|Ds\|_{k,q}) \leq \\ &\leq C(\Omega,q,\beta_0)\left[\|\mathbf{g}\|_q + \beta^{2(1-\frac{1}{q})}(\|\mathbf{g}\|_{k,q} + \|\mathbf{v}_\infty\|_{2+k-\frac{1}{q},q,\partial\Omega})\right] \qquad (2.2)\end{aligned}$$

Proof The existence and uniqueness of the solution is shown in [Ga1]. The inequality (2.1) can be found in [Ga2], the inequality (2.2) in [Po]. □

Theorem 2.5 *Let $\mathbf{u} + \mathbf{v}_\infty = \mathbf{0}$ on $\partial\Omega$. Let Ω be exterior domain with smooth boundary. Let $\mathbf{u} + \mathbf{v}_\infty \in C^k(\overline{\Omega})$, $\mathbf{F} \in W^{k,q}(\Omega)$, $k = 0,1,\cdots$, $1 < q < \infty$. Let $\vartheta_k = \|\mathbf{u} + \mathbf{v}_\infty\|_{C^k}$ be small enough (if $k = 0$, we must consider $\|\mathbf{u} + \mathbf{v}_\infty\|_{C^1}$). Then there exists exactly one solution $\mathbf{z} \in W^{k,q}(\Omega)$ to the transport equation (1.8). Moreover*
$$\|\mathbf{z}\|_{k,q} \leq \frac{1}{\mu - \alpha\vartheta_k}\|\mathbf{F}\|_{k,q}\,.$$
The constant α depends only on k, q.

Proof see [No]. □

3 Main Theorems

We proceed as follows. We first show that the mapping $\mathcal{M}$ maps for β small enough the balls (small enough) in $W^{1,q}(\Omega)$ into themselves. Then we show that the mapping is a contraction in $L^q(\Omega)$. The following classical theorem gives us the existence of a fixed point in $W^{1,q}(\Omega)$.

Theorem 3.1 *Let X be reflexive Banach space, Y Banach space, $X \hookrightarrow Y$. Let H be a closed unempty ball in the norm topology of X. Let $T: H \mapsto H$ be contraction in Y. Then there exists a unique fixed point of T in H.*

Remark 3.2 Analogously to what follows we may show that the mapping $\mathcal{M}$ maps small balls into themselves in $W^{k,q}(\Omega)$, $k \geq 2$, and is a contraction in $W^{k-1,q}(\Omega)$. Therefore there exists a fixed point in $W^{k,q}(\Omega)$ (see Theorem 3.9).

Throughout this section we shall assume that $q \in (1, \frac{6}{5})$. We shall show that if $||\mathbf{f}||_{1,q}$ is sufficiently small then

1. $\exists \delta(\beta) > 0$ such that $||\mathbf{g}||_{1,q} \leq \delta \Rightarrow ||\mathcal{M}\mathbf{g}||_{1,q} \leq \delta$
2. $\mathcal{M}$ is a contraction in $L^q(\Omega)$ i.e. $||\mathcal{M}\mathbf{g}_1 - \mathcal{M}\mathbf{g}_2||_q \leq \gamma(\delta)||\mathbf{g}_1 - \mathbf{g}_2||_q$, where $0 < \gamma < 1$ and $||\mathbf{g}_i||_{1,q} \leq \delta$, $i = 1, 2$.

We start to study the problem (1.7) i.e. the Oseen problem. From (2.1)–(2.2) we have

$$\begin{aligned} \langle \mathbf{u} \rangle_{\beta,q} &\leq C(||\mathbf{g}||_q + \beta^{1+2(1-\frac{1}{q})}|\ln \beta|^{-1}) \\ [\mathbf{u}]_1 &\leq C(||\mathbf{g}||_q + \beta^{2(1-\frac{1}{q})}||\mathbf{g}||_{1,q} + \beta^{1+2(1-\frac{1}{q})}). \end{aligned} \tag{3.1}$$

We need an estimate of $\mathbf{z} = \mathcal{M}\mathbf{g}$ in $W^{1,q}(\Omega)$ by means of the expressions on the left hand side of (3.1). Let us begin with two lemmas dealing with some auxilliary estimates.

Lemma 3.3 *Let $\mathbf{u}$ has finite norms $\langle \cdot \rangle_{\beta,q}$ and $[\cdot]_k$, $k \geq 0$. Let $\mathbf{u} = -(\beta, 0)$ on $\partial\Omega$. Then*

$$\begin{aligned} ||\mathbf{u}||_\infty &\leq C([\mathbf{u}]_0 \beta^{-2(1-\frac{1}{q})})^{\frac{3-2q}{q}}[(\langle \mathbf{u} \rangle_{\beta,q}\beta^{-\frac{2}{3}})^{\frac{3(q-1)}{q}} + \beta^{\frac{3(q-1)}{q}}] \\ ||D^k\mathbf{u}||_\infty &\leq C\beta^{-2(1-\frac{1}{q})}[\mathbf{u}]_k\,,\ k \geq 1\,. \end{aligned} \tag{3.2}$$

Proof We start with the first inequality. We denote by $\mathbf{w}$ the function which is equal to $\mathbf{u}$ inside of Ω and $-(\beta, 0)$ outside of Ω. The function $\mathbf{w}$ belongs to $W^{1,q}(\mathbb{R}^2)$ and the interpolation inequality from [Ma] gives us

$$||\mathbf{w}||_\infty \leq C||D\mathbf{w}||^a_{L^s(\mathbb{R}^2)}||\mathbf{w}||^{1-a}_{L^r(\mathbb{R}^2)}\,,$$

where $0 = a(\frac{1}{s} - \frac{1}{2}) + (1-a)\frac{1}{r}$. We put $r = \frac{3q}{3-2q}$ and $s = \frac{2q}{2-q}$; so $a = \frac{3-2q}{q}$. As $\mathbf{w} = \mathbf{u}$ on Ω and $\nabla\mathbf{w} = \mathbf{0}$ outside of Ω, we have

$$||\mathbf{u}||_\infty \leq C(||\mathbf{u}||^{\frac{3(q-1)}{q}}_{L^{\frac{3q}{3-2q}}(\Omega)} + \beta^{\frac{3(q-1)}{q}})||D\mathbf{u}||^{\frac{3-2q}{q}}_{L^{\frac{2q}{2-q}}(\Omega)}\,.$$

The inequality (3.2_1) follows by means of Theorem 2.2 and definitions of the norms.

The other inequality is even easier. Theorem 2.2 gives us

$$\|D^k\mathbf{u}\|_\infty \le C\|D^k\mathbf{u}\|_{1,\frac{2q}{2-q}} \le C\|D^{k+1}\mathbf{u}\|_{1,q} \le C\beta^{-2(1-\frac{1}{q})}[\mathbf{u}]_k$$

which finishes the proof of the lemma. □

We next estimate the quadratic terms on the right hand side of (1.8).

Lemma 3.4 *Let* $\mathbf{u}$ *be sufficiently smooth. Then we have the following estimates with* C *independent of* $\mathbf{u}$ *and* β

$$\begin{aligned}
\|\mathbf{u}\cdot\nabla\mathbf{u}\|_q &\le \langle\mathbf{u}\rangle^2_{\beta,q}\beta^{-1-2(1-\frac{1}{q})}\,,\\
\|\mathbf{u}D^k\mathbf{u}\|_q &\le C[\mathbf{u}]_0^{\frac{3-2q}{q}}[\mathbf{u}]_{k-2}\beta^{-2(1-\frac{1}{q})\frac{3-q}{q}}\cdot\\
&\qquad\cdot[\langle\mathbf{u}\rangle^{3(1-\frac{1}{q})}_{\beta,q}\beta^{-2(1-\frac{1}{q})}+\beta^{3(1-\frac{1}{q})}]\,, && k\ge 2\,,\\
\|D\mathbf{u}\|^2_{2q} &\le C\langle\mathbf{u}\rangle^{6(1-\frac{1}{q})}_{\beta,q}[\mathbf{u}]_0^{\frac{6-4q}{q}}\beta^{-6(1-\frac{1}{q})\frac{2-q}{q}}\,,\\
\|D\mathbf{u}D^k\mathbf{u}\|_q &\le C[\mathbf{u}]_1[\mathbf{u}]_{k-2}\beta^{-4(1-\frac{1}{q})}\,, && k\ge 2\,,\\
\|D^2\mathbf{u}\|^2_{2q} &\le C[\mathbf{u}]_1^2\beta^{-4(1-\frac{1}{q})}\,,\\
\|D^ksD\mathbf{u}\|_q &\le C[\mathbf{u}]_1[\mathbf{u}]_{k-1}\beta^{-4(1-\frac{1}{q})}\,, && k\ge 1\,,\\
\|DsD^k\mathbf{u}\|_q &\le C[\mathbf{u}]_1[\mathbf{u}]_{k-1}\beta^{-4(1-\frac{1}{q})}\,, && k\ge 2\,.
\end{aligned}\tag{3.3}$$

Proof The first inequality is classical and can be found e.g. in [Ga2]. We next have

$$\|\mathbf{u}D^k\mathbf{u}\|_q \le \|D^k\mathbf{u}\|_q\|\mathbf{u}\|_\infty$$

and the second inequality follows from Lemma 3.3. The third inequality is a consequence of the interpolation and imbedding inequalities

$$\|D\mathbf{u}\|^2_{2q} \le \|D\mathbf{u}\|^{6(1-\frac{1}{q})}_{\frac{3q}{3-q}}\|D\mathbf{u}\|^{\frac{6-4q}{q}}_{\frac{2q}{2-q}} \le C\|D\mathbf{u}\|^{6(1-\frac{1}{q})}_{\frac{3q}{3-q}}\|D^2\mathbf{u}\|_q^{\frac{6-4q}{q}}$$

which hold for $q\in[1,\frac{3}{2}]$. The fourth and sixth inequalities can be shown similarly as the second one. From the imbedding theorem we have

$$\|D^2\mathbf{u}\|^2_{2q} \le C\|D^2\mathbf{u}\|^2_{1,q}$$

and we get the fifth inequality. Finally

$$\|DsD^k\mathbf{u}\|_q \le \|Ds\|_{\frac{2q}{2-q}}\|D^k\mathbf{u}\|_2 \le C\|D^2s\|_q\|D^k\mathbf{u}\|_{1,q}\,.$$

The lemma is proved. □

So we are in position to show that the operator $\mathcal{M}$ maps sufficiently small balls in $W^{1,q}(\Omega)$ into themselves.

Lemma 3.5 *Let $\|\mathbf{f}\|_{1,q}$ and β be sufficiently small. Then there exists $\delta(\beta) > 0$ such that the operator $\mathcal{M}$ maps $B_\delta = \{\mathbf{g} \in W^{1,q}(\Omega); \|\mathbf{g}\|_{1,q} \le \delta\}$ into itself.*

Proof Let us take $\mathbf{g} \in W^{1,q}(\Omega)$, $1 < q < \frac{6}{5}$, $\|\mathbf{g}\|_{1,q} \le \delta$ small enough (will be precised later). For the couple $(\mathbf{u}, s)$ the estimates (3.1) are available. Now, let us assume (will be demonstrated below) that $\|\mathbf{u} + \mathbf{v}_\infty\|_{C^1}$ is small enough. Let $\mathbf{z}$ be solution of (1.8) with the right hand side depending on $(\mathbf{u}, s)$. Then

$$\|\mathbf{z}\|_{1,q} \le C\|\mathbf{F}(\mathbf{u}, s)\|_{1,q}\,.$$

We need therefore to assure the smallness of $\|\mathbf{u}+\mathbf{v}_\infty\|_{C^1}$ and to estimate $\mathbf{F}(\mathbf{u}, s)$ by means of the norms on the left hand side of (3.1). In what follows we assume that $\delta = \varepsilon\beta^\alpha$, where $\alpha > 0$ and ε is a positive small number. First we need

$$\|\mathbf{u}\|_{1,\infty} \le C\mu\,. \tag{3.4}$$

From Lemma 3.3 and estimate (3.2) we have

$$\|\mathbf{u}\|_{1,\infty} = \|\mathbf{u}\|_\infty + \|D\mathbf{u}\|_\infty \le C\Big\{\varepsilon\beta^{\alpha-2(1-\frac{1}{q})} + \varepsilon\beta^{\alpha-2(1-\frac{1}{q})\frac{3-q}{q}} + \\ + \varepsilon^{\frac{3-2q}{q}}\beta^{\alpha\frac{3-2q}{q}+2(1-\frac{1}{q})\frac{7q-6}{2q}} + \beta^{1-2(1-\frac{1}{q})\frac{3-2q}{q}} + \beta^{1+2(1-\frac{1}{q})\frac{9q-6}{2q}} + \beta\Big\}$$

Evidently, as $1 > 2(1-\frac{1}{q})\frac{3-2q}{q}$ for $q \in (1, \frac{6}{5})$, it is enough to assume β small and

$$\alpha > 2(1-\frac{1}{q})\frac{3-q}{q}\,.$$

So we get (3.4) satisfied. Let us note that it is enough to take $\alpha > \frac{1}{2}$. As will be seen later we shall need much sharper condition on α.

Now from (1.8) we see that

$$\|\mathbf{F}(\mathbf{u}, s)\|_{1,q} \le C(\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{1,q} + \|D\mathbf{u}D^2\mathbf{u}\|_{1,q} + \beta\|\mathbf{u}D^2\mathbf{u}\|_{1,q} + \\ + \|DsD\mathbf{u}\|_{1,q} + \|\mathbf{f}\|_{1,q} + \beta^2\|D^2\mathbf{u}\|_{1,q})\,.$$

Lemma 3.4 reads

$$\|\mathbf{F}\|_{1,q} \le C\Big\{\langle\mathbf{u}\rangle^2_{\beta,q}\beta^{-1-2(1-\frac{1}{q})} + \langle\mathbf{u}\rangle^{6(1-\frac{1}{q})}_{\beta,q}[\mathbf{u}]_0^{\frac{6-4q}{q}}\beta^{-6(1-\frac{1}{q})\frac{2-q}{q}} + \\ + \langle\mathbf{u}\rangle^{3(1-\frac{1}{q})}_{\beta,q}([\mathbf{u}]_0^{\frac{3-q}{q}} + [\mathbf{u}]_1^{\frac{3-q}{q}})\beta^{-2(1-\frac{1}{q})\frac{3}{q}}(1+\beta) + \\ + ([\mathbf{u}]_0^{\frac{3-q}{q}} + [\mathbf{u}]_1^{\frac{3-q}{q}})\beta^{-2(1-\frac{1}{q})\frac{5q-6}{q}}(1+\beta) + \\ + ([\mathbf{u}]_1^2 + [\mathbf{u}]_1[\mathbf{u}]_0(1+\beta))\beta^{-4(1-\frac{1}{q})} + \|\mathbf{f}\|_{1,q} + [\mathbf{u}]_1\beta^{2-2(1-\frac{1}{q})}\Big\}\,.$$

Employing Theorems 2.4 and 2.5 we get finally (we assume $|\ln\beta| > 1$)

$$\|\mathbf{z}\|_{1,q} \le C\|\mathbf{F}\|_{1,q} \le C\Big\{\|\mathbf{g}\|^2_{1,q}[\beta^{-1-2(1-\frac{1}{q})} + \beta^{-6(1-\frac{1}{q})\frac{2-q}{q}} + \\ + \beta^{-2(1-\frac{1}{q})\frac{3}{q}}(1+\beta) + \beta^{-4(1-\frac{1}{q})}(1+\beta)] + \\ + \|\mathbf{g}\|_{1,q}^{\frac{(3-q)}{q}}\beta^{-2(1-\frac{1}{q})\frac{6-5q}{q}}(1+\beta) + \|\mathbf{g}\|_{1,q}\beta^{\frac{2}{q}} + \\ + \beta^{1+2(1-\frac{1}{q})}|\ln\beta|^{-2} + \beta^{2-2(1-\frac{1}{q})\frac{6-5q}{q}} + \\ + \beta^{2-2(1-\frac{1}{q})\frac{3-2q}{q}}(1+\beta) + \beta^2(1+\beta) + \|\mathbf{f}\|_{1,q}\Big\}\,.$$

So we easily see that the smallest exponent in the terms without $||g||_{1,q}$ is exactly $1+2(1-\frac{1}{q})$. We have therefore

$$\alpha \leq 1 + 2(1 - \frac{1}{q}) .$$

On the other side, taking the terms with $||g||_{1,q}$ into accout we easily see that necesserily $2\alpha - 1 - 2(1-\frac{1}{q}) \geq \alpha$ i.e.

$$\alpha \geq 1 + 2(1 - \frac{1}{q})$$

and the only possibility is to choose $\alpha = 1 + 2(1-\frac{1}{q})$. Evidently, if ε and β are small enough, then we get

$$||\mathbf{z}||_{1,q} \leq \varepsilon \beta^{1+2(1-\frac{1}{q})} = \delta .$$

Let us emphasize that

$$||\mathbf{g}||^2_{1,q} \beta^{-1-2(1-\frac{1}{q})} \leq C\varepsilon^2 \beta^{1+2(1-\frac{1}{q})} \leq \frac{1}{10} \varepsilon \beta^{1+2(1-\frac{1}{q})}$$

for ε small enough and

$$\beta^{1+2(1-\frac{1}{q})} |\ln \beta|^{-2} \leq \frac{1}{10} \varepsilon \beta^{1+2(1-\frac{1}{q})}$$

for β small enough. Lemma 3.5 is proved. □

Now it remains to show that the operator $\mathcal{M}$ is a contraction in the space $L^q(\Omega)$. It means we are about to show that there exists δ small enough such that for all $\mathbf{g}_1, \mathbf{g}_2 \in B_\delta$ there exists $\gamma \in (0,1)$ such that

$$||\mathcal{M}\mathbf{g}_1 - \mathcal{M}\mathbf{g}_2||_q \leq \gamma ||\mathbf{g}_1 - \mathbf{g}_2||_q .$$

Let us first reformulate the problems (1.7) and (1.8). We have easily

$$\begin{gathered} -\Delta(\mathbf{u}_1 - \mathbf{u}_2) + \varrho\frac{\beta}{\mu}\frac{\partial \mathbf{u}_1 - \mathbf{u}_2}{\partial x_1} + \nabla(s_1 - s_2) = \mathbf{g}_1 - \mathbf{g}_2 \\ \nabla \cdot (\mathbf{u}_1 - \mathbf{u}_2) = 0 \\ \mathbf{u}_1 - \mathbf{u}_2 = \mathbf{0} \qquad \text{at } \partial\Omega \\ \mathbf{u}_1 - \mathbf{u}_2 \to \mathbf{0} \qquad \text{as } |\mathbf{x}| \to \infty \end{gathered} \tag{3.5}$$

$$\begin{gathered} \mu(\mathbf{z}_1 - \mathbf{z}_2) + \alpha_1(\mathbf{u}_1 + \mathbf{v}_\infty) \cdot \nabla(\mathbf{z}_1 - \mathbf{z}_2) = \\ = \mathbf{F}(\mathbf{u}_1, s_1) - \mathbf{F}(\mathbf{u}_2, s_2) - \alpha_1(\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla \mathbf{z}_2 \equiv \mathbf{G} , \end{gathered} \tag{3.6}$$

where

$$\begin{aligned} \mathbf{F}(\mathbf{u}_1, s_1) - \mathbf{F}(\mathbf{u}_2, s_2) \ &= -\varrho(\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla \mathbf{u}_1 - \varrho \mathbf{u}_2 \cdot \nabla(\mathbf{u}_1 - \mathbf{u}_2) + \\ &+ \alpha_1 \nabla \cdot \big\{ (\nabla(\mathbf{u}_1 - \mathbf{u}_2))^T [\nabla \mathbf{u}_1 + (\nabla \mathbf{u}_1)^T] + \\ &+ (\nabla \mathbf{u}_2)^T [\nabla(\mathbf{u}_1 - \mathbf{u}_2) + (\nabla(\mathbf{u}_1 - \mathbf{u}_2))^T] + \\ &+ \varrho\frac{\beta}{\mu}\frac{\partial \mathbf{u}_1}{\partial x_1} \otimes (\mathbf{u}_1 - \mathbf{u}_2) + \varrho\frac{\beta}{\mu}\frac{\partial(\mathbf{u}_1 - \mathbf{u}_2)}{\partial x_1} \otimes \mathbf{u}_2 - \\ &- (s_1 - s_2)(\nabla \mathbf{u}_1)^T - s_2(\nabla(\mathbf{u}_1 - \mathbf{u}_2))^T \big\} + \\ &+ \alpha_1 \frac{\varrho\beta^2}{\mu}\frac{\partial^2(\mathbf{u}_1 - \mathbf{u}_2)}{\partial x_1^2} . \end{aligned} \tag{3.7}$$

Our aim is to show that $\|\mathbf{z}_1 - \mathbf{z}_2\|_q \le \gamma \|\mathbf{g}_1 - \mathbf{g}_2\|_q$ with $\gamma < 1$. For (3.5) we have

$$\begin{aligned} \langle \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\beta,q} &\le C\|\mathbf{g}_1 - \mathbf{g}_2\|_q \\ [\mathbf{u}_1 - \mathbf{u}_2]_0 &\le C\|\mathbf{g}_1 - \mathbf{g}_2\|_q \end{aligned} \tag{3.8}$$

while for (3.6)

$$\|\mathbf{z}_1 - \mathbf{z}_2\|_q \le \frac{1}{\mu - \alpha\vartheta_1}\|\mathbf{G}\|_q\,. \tag{3.9}$$

Similarly as in Lemma 3.5 we can show that ϑ_1 is small if δ is small enough.

We start to estimate $\mathbf{G}$ in $L^q(\Omega)$ by means of $\langle \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\beta,q}$ and $[\mathbf{u}_1 - \mathbf{u}_2]_0$. The constants in the estimates will depend on $\langle \mathbf{u}_i \rangle_{\beta,q}$ and $[\mathbf{u}_i]_1$ and will be small for δ small. We shall give the estimates of the terms on the right hand side of (3.6).

$$\begin{aligned} \|(\mathbf{u}_1 - \mathbf{u}_2)\cdot\nabla\mathbf{z}_2\|_q &\le \|\mathbf{u}_1 - \mathbf{u}_2\|_\infty \|\nabla\mathbf{z}_2\|_q \le \\ \le \delta\beta^{-2(1-\frac{1}{q})\frac{3-q}{3}} &[\mathbf{u}_1 - \mathbf{u}_2]_{\beta,q}^{\frac{3-2q}{q}} \langle \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\beta,q}^{3(1-\frac{1}{q})} \le \\ &\le \varepsilon\beta^{1-2(1-\frac{1}{q})\frac{3-2q}{q}}\|\mathbf{g}_1 - \mathbf{g}_2\|_q \end{aligned}$$

Let us note that for $\beta \in (1, \frac{6}{5})$ the exponent by β is strictly positive.

$$\begin{aligned} \|(\mathbf{u}_1 - \mathbf{u}_2)\cdot\nabla\mathbf{u}_1\|_q &\le \beta^{-1-2(1-\frac{1}{q})}\langle \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\beta,q}\langle \mathbf{u}_1 \rangle_{\beta,q} \le \\ &\le C(|\ln\beta|^{-1} + \varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q \end{aligned}$$

The same result holds also for the term $\mathbf{u}_2 \cdot \nabla(\mathbf{u}_1 - \mathbf{u}_2)$.

$$\begin{aligned} \beta\|\mathbf{u}_2 D^2(\mathbf{u}_1 - \mathbf{u}_2)\|_q &\le \beta\|D^2(\mathbf{u}_1 - \mathbf{u}_2)\|_q\|\mathbf{u}_2\|_\infty \le \\ \le C\beta^{2-2(1-\frac{1}{q})\frac{3-q}{q}}(1+\varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q &+ \beta^2(1 + \varepsilon^{\frac{3-2q}{q}})\|\mathbf{g}_1 - \mathbf{g}_2\|_q \end{aligned}$$

Completely analogously we can estimate

$$\beta\|(\mathbf{u}_1 - \mathbf{u}_2)D^2\mathbf{u}_1\|_q \le \beta^{2-2(1-\frac{1}{q})\frac{3-q}{q}}(1+\varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q\,.$$

Moreover

$$\beta^2\|D^2(\mathbf{u}_1 - \mathbf{u}_2)\|_q \le C\beta^{\frac{2}{q}}\|\mathbf{g}_1 - \mathbf{g}_2\|_q\,.$$

All the other terms can be estimated by the same term.

$$\begin{aligned} \|D(\mathbf{u}_1 - \mathbf{u}_2)D^2\mathbf{u}_i\|_q &\le \|D(\mathbf{u}_1 - \mathbf{u}_2)\|_{\frac{2q}{2-q}}\|D^2\mathbf{u}_i\|_2 \le C\beta^{\frac{2-q}{q}}(1+\varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q \\ \|D^2(\mathbf{u}_1 - \mathbf{u}_2)D\mathbf{u}_i\|_q &\le \|D^2(\mathbf{u}_1 - \mathbf{u}_2)\|_q\|D\mathbf{u}_i\|_\infty \le C\beta^{\frac{2-q}{q}}(1+\varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q \\ \|D(s_1 - s_2)D\mathbf{u}_1\|_q &\le \|D(s_1 - s_2)\|_q\|D\mathbf{u}_1\|_\infty \le C\beta^{\frac{2-q}{q}}(1+\varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q \\ \|Ds_2 D(\mathbf{u}_1 - \mathbf{u}_2)\|_q &\le \|Ds_2\|_2\|D(\mathbf{u}_1 - \mathbf{u}_2)\|_{\frac{2q}{2-q}} \le C\beta^{\frac{2-q}{q}}(1+\varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q \end{aligned}$$

From the calculations above we conclude

Lemma 3.6 *Let β, ε be small enough, $\delta = \varepsilon\beta^{1+2(1-\frac{1}{q})}$. Then there exists $\gamma \in (0,1)$ such that*

$$||\mathcal{M}\mathbf{g}_1 - \mathcal{M}\mathbf{g}_2||_q \leq \gamma||\mathbf{g}_1 - \mathbf{g}_2||_q$$

for all $\mathbf{g}_1$, $\mathbf{g}_2 \in B_\delta$.

Combining Lemmas 3.5 and 3.6 with Theorem 3.1 we get finally

Theorem 3.7 *Let $q \in (1, \frac{6}{5})$. Let $||\mathbf{f}||_{1,q}$ be sufficiently small. Then there exists β^* such that for all $\beta \in (0, \beta^*)$ there exists at least one strong solution to (1.5). Moreover we have $D^2\mathbf{v} \in W^{1,q}(\Omega)$, $D\mathbf{v} \in L^{\frac{3q}{3-q}}(\Omega)$, $\mathbf{u} = \mathbf{v} - \mathbf{v}_\infty \in L^{\frac{3q}{3-2q}}(\Omega)$ and $Dp \in W^{1,q}(\Omega)$.*

Proof From Lemmas 3.5 and 3.6 we get existence of the fixed point $\mathbf{w} \in W^{1,q}(\Omega)$. From (1.7) we can calculate the corresponding pair $(\mathbf{u}, s)$. Now, $\mathbf{v} = \mathbf{u} + \mathbf{v}_\infty$ solves the problem (1.5) while $p = \mu s + \alpha_1(\mathbf{u} + \mathbf{v}_\infty) \cdot \nabla s$ is the corresponding pressure. We easily have

$$||Dp||_{1,q} \leq \mu||Ds||_{1,q} + ||\mathbf{u} + \mathbf{v}_\infty||_\infty ||Ds||_{1,q} + ||D\mathbf{u}||_\infty ||Ds||_q \leq C$$

and Theorem 3.7 is demonstrated. □

Remark 3.8 A similar procedure (in some sense even easier) gives us

Theorem 3.9 *Let $k \geq 1$. Let $q \in (1, \frac{6}{5})$ and let $||\mathbf{f}||_{k,q}$ be sufficiently small. Then there exists β^* such that for all $\beta \in (0, \beta^*)$ there exists at least one strong solution to (1.5). Moreover we have that $D^2\mathbf{v} \in W^{k,q}(\Omega)$, $D\mathbf{v} \in L^{\frac{3q}{3-q}}(\Omega)$, $\mathbf{u} = \mathbf{v} - \mathbf{v}_\infty \in L^{\frac{3q}{3-2q}}(\Omega)$ and $Dp \in W^{k,q}(\Omega)$.*

The details of the proof can be found in [Po].

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