Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Lubomír Kubáček; Ludmila Kubáčková

One of the calibration problems

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 36 (1997), No. 1, 117--130

Persistent URL: `http://dml.cz/dmlcz/120358`

Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica **36** (1997) 117–130

One of the Calibration Problems *

LUBOMÍR KUBÁČEK, LUDMILA KUBÁČKOVÁ

Department of Mathematical Analysis, Faculty of Sciences, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: kubacekl@risc.upol.cz

(Received January 14, 1997)

Abstract

Two measurement devices are characterized by dispersions σ_1^2 and σ_2^2 of their registration. The ratio σ_1^2/σ_2^2 is a priori unknown and approximations $\sigma_{1,0}^2$ and $\sigma_{2,0}^2$ are at our disposal only.

The relation between errorless registrations of these devices is given by the calibration curve.

One of the procedures of constructing this curve is described in the paper. Further, accuracy characteristics of the calibration curve parameters estimators and the MINQUE $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ of the dispersions σ_1^2 and σ_2^2 are given.

Key words: BLUE, MINQUE, model with constraints.

1991 Mathematics Subject Classification: 62J05, 62F10

Introduction

The aim of the paper is to find explicite formulae for estimators of parameters β_1 and β_2, which link registrations of two measurement devices and estimators of the dispersions σ_1^2 and σ_2^2 of their registrations.

*Supported by grants No. 201/96/0436 and 201/96/0665 of The Grant Agency of Czech Republic

If an errorless registration of some value v of the first device is μ, then the errorless registration of the other device is ν and

$$\nu = \beta_1 + \beta_2 \mu$$

(calibration curve).

The registration μ of the first device is realized with an error characterized by the dispersion σ_1^2, the registration ν of the other device is realized with an error characterized by the dispersion σ_2^2.

1 Preliminaries

Let n ($n \geq 2$) different values $v_1, \ldots, v_n$, of the measured quantity be registrated by both devices. The results from the first device creates a realization of the n-dimensional random vector (observation vector) $\mathbf{Y}_1$; an analogous meaning has $\mathbf{Y}_2$ for the other device.

In our case we have the model

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim \left[\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix}, \begin{pmatrix} \sigma_1^2 \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \sigma_2^2 \mathbf{I} \end{pmatrix} \right] \tag{1.1}$$

$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)'$, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)'$ and $\mathbf{I}$ denotes the $n \times n$ identity matrix (the notation $\boldsymbol{\eta} \sim (\mathbf{a}, \mathbf{W})$ means that the random vector $\boldsymbol{\eta}$ has the mean value equal to $\mathbf{a}$ and its covariance matrix is $\mathbf{W}$). Further

$$\boldsymbol{\nu} = \mathbf{1}\beta_1 + \boldsymbol{\mu}\beta_2 \tag{1.2}$$

must hold (here $\mathbf{1} = (1, \ldots, 1)' \in R^n$ and R^n denotes the n-dimensional Euclidean space).

Lemma 1.1 *Consider the model of incomplete direct measurements with a condition* $\mathbf{Y} \sim (\gamma_1, \boldsymbol{\Sigma})$,

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} : \mathbf{b} + \mathbf{B}_1 \mathbf{u} + \mathbf{B}_2 \mathbf{v} = \mathbf{0} \right\};$$

here γ_1 *is an unknown* n*-dimensional parameter,* γ_2 *an unknown* k*-dimensional parameter,* $\boldsymbol{\Sigma}$ *a given* $n \times n$ *p.d.matrix,* $\mathbf{B}_1$ *a given* $q \times n$ *matrix and* $\mathbf{B}_2$ *is a given* $q \times k$ *matrix; the ranks of the matrices* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *are characterized by the relations*

$$r(\mathbf{B}_1, \mathbf{B}_2) = q, \;\; r(\mathbf{B}_2) = k, \;\; k < q < n + k.$$

Then the BLUE (best linear unbiased estimator) of γ_1 *and* γ_2 *is*

$$\begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I} - \boldsymbol{\Sigma}\mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1 \\ -\mathbf{Q}_{2,1} \mathbf{B}_1 \end{pmatrix} \mathbf{Y} + \begin{pmatrix} -\boldsymbol{\Sigma}\mathbf{B}_1' \mathbf{Q}_{1,1} \\ -\mathbf{Q}_{2,1} \end{pmatrix} \mathbf{b}, \tag{1.3}$$

and the covariance matrix of this estimator is

$$\operatorname{Var}\begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1 \boldsymbol{\Sigma}, & -\boldsymbol{\Sigma}\mathbf{B}_1' \mathbf{Q}_{1,2} \\ -\mathbf{Q}_{2,1} \mathbf{B}_1 \boldsymbol{\Sigma}, & -\mathbf{Q}_{2,2} \end{pmatrix},$$

where

$$\begin{pmatrix} \mathbf{Q}_{1,1}, & \mathbf{Q}_{1,2} \\ \mathbf{Q}_{2,1}, & \mathbf{Q}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1\mathbf{\Sigma}\mathbf{B}_1', & \mathbf{B}_2 \\ \mathbf{B}_2', & \mathbf{0} \end{pmatrix}^{-1}. \tag{1.4}$$

Proof cf. in [2], pp. 138–143.

Remark 1.2 If in Lemma 1.1 $\mathbf{\Sigma} = \sigma^2\mathbf{V}$, where $\mathbf{V}$ is a given $n \times n$ p.d. matrix and $\sigma^2 \in (0, \infty)$ is an unknown parameter, then

$$\begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I} - \mathbf{V}\mathbf{B}_1'\tilde{\mathbf{Q}}_{1,1}\mathbf{B}_1 \\ -\tilde{\mathbf{Q}}_{2,1}\mathbf{B}_1 \end{pmatrix} \mathbf{Y} + \begin{pmatrix} -\mathbf{V}\mathbf{B}_1'\tilde{\mathbf{Q}}_{1,1} \\ -\tilde{\mathbf{Q}}_{2,1} \end{pmatrix} \mathbf{b},$$

$$\mathrm{Var}\begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = \sigma^2 \begin{pmatrix} \mathbf{V} - \mathbf{V}\mathbf{B}_1'\tilde{\mathbf{Q}}_{1,1}\mathbf{B}_1\mathbf{V}, & -\mathbf{V}\mathbf{B}_1'\tilde{\mathbf{Q}}_{1,2} \\ -\tilde{\mathbf{Q}}_{2,1}\mathbf{B}_1\mathbf{V}, & -\tilde{\mathbf{Q}}_{2,2} \end{pmatrix},$$

where

$$\begin{pmatrix} \tilde{\mathbf{Q}}_{1,1}, & \tilde{\mathbf{Q}}_{1,2} \\ \tilde{\mathbf{Q}}_{2,1}, & \tilde{\mathbf{Q}}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1\mathbf{V}\mathbf{B}_1', & \mathbf{B}_2 \\ \mathbf{B}_2', & \mathbf{0} \end{pmatrix}^{-1}.$$

In this case the parameter σ^2 can be estimated by the statistic

$$\hat{\sigma}^2 = (\mathbf{Y} - \hat{\gamma}_1)'\mathbf{V}^{-1}(\mathbf{Y} - \hat{\gamma}_1)/(q-k).$$

In the case of normality of the vector $\mathbf{Y}$, the distribution of the statistic $\hat{\sigma}^2$ is

$$\hat{\sigma}^2 \sim \sigma^2\chi^2_{q-k}(0)/(q-k),$$

where $\chi^2_{q-k}(0)$ is the random variable with the central chi-square distribution and $q-k$ degrees of freedom (in more detail cf. [2]).

The notation $\mathbf{P}_A^W$ (used in the following) means an orthogonal projection matrix $\mathbf{A}(\mathbf{A}'\mathbf{W}\mathbf{A})^{-1}\mathbf{A}'\mathbf{W}$ on the subspace $\mathcal{M}(\mathbf{A}_{n,k}) = \{\mathbf{A}\mathbf{u},\ \mathbf{u} \in R^k\}$ in the norm given by the p.d. $n \times n$ matrix $\mathbf{W}$, i.e. $||\mathbf{y}||_W = \sqrt{\mathbf{y}'\mathbf{W}\mathbf{y}}$, $\mathbf{y} \in R^n$. Futhermore $\mathbf{M}_A^W = \mathbf{I} - \mathbf{P}_A^W$. If $\mathbf{W} = \mathbf{I}$, then instead of $\mathbf{P}_A^I$ and $\mathbf{M}_A^I$ the notation $\mathbf{P}_A$ and $\mathbf{M}_A$, respectively, is used.

2 Estimators of μ, ν, β_1 and β_2

Let $\boldsymbol{\mu}_0$ and $\beta_{2,0}$ be approximate values of $\boldsymbol{\mu}$ and β_2, respectively, such that the vector $(\boldsymbol{\mu} - \boldsymbol{\mu}_0)(\beta_2 - \beta_{2,0})$ can be neglected. Then the relation (1.2) can be substituted by

$$\beta_{2,0}\boldsymbol{\mu}_0 + (\beta_{2,0}\mathbf{I}, -\mathbf{I})\begin{pmatrix} \delta\boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix} + (\mathbf{1}, \boldsymbol{\mu}_0)\begin{pmatrix} \beta_1 \\ \delta\beta_2 \end{pmatrix} = \mathbf{0},$$

where $\delta\boldsymbol{\mu} = \boldsymbol{\mu} - \boldsymbol{\mu}_0$ and $\delta\beta_2 = \beta_2 - \beta_{2,0}$.

In order to obtain a little more general solution of our problem, let $\sigma_1^2\mathbf{I}$ be substituted by an $n \times n$ p.d. matrix $\mathbf{\Sigma}_{1,1}$ and $\sigma_2^2\mathbf{I}$ by the $n \times n$ p.d. matrix $\mathbf{\Sigma}_{2,2}$.

Under this assumptions, with respect to Lemma 1.1. our model can be rewritten under the following scheme

$$
\begin{aligned}
\mathbf{Y} &\to \begin{pmatrix} \mathbf{Y}_1 - \boldsymbol{\mu}_0 \\ \mathbf{Y}_2 \end{pmatrix}, \quad \gamma_1 \to \begin{pmatrix} \delta\boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix}, \quad \gamma_2 \to \begin{pmatrix} \beta_1 \\ \delta\beta_2 \end{pmatrix}, \\
\mathbf{b} &\to \beta_{2,0}\boldsymbol{\mu}_0, \quad \mathbf{B}_1 \to (\beta_{2,0}\mathbf{I}, -\mathbf{I}), \quad \mathbf{B}_2 \to (\mathbf{1}, \boldsymbol{\mu}_0), \qquad (2.1) \\
\boldsymbol{\Sigma} &\to \begin{pmatrix} \boldsymbol{\Sigma}_{1,1}, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_{2,2} \end{pmatrix}.
\end{aligned}
$$

Lemma 2.1 *Let* $\mathbf{W}$ *be an* $n \times n$ *p.d. matrix and* $\mathbf{A}$ *an* $n \times k$ *matrix with the rank* $r(\mathbf{A}) = k$*. Then the matrix*

$$\begin{pmatrix} \mathbf{W}, & \mathbf{A} \\ \mathbf{A}', & \mathbf{0} \end{pmatrix}$$

is regular and its inverse is

$$
\begin{pmatrix} \mathbf{W}, & \mathbf{A} \\ \mathbf{A}', & \mathbf{0} \end{pmatrix}^{-1} =
$$
$$
= \begin{pmatrix} \mathbf{W}^{-1} - \mathbf{W}^{-1}\mathbf{A}(\mathbf{A}'\mathbf{W}^{-1}\mathbf{A})^{-1}\mathbf{A}'\mathbf{W}^{-1}, & \mathbf{W}^{-1}\mathbf{A}(\mathbf{A}'\mathbf{W}^{-1}\mathbf{A})^{-1} \\ (\mathbf{A}'\mathbf{W}^{-1}\mathbf{A})^{-1}\mathbf{A}'\mathbf{W}^{-1}, & -(\mathbf{A}'\mathbf{W}^{-1}\mathbf{A})^{-1} \end{pmatrix}.
$$

Proof is obvious.

The matrix $\mathbf{W}^{-1} - \mathbf{W}^{-1}\mathbf{A}(\mathbf{A}'\mathbf{W}^{-1}\mathbf{A})^{-1}\mathbf{A}'\mathbf{W}^{-1}$ can be written in the form $(\mathbf{M}_A\mathbf{W}\mathbf{M}_A)^+$ ($^+$ denotes the Moore–Penrose g-inverse; in more detail cf. [3]).

Theorem 2.2 *In the model (cf. (2.1))*

$$
\begin{pmatrix} \mathbf{Y}_1 - \boldsymbol{\mu}_0 \\ \mathbf{Y}_2 \end{pmatrix} \sim \left[\begin{pmatrix} \delta\boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{1,1}, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_{2,2} \end{pmatrix} \right],
$$
$$
\beta_{2,0}\boldsymbol{\mu}_0 + (\beta_{2,0}\mathbf{I}, -\mathbf{I}) \begin{pmatrix} \delta\boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix} + (\mathbf{1}, \boldsymbol{\mu}_0) \begin{pmatrix} \beta_1 \\ \delta\beta_2 \end{pmatrix} = \mathbf{0},
$$

if we denote $\mathbf{M}_{1,\mu_0} = \mathbf{I} - \mathbf{P}_{1,\mu_0}$*, where*

$$
\mathbf{P}_{1,\mu_0} = (\mathbf{1}, \boldsymbol{\mu}_0) \begin{pmatrix} n, & \mathbf{1}'\boldsymbol{\mu}_0 \\ \mathbf{1}'\boldsymbol{\mu}_0, & \boldsymbol{\mu}_0'\boldsymbol{\mu} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{1}' \\ \boldsymbol{\mu}_0' \end{pmatrix}
$$

the following is valid

$$
\begin{aligned}
\mathbf{Q}_{1,1} &= [\mathbf{M}_{1,\mu_0}(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})\mathbf{M}_{1,\mu_0}]^+, \\
\mathbf{Q}_{2,1} &= -\mathbf{Q}_{2,2} \begin{pmatrix} \mathbf{1}'(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})^{-1} \\ \boldsymbol{\mu}_0'(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})^{-1} \end{pmatrix}, \\
\mathbf{Q}_{2,2} &= -\begin{pmatrix} \mathbf{1}'(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})^{-1}\mathbf{1}, & \mathbf{1}'(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})^{-1}\boldsymbol{\mu}_0 \\ \boldsymbol{\mu}_0'(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})^{-1}\mathbf{1}, & \boldsymbol{\mu}_0'(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})^{-1}\boldsymbol{\mu}_0 \end{pmatrix}^{-1}, \\
\hat{\boldsymbol{\mu}} &= \boldsymbol{\mu}_0 + \widehat{\delta\boldsymbol{\mu}}
\end{aligned}
$$

$$\begin{aligned}
&= \mathbf{Y}_1 + \beta_{2,0}\boldsymbol{\Sigma}_{1,1}[\mathbf{M}_{1,\mu_0}(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})\mathbf{M}_{1,\mu_0}]^+(\mathbf{Y}_2 - \beta_{2,0}\mathbf{Y}_1),\\
\hat{\boldsymbol{\nu}} &= \mathbf{Y}_2 - \boldsymbol{\Sigma}_{2,2}[\mathbf{M}_{1,\mu_0}(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})\mathbf{M}_{1,\mu_0}]^+(\mathbf{Y}_2 - \beta_{2,0}\mathbf{Y}_1),\\
\begin{pmatrix} \hat{\beta}_1 \\ \widehat{\delta\beta_2} \end{pmatrix} &= -\mathbf{Q}_{2,2}\begin{pmatrix} \mathbf{1}'(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})^{-1}(\mathbf{Y}_2 - \beta_{2,0}\mathbf{Y}_1) \\ \boldsymbol{\mu}_0'(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})^{-1}(\mathbf{Y}_2 - \beta_{2,0}\mathbf{Y}_1) \end{pmatrix},\\
\operatorname{Var}(\hat{\boldsymbol{\mu}}) &= \boldsymbol{\Sigma}_{1,1} - \beta_{2,0}^2\boldsymbol{\Sigma}_{1,1}\mathbf{Q}_{1,1}\boldsymbol{\Sigma}_{1,1},\\
\operatorname{cov}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\nu}}) &= \beta_{2,0}\boldsymbol{\Sigma}_{1,1}\mathbf{Q}_{1,1}\boldsymbol{\Sigma}_{2,2},\\
\operatorname{Var}(\hat{\boldsymbol{\nu}}) &= \boldsymbol{\Sigma}_{2,2} - \boldsymbol{\Sigma}_{2,2}\mathbf{Q}_{1,1}\boldsymbol{\Sigma}_{2,2},\\
\operatorname{Var}\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} &= -\mathbf{Q}_{2,2}.
\end{aligned}$$

and

$$\operatorname{cov}\left[\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}, \begin{pmatrix} \hat{\boldsymbol{\mu}} \\ \hat{\boldsymbol{\nu}} \end{pmatrix}\right] =$$

$$= \mathbf{Q}_{2,2}\begin{pmatrix} \mathbf{1}'(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})^{-1}\beta_{2,0}\boldsymbol{\Sigma}_{1,1}, & -\mathbf{1}'(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})^{-1}\boldsymbol{\Sigma}_{2,2} \\ \boldsymbol{\mu}_0'(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})^{-1}\beta_{2,0}\boldsymbol{\Sigma}_{1,1}, & -\boldsymbol{\mu}_0'(\beta_{2,0}^2\boldsymbol{\Sigma}_{1,1} + \boldsymbol{\Sigma}_{2,2})^{-1}\boldsymbol{\Sigma}_{2,2} \end{pmatrix}.$$

Proof The expression for $\mathbf{Q}_{1,1}$ is obvious with respect to Lemma 2.1 and the remark after it. Further it is necessary to use Lemma 1.1 under the scheme (2.1). Since this is simple however tedious, it is omitted. □

Corollary 2.3 *If in Theorem 2.2.* $\boldsymbol{\Sigma}_{1,1} = \sigma_1^2\mathbf{I}$ *and* $\boldsymbol{\Sigma}_{2,2} = \sigma_2^2\mathbf{I}$, *then*

$$\mathbf{Q}_{1,1} = \frac{1}{\beta_{2,0}^2\sigma_1^2 + \sigma_2^2}\mathbf{M}_{1,\mu_0}, \quad \mathbf{Q}_{2,1} = \begin{pmatrix} n, & \mathbf{1}'\boldsymbol{\mu}_0 \\ \boldsymbol{\mu}_0'\mathbf{1}, & \boldsymbol{\mu}_0'\boldsymbol{\mu}_0 \end{pmatrix}\begin{pmatrix} \mathbf{1}' \\ \boldsymbol{\mu}_0' \end{pmatrix}$$

$$\mathbf{Q}_{2,2} = -(\beta_{2,0}^2\sigma_1^2 + \sigma_2^2)\begin{pmatrix} n, & \mathbf{1}'\boldsymbol{\mu}_0 \\ \boldsymbol{\mu}_0'\mathbf{1}, & \boldsymbol{\mu}_0'\boldsymbol{\mu}_0 \end{pmatrix}^{-1}.$$

Thus

$$\begin{aligned}
\hat{\boldsymbol{\mu}} &= \mathbf{Y}_1 + \frac{\beta_{2,0}\sigma_1^2}{\beta_{2,0}^2\sigma_1^2 + \sigma_2^2}\mathbf{M}_{1,\mu_0}(\mathbf{Y}_2 - \beta_{2,0}\mathbf{Y}_1),\\
\hat{\boldsymbol{\nu}} &= \mathbf{Y}_2 - \frac{\sigma_2^2}{\beta_{2,0}^2\sigma_1^2 + \sigma_2^2}\mathbf{M}_{1,\mu_0}(\mathbf{Y}_2 - \beta_{2,0}\mathbf{Y}_1),\\
\begin{pmatrix} \hat{\beta}_1 \\ \widehat{\delta\beta_2} \end{pmatrix} &= \begin{pmatrix} n, & \mathbf{1}'\boldsymbol{\mu}_0 \\ \boldsymbol{\mu}_0'\mathbf{1}, & \boldsymbol{\mu}_0'\boldsymbol{\mu}_0 \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{1}'(\mathbf{Y}_2 - \beta_{2,0}\mathbf{Y}_1) \\ \boldsymbol{\mu}_0'(\mathbf{Y}_2 - \beta_{2,0}\mathbf{Y}_1) \end{pmatrix},\\
\operatorname{Var}(\hat{\boldsymbol{\mu}}) &= \sigma_1^2\mathbf{I} - \frac{\beta_{2,0}^2\sigma_1^4}{\beta_{2,0}^2\sigma_1^2 + \sigma_2^2}\mathbf{M}_{1,\mu_0},\\
\operatorname{cov}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\nu}}) &= \frac{\beta_{2,0}\sigma_1^2\sigma_2^2}{\beta_{2,0}^2\sigma_1^2 + \sigma_2^2}\mathbf{M}_{1,\mu_0},\\
\operatorname{Var}(\hat{\boldsymbol{\nu}}) &= \sigma_2^2\mathbf{I} - \frac{\sigma_2^4}{\beta_{2,0}^2\sigma_1^2 + \sigma_2^2}\mathbf{M}_{1,\mu_0},
\end{aligned}$$

$$\mathrm{Var}\begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{pmatrix} = (\beta_{2,0}^2\sigma_1^2+\sigma_2^2)\begin{pmatrix} n, & \mathbf{1}'\boldsymbol{\mu}_0\\ \boldsymbol{\mu}_0'\mathbf{1}, & \boldsymbol{\mu}_0'\boldsymbol{\mu}_0\end{pmatrix}^{-1} = \begin{pmatrix} v_{1,1}, & v_{1,2}\\ v_{2,1}, & v_{2,2}\end{pmatrix}$$

and

$$\mathrm{cov}\left[\begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{pmatrix},\begin{pmatrix}\hat{\boldsymbol{\mu}}\\ \hat{\boldsymbol{\nu}}\end{pmatrix}\right] = -\begin{pmatrix} n, & \mathbf{1}'\boldsymbol{\mu}_0\\ \boldsymbol{\mu}_0'\mathbf{1}, & \boldsymbol{\mu}_0'\boldsymbol{\mu}_0\end{pmatrix}^{-1}\begin{pmatrix}\beta_{2,0}\sigma_1^2\mathbf{1}', & -\sigma_2^2\mathbf{1}'\\ \beta_{2,0}\sigma_1^2\boldsymbol{\mu}_0', & -\sigma_2^2\boldsymbol{\mu}_0'\end{pmatrix}.$$

Remark 2.4 The formulae given in Corollary 2.3 can be considered as a solution of the first problem, i.e. how to determine parameters of the calibration curve. It is to be remarked that for $\sigma_1 = \sigma_2$ and $\widehat{\delta\beta}_2 = 0$ (i.e. $\beta_{2,0} = \hat{\beta}_2$) the equality

$$\forall\{i=1,\dots,n\}(y_{2,i}-\hat{\nu}_i)/(y_{1,i}-\hat{\mu}_i) = -1/\beta_{2,0}$$

is valid. This is a reason that the treated problem is sometimes called "orthogonal regression problem".

It is not necessary to emphasize that in the case $\widehat{\delta\beta}_2 \neq 0$ (or a negligible value) or $\widehat{\delta\boldsymbol{\mu}} \neq \mathbf{0}$, the calculation must be repeated with the new starting vectors $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}_0 + \widehat{\delta\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\nu}}$ instead of $\mathbf{Y}_1$ and $\mathbf{Y}_2$, respectively; also the value $\beta_{2,0}$ must be substituted by $\beta_{2,0} + \widehat{\delta\beta}_2$.

Remark 2.5 If it is a priori known that $\sigma = \sigma_1 = \sigma_2$, where σ is unknown, then there is no problem to estimate it. With respect to Remark 1.2

$$\hat{\sigma}^2 = [(\mathbf{Y}_1-\hat{\boldsymbol{\mu}})'(\mathbf{Y}_1-\hat{\boldsymbol{\mu}}) + (\mathbf{Y}_2-\hat{\boldsymbol{\nu}})'(\mathbf{Y}_2-\hat{\boldsymbol{\nu}})]/(n-2).$$

In the next section we shall see that the estimator of the vector $(\sigma_1^2,\sigma_2^2)'$ in the case $\sigma_1 \neq \sigma_2$ when simultaneously σ_1/σ_2 is unknown, cannot be found so simply.

3 MINQUE of σ_1^2 and σ_2^2

In this section the model of incomplete direct measurement with a condition

$$\begin{pmatrix}\mathbf{Y}_1-\boldsymbol{\mu}_0\\ \mathbf{Y}_2\end{pmatrix} \sim \left[\begin{pmatrix}\delta\boldsymbol{\mu}\\ \boldsymbol{\nu}\end{pmatrix},\begin{pmatrix}\sigma_1^2\mathbf{I}, & \mathbf{0}\\ \mathbf{0}, & \sigma_2^2\mathbf{I}\end{pmatrix}\right], \tag{3.1}$$

$$\beta_{2,0}\boldsymbol{\mu}_0 + (\beta_{2,0}\mathbf{I},-\mathbf{I})\begin{pmatrix}\delta\boldsymbol{\mu}\\ \boldsymbol{\nu}\end{pmatrix} + (\mathbf{1},\boldsymbol{\mu}_0)\begin{pmatrix}\beta_1\\ \delta\beta_2\end{pmatrix} = \mathbf{0}$$

is under consideration.

Lemma 3.1 *Let in the model of incomplete direct measurement with a condition from Lemma 1.1 the matrix* $\boldsymbol{\Sigma}$ *be of the form* $\boldsymbol{\Sigma} = \sum_{i=1}^p \vartheta_i\mathbf{V}_i$, *where* $\mathbf{V}_1,\dots\mathbf{V}_p$, *are given symmetric* $n\times n$ *matrices and* $\boldsymbol{\vartheta} = (\vartheta_1,\dots,\vartheta_p)' \in \underline{\vartheta} \subset R^p$

is an unknown vector parameter; $\underline{\vartheta}$ *is an open set in* R^p *such that* $\boldsymbol{\vartheta} \in \underline{\vartheta}$ *implies* $\sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$ *is p.d. Let* $\mathcal{K}er(\mathbf{B}_1, \mathbf{B}_2) = \mathcal{M}\begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix}$, *where* $\mathbf{K}_1$ *is an* $n \times (n+k-q)$ *matrix and* $\mathbf{K}_2$ *is a* $k \times (n+k-q)$ *matrix. Let* $\gamma_{1,0}$ *be any solution of the equation* $\mathbf{b} + \mathbf{B}_1\gamma_1 + \mathbf{B}_2\gamma_2 = \mathbf{0}$.

Then $\boldsymbol{\vartheta}_0$*-MINQUE (for more detail cf. [4]) of a function* $\mathbf{g}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\vartheta}$, *exists iff*

$$\mathbf{g} \in \mathcal{M}\left(\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}\right),$$

where

$$\left\{\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}\right\}_{i,j} = \mathrm{Tr}\left[(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+\mathbf{V}_i(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+\mathbf{V}_j\right],$$

$i,j = 1,\ldots,p$, *and* $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) = \sum_{i=1}^{p} \vartheta_{0,i}\mathbf{V}_i$, $\boldsymbol{\vartheta}_0 \in \underline{\vartheta}$.

If this condition is fulfilled, then $\boldsymbol{\vartheta}_0$*-MINQUE of the function* $\mathbf{g}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\vartheta}$, *is*

$$\widehat{\mathbf{g}'\boldsymbol{\vartheta}} = \sum_{i=1}^{p} \lambda_i (\mathbf{Y} - \gamma_{1,0})'(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+\mathbf{V}_i(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+(\mathbf{Y} - \gamma_{1,0}),$$

where the vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)'$ *is a solution of the equation*

$$\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}\boldsymbol{\lambda} = \mathbf{g}.$$

Proof It is a consequence of Theorem 5.2.1 in [4].

Corollary 3.2 *Let the matrix* $\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}$ *be regular; thus there exists the* $\boldsymbol{\vartheta}_0$*-MINQUE of the whole vector* $\boldsymbol{\vartheta}$ *and it is*

$$\hat{\boldsymbol{\vartheta}} =$$
$$= \mathbf{S}^{-1}_{(M_{K_1}\Sigma_0 M_{K_1})^+}\begin{pmatrix} (\mathbf{Y} - \gamma_{1,0})'(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+\mathbf{V}_1(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+(\mathbf{Y} - \gamma_{1,0}) \\ \vdots \\ (\mathbf{Y} - \gamma_{1,0})'(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+\mathbf{V}_p(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+(\mathbf{Y} - \gamma_{1,0}) \end{pmatrix}.$$

Theorem 3.3 *(i) The matrix* $(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+$ *can be expressed as*

$$\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1$$

(in (1.4) the matrix $\boldsymbol{\Sigma}_0$ *is used instead of* $\boldsymbol{\Sigma}$*).*

(ii) The expression $(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+(\mathbf{Y} - \gamma_{1,0})$ *is equal to* $\boldsymbol{\Sigma}_0^{-1}(\mathbf{Y} - \hat{\gamma}_1)$, *with* $\hat{\gamma}_1$ *given by (1.3), where the matrix* $\boldsymbol{\Sigma}_0$ *is used instead of* $\boldsymbol{\Sigma}$.

Proof (i) Obviously $\mathbf{B}_1\mathbf{K}_1 + \mathbf{B}_2\mathbf{K}_2 = \mathbf{0} \Rightarrow \mathbf{K}_2 = -(\mathbf{B}_2'\mathbf{B}_2)^{-1}\mathbf{B}_2'\mathbf{B}_1\mathbf{K}_1$. Thus $[\mathbf{I} - \mathbf{B}_2(\mathbf{B}_2'\mathbf{B}_2)^{-1}\mathbf{B}_2']\mathbf{B}_1\mathbf{K}_1 = \mathbf{M}_{B_2}\mathbf{B}_1\mathbf{K}_1 = \mathbf{0}$. If $\gamma_1 = \gamma_{1,0} + \mathbf{K}_1\boldsymbol{\tau}$, then the BLUE of $\boldsymbol{\beta}_1$ is

$$\hat{\gamma}_1 = \gamma_{0,1} + \widehat{\mathbf{K}_1\boldsymbol{\tau}} = \gamma_{1,0} + \mathbf{P}_{K_1}^{\Sigma_0^{-1}}(\mathbf{Y} - \gamma_{1,0})$$

and with respect to (1.3) also

$$\hat{\gamma}_1 = (\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1)\mathbf{Y} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{b}.$$

Thus $\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1 = \mathbf{P}_{K_1}^{\Sigma_0^{-1}}$.

To prove directly this equality it is necessary and sufficient to show that

(a) $(\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1)^2 = \mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1$,

(b) $\mathcal{M}(\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1) = \mathcal{M}(\mathbf{K}_1)$,

(c) $\mathbf{\Sigma}_0^{-1}(\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1) = (\mathbf{I} - \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1 \mathbf{\Sigma}_0)\mathbf{\Sigma}_0^{-1}$.

The equality (a) is a direct consequence of (1.4).

(b) Obviously

$$\mathbf{M}_{B_2} \mathbf{B}_1 (\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1) = \mathbf{M}_{B_2} (\mathbf{B}_1 - \mathbf{B}_1 \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1) = \mathbf{0},$$

since $\mathbf{B}_1 - \mathbf{B}_1 \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1 = \mathbf{B}_2 \mathbf{Q}_{2,1}$ (cf. (1.4)). Thus

$$\mathcal{M}(\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1) \subset \mathcal{M}(\mathbf{K}_1).$$

Since $\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1$ is idempotent,

$$\begin{aligned} r(\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1) &= \mathrm{Tr}(\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1) \\ &= \mathrm{Tr}(\mathbf{I}) - \mathrm{Tr}(\mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1). \end{aligned}$$

Further $\mathbf{B}_1 \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} + \mathbf{B}_2 \mathbf{Q}_{2,1} = \mathbf{I}$ and thus

$$\begin{aligned} \mathrm{Tr}(\mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1) &= \mathrm{Tr}(\mathbf{B}_1 \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1}) = \mathrm{Tr}(\mathbf{I} - \mathbf{B}_2 \mathbf{Q}_{2,1}) \\ &= q - \mathrm{Tr}(\mathbf{Q}_{2,1} \mathbf{B}_2) = q - \mathrm{Tr}(\mathbf{I}_{k,k}) = q - k. \end{aligned}$$

Thus $r(\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1) = n + k - q$.

Further

$$\begin{aligned} & n + k - q = \\ & = \dim \mathcal{M} \begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix} = \dim \mathcal{M} \left[\begin{pmatrix} \mathbf{I} \\ -(\mathbf{B}_2' \mathbf{B}_2)^{-1} \mathbf{B}_2' \mathbf{B}_1 \end{pmatrix} \mathbf{K}_1 \right] = \dim \mathcal{M}(\mathbf{K}_1), \end{aligned}$$

since the matrix $\begin{pmatrix} \mathbf{I} \\ -(\mathbf{B}_2' \mathbf{B}_2)^{-1} \mathbf{B}_2' \mathbf{B}_1 \end{pmatrix}$ is of the full rank in columns.

The implication

$$\begin{aligned} \mathcal{M}(\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1) \subset \mathcal{M}(\mathbf{K}_1) \quad \& \quad \dim \mathcal{M}(\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1) = \dim \mathcal{M}(\mathbf{K}_1) \\ \Rightarrow \quad \mathcal{M}(\mathbf{I} - \mathbf{\Sigma}_0 \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1) = \mathcal{M}(\mathbf{K}_1) \end{aligned}$$

proves (b).

(c) is obvious.

Since

$$(\mathbf{M}_{K_1} \mathbf{\Sigma}_0 \mathbf{M}_{K_1})^+ = \mathbf{\Sigma}_0^{-1} \mathbf{M}_{K_1}^{\Sigma_0^{-1}} = \mathbf{\Sigma}_0^{-1}(\mathbf{I} - \mathbf{P}_{K_1}^{\Sigma_0^{-1}}) = \mathbf{B}_1' \mathbf{Q}_{1,1} \mathbf{B}_1,$$

the statement (i) is proved.

(ii) The expression $(\mathbf{M}_{K_1}\Sigma_0\mathbf{M}_{K_1})^+(\mathbf{Y}-\gamma_{1,0})$ can be rewritten as

$$\Sigma_0^{-1}(\mathbf{I}-\mathbf{P}_{K_1}^{\Sigma_0^{-1}})(\mathbf{Y}-\gamma_{1,0}) = \Sigma_0^{-1}(\mathbf{Y}-\gamma_{1,0}-\widehat{\mathbf{K}_1\tau}) = \Sigma_0^{-1}(\mathbf{Y}-\hat{\gamma}_1).$$

□

Theorem 3.4 *In the framework of the model (3.1) we have*

$$\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+} = \frac{n-2}{(\beta_{2,0}^2\sigma_{1,0}^2+\sigma_{2,0}^2)^2}\begin{pmatrix}\beta_{2,0}^4, & \beta_{2,0}^2\\ \beta_{2,0}^2, & 1\end{pmatrix},$$

and for $\mathbf{g}\in\mathcal{M}(\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+})$

$$\mathbf{g}'\widehat{\begin{pmatrix}\sigma_1^2\\ \sigma_2^2\end{pmatrix}} = \lambda_1\frac{1}{\sigma_{1,0}^4}(\mathbf{Y}_1-\hat{\boldsymbol{\mu}})'(\mathbf{Y}_1-\hat{\boldsymbol{\mu}}) + \lambda_2\frac{1}{\sigma_{2,0}^4}(\mathbf{Y}_2-\hat{\boldsymbol{\nu}})'(\mathbf{Y}_2-\hat{\boldsymbol{\nu}}),$$

where

$$\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix}g_1\\ g_2\end{pmatrix}.$$

Proof With respect to Theorem 2.2 and Corollary 2.3 we have

$$\mathbf{Q}_{1,1} = \left(\mathbf{M}_{1,\mu_0}(\beta_{2,0}^2\Sigma_{1,1}+\Sigma_{2,2})\mathbf{M}_{1,\mu_0}\right)^+ = \frac{1}{\beta_{2,0}^2\sigma_{1,0}^2+\sigma_{2,0}^2}\mathbf{M}_{1,\mu_0}$$

(here the obvious equality $\mathbf{M}_{1,\mu_0}^+ = \mathbf{M}_{1,\mu_0}$ is utilized) and thus

$$(\mathbf{M}_{K_1}\Sigma_0\mathbf{M}_{K_1})^+ = \mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1 = \frac{1}{\beta_{2,0}^2\sigma_{1,0}^2+\sigma_{2,0}^2}\begin{pmatrix}\beta_{2,0}^2\mathbf{M}_{1,\mu_0}, & -\beta_{2,0}\mathbf{M}_{1,\mu_0}\\ -\beta_{2,0}\mathbf{M}_{1,\mu_0}, & \mathbf{M}_{1,\mu_0}\end{pmatrix}.$$

Since $\mathrm{Tr}(\mathbf{M}_{1,\mu_0}) = n-2$, we obtain

$$\begin{aligned}
\mathrm{Tr}(\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1\mathbf{V}_1\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1\mathbf{V}_1) &= \{\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}\}_{1,1} = \frac{n-2}{(\beta_{2,0}^2\sigma_{1,0}^2+\sigma_{2,0}^2)^2}\beta_{2,0}^4\\
\mathrm{Tr}(\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1\mathbf{V}_1\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1\mathbf{V}_2) &= \{\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}\}_{1,2} = \{\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}\}_{2,1}\\
&= \frac{n-2}{(\beta_{2,0}^2\sigma_{1,0}^2+\sigma_{2,0}^2)^2}\beta_{2,0}^2\\
\mathrm{Tr}(\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1\mathbf{V}_2\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1\mathbf{V}_2) &= \{\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}\}_{2,2} = \frac{n-2}{(\beta_{2,0}^2\sigma_{1,0}^2+\sigma_{2,0}^2)^2}.
\end{aligned}$$

As far as the terms

$$(\mathbf{Y}-\gamma_{1,0})'(\mathbf{M}_{K_1}\Sigma_0\mathbf{M}_{K_1})^+\mathbf{V}_i(\mathbf{M}_{K_1}\Sigma_0\mathbf{M}_{K_1})^+(\mathbf{Y}-\gamma_{1,0}),\quad i=1,\dots,p,$$

are concerned, it is sufficient to realize the relationships (implied by Theorem 3.3. (ii))

$$\begin{aligned}
&(\mathbf{Y}-\gamma_{1,0})'(\mathbf{M}_{K_1}\Sigma_0\mathbf{M}_{K_1})^+\mathbf{V}_1(\mathbf{M}_{K_1}\Sigma_0\mathbf{M}_{K_1})^+(\mathbf{Y}-\gamma_{1,0}) \rightarrow\\
&\rightarrow \begin{pmatrix}\mathbf{Y}_1-\hat{\boldsymbol{\mu}}\\ \mathbf{Y}_2-\hat{\boldsymbol{\nu}}\end{pmatrix}'\begin{pmatrix}\sigma_{1,0}^{-2}\mathbf{I}, & \mathbf{0}\\ \mathbf{0}, & \sigma_{2,0}^{-2}\mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{I}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\begin{pmatrix}\sigma_{1,0}^{-2}\mathbf{I}, & \mathbf{0}\\ \mathbf{0}, & \sigma_{2,0}^{-2}\mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{Y}_1-\hat{\boldsymbol{\mu}}\\ \mathbf{Y}_2-\hat{\boldsymbol{\nu}}\end{pmatrix} =\\
&= \frac{1}{\sigma_{1,0}^4}(\mathbf{Y}_1-\hat{\boldsymbol{\mu}})'(\mathbf{Y}_1-\hat{\boldsymbol{\mu}})
\end{aligned}$$

and

$$(\mathbf{Y}-\gamma_{1,0})'(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+\mathbf{V}_2(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+(\mathbf{Y}-\gamma_{1,0}) \quad \rightarrow$$

$$\rightarrow \begin{pmatrix} \mathbf{Y}_1-\hat{\boldsymbol{\mu}} \\ \mathbf{Y}_2-\hat{\boldsymbol{\nu}} \end{pmatrix}' \begin{pmatrix} \sigma_{1,0}^{-2}\mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \sigma_{2,0}^{-2}\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{0}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{I} \end{pmatrix} \begin{pmatrix} \sigma_{1,0}^{-2}\mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \sigma_{2,0}^{-2}\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{Y}_1-\hat{\boldsymbol{\mu}} \\ \mathbf{Y}_2-\hat{\boldsymbol{\nu}} \end{pmatrix} =$$

$$= \frac{1}{\sigma_{2,0}^4}(\mathbf{Y}_2-\hat{\boldsymbol{\nu}})'(\mathbf{Y}_2-\hat{\boldsymbol{\nu}}).$$

□

Corollary 3.5 *The $(\sigma_{1,0}^2,\sigma_{2,0}^2)$-MINQUE of the function*

$$\mathbf{g}'(\sigma_1^2,\sigma_2^2)', (\sigma_1^2,\sigma_2^2)' \in (0,\infty)\times(0,\infty),$$

exists iff $\mathbf{g} = c(\beta_{2,0}^2, 1)'$, where $c \in R^1$; in this case the estimator is $(c = 1)$

$$\widehat{(\beta_{2,0}^2\sigma_1^2+\sigma_2^2)} =$$

$$= \frac{(\beta_{2,0}^2\sigma_{1,0}^2+\sigma_{2,0}^2)^2}{(n-2)(1+\beta_{2,0}^4)} \left\{ \frac{\beta_{2,0}^2}{\sigma_{1,0}^4}(\mathbf{Y}_1-\hat{\boldsymbol{\mu}})'(\mathbf{Y}_1-\hat{\boldsymbol{\mu}}) + \frac{1}{\sigma_{2,0}^4}(\mathbf{Y}_2-\hat{\boldsymbol{\nu}})'(\mathbf{Y}_2-\hat{\boldsymbol{\nu}}) \right\}.$$

Proof With respect to Lemma 3.1. and Theorem 3.3.

$$\mathbf{g} \in \mathcal{M}(\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}) \Rightarrow \widehat{\mathbf{g}'\boldsymbol{\vartheta}} = \mathbf{g}'\mathbf{S}^-_{(M_{K_1}\Sigma_0 M_{K_1})^+}\hat{\boldsymbol{\kappa}},$$

where

$$\hat{\kappa}_i = (\mathbf{Y}-\hat{\gamma}_1)'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}(\mathbf{Y}-\hat{\gamma}_1), \quad i=1,\dots,p.$$

In our case

$$\mathbf{S}^-_{(M_{K_1}\Sigma_0 M_{K_1})^+} = \frac{(\beta_{2,0}^2\sigma_{1,0}^2+\sigma_{2,0}^2)^2}{n-2}\begin{pmatrix} \beta_{2,0}^4, & \beta_{2,0}^2 \\ \beta_{2,0}^2, & 1 \end{pmatrix}^-$$

$$= \frac{(\beta_{2,0}^2\sigma_{1,0}^2+\sigma_{2,0}^2)^2}{n-2}\frac{1}{(1+\beta_{2,0}^4)^2}\begin{pmatrix} 2\beta_{2,0}^4+1, & -\beta_{2,0}^6 \\ -\beta_{2,0}^6, & 1+\beta_{2,0}^4+\beta_{2,0}^8 \end{pmatrix}.$$

(Here a special version of a g-inverse of the matrix $\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}$ is given, i.e. the Moore - Penrose g-inverse.) Since $\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_1\boldsymbol{\Sigma}_0^{-1} = \frac{1}{\sigma_{1,0}^4}\begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{pmatrix}$ and $\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_2\boldsymbol{\Sigma}_0^{-1} = \frac{1}{\sigma_{2,0}^4}\begin{pmatrix} \mathbf{0}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{I} \end{pmatrix}$, we obtain the equality in the statement. □

The estimator $\widehat{(\beta_{2,0}^2\sigma_1^2+\sigma_2^2)}$ is a good check of the value $\beta_{2,0}^2\sigma_{1,0}^2+\sigma_{2,0}^2$ in the case that in Corollary 2.3 the value $\beta_{2,0}^2\sigma_{1,0}^2+\sigma_{2,0}^2$ must be used instead of $\beta_{2,0}^2\sigma_1^2+\sigma_2^2$.

Under the design of experiment till now considered, the vector $(\sigma_1^2,\sigma_2^2)'$ cannot be estimated. Thus it is necessary to change the design. The simplest way how to do it is to replicate the measurement.

Let $\mathbf{Y}_{(1)}, \dots, \mathbf{Y}_{(m)}$ be m replications of the observation vector $\mathbf{Y}$ in the model

$$(\mathbf{Y}, \gamma_1, \mathbf{\Sigma}(\boldsymbol{\vartheta})), \quad \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in \mathcal{V},$$
$$\mathcal{V} = \left\{ \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} : \mathbf{u} \in R^n, \mathbf{v} \in R^k, \mathbf{b} + \mathbf{B}_1\mathbf{u} + \mathbf{B}_2\mathbf{v} = \mathbf{0} \right\}.$$

If $\underline{\mathbf{Y}} = (\mathbf{Y}'_{(1)}, \dots, \mathbf{Y}'_{(m)})'$ is the observation vektor after m replications, we have the model

$$\left(\underline{\mathbf{Y}}, (\mathbf{1}_m \otimes \mathbf{I})\gamma_1, \mathbf{I} \otimes \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i \right), \quad \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in \mathcal{V},$$

which is equivalent to

$$\left(\underline{\mathbf{Y}} - (\mathbf{1} \otimes \mathbf{I})\gamma_{1,0}, (\mathbf{1} \otimes \mathbf{K}_1)\boldsymbol{\tau}, \mathbf{I} \otimes \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i \right), \quad \boldsymbol{\tau} \in R^{n+k-q}, \tag{3.2}$$

where $\gamma_{1,0}$ is any vector satisfying the equality $\mathbf{b} + \mathbf{B}_1\gamma_{1,0} + \mathbf{B}_2\gamma_{2,0} = \mathbf{0}$ and the matrix $\mathbf{K}_1$ is the same as in Lemma 3.1.

Lemma 3.6 *In the model (3.2) the $\boldsymbol{\vartheta}_0$-MINQUE of a function $\mathbf{g}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\vartheta}$, is*

$$\begin{aligned} \widehat{\mathbf{g}'\boldsymbol{\vartheta}} &= \sum_{i=1}^{p} \lambda_i \left\{ \operatorname{Tr} \left[\mathbf{\Sigma}_0^{-1}\mathbf{V}_i\mathbf{\Sigma}_0^{-1} \sum_{j=1}^{m} (\mathbf{Y}_{(j)} - \overline{\mathbf{Y}})(\mathbf{Y}_{(j)} - \overline{\mathbf{Y}})' \right] + \right. \\ &\left. + m(\overline{\mathbf{Y}} - \gamma_{1,0})'(\mathbf{M}_{K_1}\mathbf{\Sigma}_0\mathbf{M}_{K_1})^+\mathbf{V}_i(\mathbf{M}_{K_1}\mathbf{\Sigma}_0\mathbf{M}_{K_1})^+(\overline{\mathbf{Y}} - \gamma_{1,0}) \right\}, \end{aligned}$$

where $\overline{\mathbf{Y}} = \frac{1}{m}\sum_{j=1}^{m} \mathbf{Y}_{(j)}$ and the vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_p)'$ is a solution of the equation

$$\left[(m-1)\mathbf{S}_{\Sigma_0^{-1}} + \mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+} \right] \boldsymbol{\lambda} = \mathbf{g}.$$

If $\mathbf{V}_1, \dots, \mathbf{V}_p$ are linearly independent, then $\mathbf{S}_{\Sigma_0^{-1}}$ is p.d. (thus regular).

Proof It is a consequence of Theorem 6.1.1 in [4].

Theorem 3.7 *Let the model (3.1) be m-times replicated. Then the $(\sigma_{1,0}^2, \sigma_{2,0}^2)$-MINQUE of the vector $(\sigma_1^2, \sigma_2^2)'$ is*

$$\begin{pmatrix} \hat{\sigma}_1^2 \\ \hat{\sigma}_2^2 \end{pmatrix} = \frac{1}{(m-1)n} \left[\mathbf{I} - c(m, n, \beta_{2,0}^2, \sigma_{1,0}^2, \sigma_{2,0}^2) \begin{pmatrix} \sigma_{1,0}^4\beta_{2,0}^4, & \sigma_{1,0}^4\beta_{2,0}^2 \\ \sigma_{2,0}^4\beta_{2,0}^2, & \sigma_{2,0}^4 \end{pmatrix} \right] \begin{pmatrix} \hat{\kappa}_1 \\ \hat{\kappa}_2 \end{pmatrix},$$

where

$$c(m, n, \beta_{2,0}^2, \sigma_{1,0}^2, \sigma_{2,0}^2) =$$
$$= \frac{n-2}{(\beta_{2,0}^4\sigma_{1,0}^4 + \sigma_{2,0}^4)[(m-1)n + n - 2] + 2\beta_{2,0}^2\sigma_{1,0}^2\sigma_{2,0}^2(m-1)n}$$

and

$$\hat{\kappa}_1 = \sum_{j=1}^{m}(\mathbf{Y}_{1,(j)} - \overline{\mathbf{Y}}_1)'(\mathbf{Y}_{1,(j)} - \overline{\mathbf{Y}}_1) + m(\overline{\mathbf{Y}}_1 - \hat{\boldsymbol{\mu}})'(\overline{\mathbf{Y}}_1 - \hat{\boldsymbol{\mu}}),$$
$$\hat{\kappa}_2 = \sum_{j=1}^{m}(\mathbf{Y}_{2,(j)} - \overline{\mathbf{Y}}_2)'(\mathbf{Y}_{2,(j)} - \overline{\mathbf{Y}}_2) + m(\overline{\mathbf{Y}}_2 - \hat{\boldsymbol{\mu}})'(\overline{\mathbf{Y}}_2 - \hat{\boldsymbol{\mu}}).$$

Proof Since $\mathbf{V}_1 = \begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{pmatrix}$ and $\mathbf{V}_2 = \begin{pmatrix} \mathbf{0}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{I} \end{pmatrix}$, are linearly independent, the matrix $\mathbf{S}_{\Sigma_0^{-1}}$ is regular and since both matrices $\mathbf{S}_{\Sigma_0^{-1}}$ and $\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}$ are at least p.s.d., the criterion matrix

$$(m-1)\mathbf{S}_{\Sigma_0^{-1}} + \mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}$$

is p.d.

The matrix $\mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}$ is of the form $\mathbf{hh}'$, where

$$\mathbf{h} = \frac{\sqrt{n-2}}{\beta_{2,0}^2\sigma_{1,0}^2 + \sigma_{2,0}^2}\begin{pmatrix} \beta_{2,0}^2 \\ 1 \end{pmatrix};$$

thus the relationship

$$[(m-1)\mathbf{S}_{\Sigma_0^{-1}} + \mathbf{hh}']^{-1} = \frac{1}{m-1}\mathbf{S}_{\Sigma_0^{-1}}^{-1} - \frac{1}{(m-1)}\frac{\mathbf{S}_{\Sigma_0^{-1}}^{-1}\mathbf{hh}'\mathbf{S}_{\Sigma_0^{-1}}^{-1}}{m-1+\mathbf{h}'\mathbf{S}_{\Sigma_0^{-1}}^{-1}\mathbf{h}}$$

enables us to express the matrix $[(m-1)\mathbf{S}_{\Sigma_0^{-1}} + \mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}]^{-1}$ in our case in the form

$$[(m-1)\mathbf{S}_{\Sigma_0^{-1}} + \mathbf{S}_{(M_{K_1}\Sigma_0 M_{K_1})^+}]^{-1} = \frac{1}{(m-1)n}\begin{pmatrix} \sigma_{1,0}^4, & 0 \\ 0, & \sigma_{2,0}^4 \end{pmatrix} -$$
$$- \frac{1}{(m-1)n}c(m,n,\beta_{2,0}^2,\sigma_{1,0}^2,\sigma_{2,0}^2)\begin{pmatrix} \sigma_{1,0}^4\beta_{2,0}^4, & \sigma_{1,0}^4\beta_{2,0}^2 \\ \sigma_{2,0}^4\beta_{2,0}^2, & \sigma_{2,0}^4 \end{pmatrix}\begin{pmatrix} \sigma_{1,0}^4, & 0 \\ 0, & \sigma_{2,0}^4 \end{pmatrix}.$$

Further Theorem 3.3 (ii) must be used; with respect to it we obtain

$$(\overline{\mathbf{Y}} - \gamma_{1,0})'(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+\mathbf{V}_i(\mathbf{M}_{K_1}\boldsymbol{\Sigma}_0\mathbf{M}_{K_1})^+(\overline{\mathbf{Y}} - \gamma_{1,0}) \to$$
$$\to \begin{pmatrix} \mathbf{Y}_1 - \hat{\boldsymbol{\mu}} \\ \mathbf{Y}_2 - \hat{\boldsymbol{\nu}} \end{pmatrix}'\begin{pmatrix} \sigma_{1,0}^{-2}\mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \sigma_{2,0}^{-2}\mathbf{I} \end{pmatrix}\mathbf{V}_i\begin{pmatrix} \sigma_{1,0}^{-2}\mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \sigma_{2,0}^{-2}\mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{Y}_1 - \hat{\boldsymbol{\mu}} \\ \mathbf{Y}_2 - \hat{\boldsymbol{\nu}} \end{pmatrix}$$

and obviously

$$\mathrm{Tr}[\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_1\boldsymbol{\Sigma}_0^{-1}\sum_{j=1}^{m}(\mathbf{Y}_{(j)} - \overline{\mathbf{Y}})(\mathbf{Y}_{(j)} - \overline{\mathbf{Y}})'] \to \frac{1}{\sigma_{1,0}^4}\sum_{j=1}^{m}(\mathbf{Y}_{1,(j)} - \overline{\mathbf{Y}}_1)'(\mathbf{Y}_{1,(j)} - \overline{\mathbf{Y}}_1)$$
$$\mathrm{Tr}[\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_2\boldsymbol{\Sigma}_0^{-1}\sum_{j=1}^{m}(\mathbf{Y}_{(j)} - \overline{\mathbf{Y}})(\mathbf{Y}_{(j)} - \overline{\mathbf{Y}})'] \to \frac{1}{\sigma_{2,0}^4}\sum_{j=1}^{m}(\mathbf{Y}_{2,(j)} - \overline{\mathbf{Y}}_2)'(\mathbf{Y}_{2,(j)} - \overline{\mathbf{Y}}_2).$$

Now it is obvious how to finish the proof. □

4 Conclusion remarks

If it is necessary to characterize the random variable ξ given by the expression $(\eta - \hat{\beta}_1)/\hat{\beta}_2$, where η is a registration of the value ν given by the second measurement device and $\hat{\beta}_1$ and $\hat{\beta}_2$ are given by Corollary 2.3, then it is necessary to determine the bias

$$b = E(\xi) - \frac{\nu - \beta_1}{\beta_2} = E(\xi) - f(\nu, \beta_1, \beta_2)$$

and the variance $\mathrm{Var}(\xi)$.

Here the formulae given in [1] can be used. Since η is stochastically independent of $\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$ we obtain an approximate value of b in the form [1], Corollary 3.5 (the value $v_{i,j}$ cf. Corollary 2.3)

$$\begin{aligned} b &= \frac{1}{2}\mathrm{Tr}\left[\frac{\partial^2 f(\nu,\beta_1,\beta_2)}{\partial\begin{pmatrix}\nu\\ \beta_1\\ \beta_2\end{pmatrix}\partial(\nu,\beta_1,\beta_2)}\begin{pmatrix}\sigma_2^2, & 0, & 0\\ 0, & v_{1,1}, & v_{1,2}\\ 0, & v_{2,1}, & v_{2,2}\end{pmatrix}\right] + \dots \\ &= \frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} 0, & 0, & -1/\beta_2^2\\ 0, & 0, & 1/\beta_2^2\\ -1/\beta_2^2, & 1/\beta_2^2, & 2\frac{\nu-\beta_1}{\beta_2^3}\end{pmatrix}\begin{pmatrix}\sigma_2^2, & 0, & 0\\ 0, & v_{1,1}, & v_{1,2}\\ 0, & v_{2,1}, & v_{2,2}\end{pmatrix}\right] + \dots \\ &= \frac{v_{1,2}}{\beta_2^2} + \frac{\nu-\beta_1}{\beta_2^3}v_{2,2} + \dots . \end{aligned}$$

Further (in the case of normally distributed errors, cf. Corollary 3.6 in [1]) an approximate value of the variance is

$$\begin{aligned} &\mathrm{Var}\left(\frac{\eta - \hat{\beta}_1}{\hat{\beta}_2}\right) = \\ &= \frac{\partial f(\nu,\beta_1,\beta_2)}{\partial(\nu,\beta_1,\beta_2)}\begin{pmatrix}\sigma_2^2, & 0, & 0\\ 0, & v_{1,1}, & v_{1,2}\\ 0, & v_{2,1}, & v_{2,2}\end{pmatrix}\frac{\partial f(\nu,\beta_1,\beta_2)}{\partial\begin{pmatrix}\nu\\ \beta_1\\ \beta_2\end{pmatrix}} \\ &\quad + \frac{1}{2}\mathrm{Tr}\left\{\left[\frac{\partial^2 f(\nu,\beta_1,\beta_2)}{\partial\begin{pmatrix}\nu\\ \beta_1\\ \beta_2\end{pmatrix}\partial(\nu,\beta_1,\beta_2)}\begin{pmatrix}\sigma_2^2, & 0, & 0\\ 0, & v_{1,1}, & v_{1,2}\\ 0, & v_{2,1}, & v_{2,2}\end{pmatrix}\right]^2\right\} + \dots \\ &= \frac{\sigma_2^2}{\beta_2^2} + v_{1,1}\frac{1}{\beta_2^2} + 2v_{1,2}\frac{\nu-\beta_1}{\beta_2^3} + v_{2,2}\frac{(\nu-\beta_1)^2}{\beta_2^4} \end{aligned}$$

$$+\frac{\sigma_2^2}{\beta_2^4}v_{2,2}+\frac{v_{1,1}v_{2,2}+v_{1,2}^2}{\beta_2^4}+4\frac{\nu-\beta_1}{\beta_2^5}v_{1,2}v_{2,2}+2\frac{(\nu-\beta_1)^2}{\beta_2^6}v_{2,2}^2+\dots.$$

The given formula reflects reality adequately if the distribution of the errors can be approximated by the normal distribution however the support of the probability measure of the actual errors is included into the domain of convergence of the Taylor series considered. By a simulation it was verified that in the case $\frac{\sqrt{v_{2,2}}}{\beta_2}<0.1$ the given approximation is sufficient for practical purposes (in more detail cf. [1]).

In the practice the polynomial

$$\nu_i=\beta_1+\mu_i\beta_2+\mu_i^2\beta_3\dots+\mu_i^{k-1}\beta_k$$

$$i=1,\dots,n,\quad k<n,\quad \text{and}\quad \mu_i\neq\mu_j\quad \text{for more than}\quad k\quad \text{couples}$$

is frequently used.

If there exists a special reason to use a nonlinear (in parameters) calibration curve, then the procedures demonstrated in preceding sections can also be used.

Let, e.g. the calibration curve be of the form

$$\nu=f(\mu,\beta_1,\beta_2).$$

Then the linearized model can be written in the form

$$\begin{pmatrix}\mathbf{Y}_1-\boldsymbol{\mu}_0\\ \mathbf{Y}_2\end{pmatrix}\sim\left[\begin{pmatrix}\delta\boldsymbol{\mu}\\ \boldsymbol{\nu}\end{pmatrix},\begin{pmatrix}\boldsymbol{\Sigma}_{1,1}, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_{2,2}\end{pmatrix}\right],$$

$$\begin{pmatrix}f(\mu_{0,1},\beta_{0,1},\beta_{0,2})\\ \vdots\\ f(\mu_{0,n},\beta_{0,1},\beta_{0,2})\end{pmatrix}+(\mathbf{A},-\mathbf{I})\begin{pmatrix}\delta\boldsymbol{\mu}\\ \boldsymbol{\nu}\end{pmatrix}+$$

$$+\left.\begin{pmatrix}\partial f(\mu_{0,1},\beta_1,\beta_2)/\partial(\beta_1,\beta_2)\\ \vdots\\ \partial f(\mu_{0,n},\beta_1,\beta_2)/\partial(\beta_1,\beta_2)\end{pmatrix}\right|_{\beta_1=\beta_{0,1};\beta_2=\beta_{0,2}}\begin{pmatrix}\delta\beta_1\\ \delta\beta_2\end{pmatrix}=\mathbf{0},$$

$$\begin{pmatrix}\delta\beta_1\\ \delta\beta_2\end{pmatrix}=\begin{pmatrix}\beta_1-\beta_{0,1}\\ \beta_2-\beta_{0,2}\end{pmatrix}$$

where

$$\mathbf{A}=\mathrm{Diag}(\partial f(\mu_1,\beta_{0,1},\beta_{0,2})/\partial\mu_1|_{\mu_1=\mu_{0,1}},\dots,\partial f(\mu_n,\beta_{0,1},\beta_{0,2})/\partial\mu_n|_{\mu_n=\mu_{0,n}})$$

and now we can procede analogously as it was already demonstrated.

References

[1] Kubáček, L.: *Nonlinear error propagation law.* Appl. of Math. **41** (1996), 329–345.

[2] Kubáčková, L.: *Foundations of Experimental Data Analysis.* CRC Press, Boca Raton–Ann Arbor–London–Tokyo 1992.

[3] Rao, C. R., Mitra, K. S.: *Generalized Inverse of Matrices and Its Application.* J. Wiley, New York–London–Sydney–Toronto, 1971.

[4] Rao, C. R., Kleffe, J.: *Estimation of Variance Components and Applications.* North-Holland, Amsterdam–Oxford–New York–Tokyo 1988.