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Quartic and Biquartic Interpolatory Splines on Simple Grid *

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Abstract

Continuity conditions for quartic splines as recurrence relations between function values and first derivatives of the spline at the knots are used in the algorithms for computing its local parameters. Proper local parameters of the biquartic splines on the rectangular mesh are investigated for the use of tensor product technique. The appropriate boundary conditions for interpolatory biquartic splines are given and the algorithm for computing needed local parameters is described.

Key words: Splines, quartic and biquartic interpolatory splines.

1991 Mathematics Subject Classification: 41A15, 65D05

1 Quartic splines

Let us have the set of simple spline knots

$$(\triangle x): \qquad x_0 < x_1 < \ldots < x_n < x_{n+1}$$

on the real axis with stepsizes $h_i = x_{i+1} - x_i$. Denote $S_4(\triangle x)$ the linear space of quartic polynomial splines with the defect one $s_4(x) = s(x) \in \mathbb{C}^3$ on the

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knotset (Δx) with $\dim S_4(\Delta x) = n+5$ (see [2], [4]). When we want to use information placed at knots of a spline only, such a quartic spline can be uniquely determined by
— *conditions of interpolation* $s(x_i) = s_i,\ i = 0(1)n+1$ (CI)
— *some boundary conditions; the simplest case is to prescribe*
$m_0 = s'(x_0),\ m_{n+1} = s'(x_{n+1}),\ M_k = s''(x_k),\ k \in \{0, \dots, n+1\}.$ (BC)
As in the similar quadratic case on the knotset (Δx) some unsymmetry in the local representation of such interpolatory spline and boundary conditions is unavoidable (see [2]). Some another types of boundary conditions can be considered too (see [3]).

1.1 Local representation

The Taylor's representation of the spline $s(x)$ is often used

$$s(x) = s_i + m_i(x-x_i) + \tfrac{1}{2}M_i(x-x_i)^2 + \tfrac{1}{6}T_i(x-x_i)^3 + \tfrac{1}{24}Q_i(x-x_i)^4 \quad (1)$$

for $x \in \langle x_i, x_{i+1}\rangle$ with the notation $s_i = s(x_i),\ m_i = s'(x_i)$,

$$M_i = s''(x_i),\ T_i = s'''(x_i),\ Q_i = s^{(4)}(x_i) = (T_{i+1} - T_i)/h_i, \quad i = 0(1)n. \quad (2)$$

We will use the simplest pieces of information given in the knots of the spline for its local representation—the function values and the values of the first and second derivatives. For $x \in \langle x_i, x_{i+1}\rangle$, we can write then a spline $s(x) \in S_4(\Delta x)$ in the local representation

$$s(x) = \varphi_0(u)s_i + \varphi_1(u)s_{i+1} + h_i[\varphi_0^1(u)m_i + \varphi_1^1(u)m_{i+1}] + \tfrac{1}{2}h_i^2\varphi_0^2(u)M_i \quad (LR)$$

with basis functions

$$\begin{aligned} &\varphi_0(u) = (u-1)^2(1+2u+3u^2), && \varphi_1(u) = u^3(4-3u),\\ &\varphi_0^1(u) = u(u-1)^2(1+2u), && \varphi_1^1(u) = u^3(u-1),\\ &\varphi_0^2(u) = u^2(1-u)^2. \end{aligned} \quad (BF)$$

1.2 Continuity conditions

The quartic pieces of $s(x)$ on $\langle x_i, x_{i+1}\rangle$ are connected together with the $\mathbb{C}^3$ continuity. These *continuity conditions* (CC) we can express in various local parameters of the spline. The most simple way is to use the Taylor's parameters (2)—we obtain the system of conditions

$$\begin{aligned} s_j + h_jm_j + \tfrac{1}{2}h_j^2M_j + \tfrac{1}{6}h_j^3T_j + \tfrac{1}{24}h_j^4Q_j &= s_{j+1}\\ m_j + h_jM_j + \tfrac{1}{2}h_j^2T_j + \tfrac{1}{6}h_j^3Q_j &= m_{j+1}\\ M_j + h_jT_j + \tfrac{1}{2}h_j^2Q_j &= M_{j+1}\\ T_j + h_jQ_j &= T_{j+1}, \qquad j = 1(1)n-1. \end{aligned} \quad (CC)$$

Computing the local parameters (2) directly from this simple form of (CC) leads to some large block systems of linear equations (see e.g.[2]). In the local representation (LR) the continuity of $s(x), s'(x)$ is implicitly included. The continuity of the second and third derivatives at knots can be expressed as

$$\frac{1}{2}(m_i + m_{i+1}) + \frac{1}{12}h_i(M_i - M_{i+1}) = (s_{i+1} - s_i)/h_i, \tag{3}$$

$$5h_i p_i^3 m_i + 3(h_i p_i^3 + h_{i+1})m_{i+1} + h_{i+1}m_{i+2} + h_i^2 p_i^3 M_i + h_{i+1}^2 M_{i+1} =$$
$$= -8p_i^3 s_i + (8p_i^3 - 4)s_{i+1} + 4s_{i+2}, \quad p_i = h_{i+1}/h_i, \quad i = 0(1)n-1.$$

Given the boundary values m_0, m_{n+1}, M_k we can all remaining $2n$ local parameters needed in (LR) compute from the system of $2n$ equations with the more simple block structure. Another relations follow from the fact, that the derivatives $s^{(j)}(x)$ of $s(x)$ are splines from $S_{4-j}(\Delta x)$ (see [3]—we can obtain them also as consequences of some subset of relations (CC)). In case of the equidistant mesh, the recursions between parameters T_j, s_j obtained by divided differences technique are mentioned yet in [1]. Using symbolic computing means (e.g. MATHEMATICA), we can choose some proper subsets of (CC) for elimination of chosen subset of parameters to obtain recurrence relations between remaining local parameters of $s(x)$ (see [3] for more details). So we have obtained the relation (which follows also from known Hermite interpolation formula, hidden yet in (3))

$$M_1 = (6/h_0)(m_0 + m_1) + M_0 + (12/h_0^2)(s_0 - s_1). \tag{4}$$

The following recurrence relation

$$a_i M_{i-1} + 2M_i + c_i M_{i+1} = f_i \quad i = 1(1)n \tag{5}$$

$$a_i = h_i/(h_{i-1} + h_i), \ c_i = 1 - a_i, \ f_i = 3[c_i(m_{i+1} - m_i)/h_i + a_i(m_i - m_{i-1})/h_i]$$

is the consequence of the fact that $s'(x) \in S_3(\Delta x)$ and the corresponding (CC) for the cubic splines on the knotset (Δx) (see [3]). We can compute then

$M_i, \ i = 2(1)n+1$ by recursion using (4), (5) from know values

$\{s_i, m_i, i = 0(1)n+1\}$ and some given $M_k, \ k \in \{0, \ldots, n+1\}$.

The values $\{m_i, i = 1(1)n\}$ we can compute from continuity conditions expressed in terms of local parameters s_i, m_i by elimination of another local parameters from some proper subset of (CC) (or by elimination of M_i from the first equation in (3) and substitution into the second equation in (3)) as

$$p_0(6 + 5p_0)m_0 + 3(1 + p_0)^2 m_1 + m_2 = \tag{6}$$
$$= (4/h_1)[-p_0^2(2p_0 + 3)s_0 + (2p_0^3 + 3p_0^2 - 1)s_1 + s_2] - h_1 M_0(1 + p_0),$$

$$p_{i-1}^2 p_i m_{i-1} + \overline{a_i} p_i[(1 + p_{i-1})(3 + 2p_i) + p_{i-1}p_i]m_i +$$
$$+[1 + (2 + 3p_{i-1})(1 + p_i)]m_{i+1} + \overline{b_i} m_{i+2} =$$
$$= \frac{4}{h_i}\left\{-p_{i-1}^3 p_i s_{i-1} - p_i[2\overline{b_i} + \overline{c_i} - p_{i-1}^3]s_i + p_{i-1}\left[\overline{b_i}\left(2 - \frac{1}{p_i^2}\right) + \overline{c_i}\right]s_{i+1} + \frac{1}{p_i}\overline{b_i}s_{i+2}\right\},$$

$$\overline{a_i} = \frac{1 + p_{i-1}}{1 + p_i}, \ \overline{b_i} = \frac{1 + p_{i-1}p_i}{1 + p_i}, \ \overline{c_i} = \frac{3p_{i-1} + p_i}{1 + p_i}, \ p_i = \frac{h_{i+1}}{h_i}; \quad i = 1(1)n-1.$$

Given m_0, m_{n+1}, M_0, the system (6) consist of n linear relations for n unknown parameters m_i, $i = 1(1)n$. The matrix of that system is nonsymmetric, with four nonzero diagonal band.

In the equidistant case $h_i = h$ we obtain the following system

$$\frac{1}{24}(12m_1 + m_2) = \frac{1}{24h}(-20s_0 + 16s_1 + 4s_2) - \frac{h}{24}M_0 - \frac{11}{24}m_0$$

$$\frac{1}{24}(11m_1 + 11m_2 + m_3) = \frac{1}{6h}(-s_0 - 3s_1 + 3s_2 + s_3) - \frac{1}{24}m_0$$

$$\frac{1}{24}(m_{i-1} + 11m_i + 11m_{i+1} + m_{i+2}) = \frac{1}{6h}(-s_{i-1} - 3s_i + 3s_{i+1} + s_{i+2}) \ ,$$

$$i = 2(1)n-2 \ , \tag{7}$$

$$\frac{1}{24}(m_{n-2} + 11m_{n-1} + 11m_n) = \frac{1}{6h}(-s_{n-2} - 3s_{n-1} + 3s_n + s_{n+1}) - \frac{1}{24}m_{n+1}.$$

1.3 Algorithms for computing 1D local parameters

Let us summarize the results of the Sections 1.1 and 1.2 into two algorithms for computing the local parameters of the one-dimensional quartic spline under boundary conditions (BC).

Algorithm (m,M)

Given the data $\{(x_i, s_i),\ i = 0(1)n+1\}$ *on the knotset* (Δx) *and boundary values* $m_0,\ m_{n+1}, M_k$

1. *compute* h_i, p_i *for given knotset* (Δx)*;*
2. *compute the components of the matrix and of the right-hand side of the system (3);*
3. *solve the block system of equations (3) for the parameters (m,M);*
4. *use the computed local parameters {s,m,M} in the local representation (LR) for the full description of the spline s(x) (e.g. graphic visualisation, applications).*

Algorithm (m)

Given the data $\{(x_i, s_i),\ i = 0(1)n+1\};\ m_0, m_{n+1}, M_0$

1. *compute* h_i, $p_i \in (6)$ *(in the general case);*
2. *compute components of the matrix and right hand side in (6) or (7);*
3. *solve the system (6) or (7) for* m_i*;*
4. *compute* M_1 *from (4) and* M_i *recursively according to (5);*
5. *use the computed local parameters in* (LR) *for computations with s(x).*

2 Biquartic splines

We use the tensor product technique now for construction of biquartic spline on the rectangle D with the grid (Δ):

$$D = \langle x_0, x_{n+1}\rangle \times \langle y_0, y_{m+1}\rangle, \quad (\Delta) = (\Delta x) \times (\Delta y), \qquad (\Delta)$$

$$(\Delta y) = y_0 < y_1 < \ldots < y_m < y_{m+1}, \quad l_j = y_{j+1} - y_j .$$

Let us denote further

$D_{ij} = \langle x_i, x_{i+1}\rangle \times \langle y_j, y_{j+1}\rangle$, $i = 0(1)n$, $j = 0(1)m$ (the local rectangles),

$s(x,y) = \sum_{i,j=0}^{4} a_{ij} x^i y^j$—the biquartic spline $s \in \mathrm{C}^{3,3}(D)$,

$S_{44}(\Delta)$ the linear space of biquartic splines on (Δ),

$s^{kl}(x,y) = \frac{\partial^{k+l}}{\partial x^k \partial y^l} s(x,y) \quad k,l = 0,1,2,\ldots$ the derivatives of $s(x,y)$.

To apply the tensor product technique to the construction of biquartic spline $s(x,y)$ on D, we have first to consider some appropriate local parameters of $s(x,y)$ on the subrectangle D_{ij}.

2.1 Local parameters

It is preferred usually to choose 25 local parameters of the biquartic spline in the vertices of the subrectangle D_{ij}; we have to do it in a proper way to guarantee

- existence and uniqueness of $s(x,y)$ determined by such a parameters,
- storage economy (a low number of local parameters over the whole grid),
- the possibility of repeated use of the one-dimensional algorithms for the calculation of parameters and of the spline function values for $(x,y) \in D_{ij}$.

Theorem 1 *The following 25 local parameters uniquely determine biquartic spline $s(x,y)$ over $D_{ij} = \langle x_i, x_{i+1}\rangle \times \langle y_j, y_{j+1}\rangle$, $i \in \{0,1,\ldots,n\}$, $j \in \{0,1,\ldots,m\}$*

$$\begin{array}{llll} 1^\circ \textit{ the values of} & s, s^{10}, s^{01}, s^{20}, s^{11}, s^{02}, s^{21}, s^{12}, s^{22} & \textit{at} & (x_i, y_i); \\ 2^\circ & s, s^{10}, s^{01}, s^{11}, s^{02}, s^{12} & & (x_{i+1}, y_j); \\ 3^\circ & s, s^{10}, s^{01}, s^{11}, s^{20}, s^{21} & & (x_i, y_{j+1}); \\ 4^\circ & s, s^{10}, s^{01}, s^{11} & & (x_{i+1}, y_{j+1}) \end{array} \qquad (LP2)$$

(see Fig. 1a).

Proof The matrix of linear relations between coefficients a_{ij} and parameters $(LP2)$ has a nonzero determinant, as can be easily verified. Let us denote by

$$\mathbf{M}_{ij} = \begin{bmatrix} s_{ij} & s_{i+1,j} & s_{ij}^{10} & s_{i+1,j}^{10} & s_{ij}^{20} \\ s_{i,j+1} & s_{i+1,j+1} & s_{i,j+1}^{10} & s_{i+1,j+1}^{10} & s_{i,j+1}^{20} \\ s_{ij}^{01} & s_{i+1,j}^{01} & s_{ij}^{11} & s_{i+1,j}^{11} & s_{ij}^{21} \\ s_{i,j+1}^{01} & s_{i+1,j+1}^{01} & s_{i,j+1}^{11} & s_{i+1,j+1}^{11} & s_{i,j+1}^{21} \\ s_{ij}^{02} & s_{i+1,j}^{02} & s_{ij}^{12} & s_{i+1,j}^{12} & s_{ij}^{22} \end{bmatrix} \qquad (8)$$

the "mapping matrix" of 25 chosen local parameters of $s(x,y)$ on D_{ij}.

Remark 1 We need to store only 9 local parameters $(LP2)$ at every grid point for the full description of $s(x,y)$ on D.

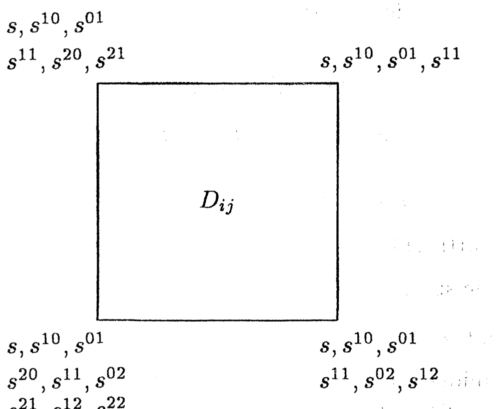

Fig. 1a

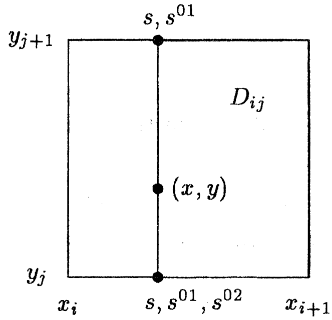

Fig. 1b

2.2 Algorithm for computing function values in 2D

We can use repeatedly the one-dimensional local representation (LP) for computing $s(x,y)$ from two-dimensional local parameters $(LP2)$ in D_{ij}.

Let us denote $u=(x-x_i)/h_i$, $v=(y=y_j)/l_j$ for $(x,y)\in D_{ij}$.

With the one-dimensional basis functions φ_0, φ_1, φ_0^1, φ_1^1, φ_0^2 given in (BF) we can compute stepwise the values

$$1^\circ \quad s(x,y_k)=\varphi_0(u)s_{ik}+\varphi_1(u)s_{i+1,k}+h_i\varphi_0^1(u)s_{ik}^{10}+h_i\varphi_1^1(u)s_{i+1,k}^{10}+\tfrac{1}{2}h_i^2\varphi_0^2(u)s_{ik}^{20};$$

$$2^\circ \quad s^{01}(x,y_k)=\varphi_0(u)s_{ik}^{01}+\varphi_1(u)s_{i+1,k}^{01}+h_i\varphi_0^1(u)s_{ik}^{11}+h_i\varphi_1^1(u)s_{i+1,k}^{11}+ \\ +\tfrac{1}{2}h_i^2\varphi_0^2(u)s_{ik}^{21}; \qquad k=j,j+1;$$

$$3^\circ \quad s^{02}(x,y_j)=\varphi_0(u)s_{ij}^{02}+\varphi_1(u)s_{i+1,j}^{02}+h_i\varphi_0^1(u)s_{ij}^{12}+h_i\varphi_1^1(u)s_{i+1,j}^{12}+ \\ +\tfrac{1}{2}h_i^2\varphi_0^2(u)s_{ij}^{22};$$

$$4^\circ \quad s(x,y)=\varphi_0(v)s(x,y_j)+\varphi_1(v)s(x,y_{j+1})+l_j\varphi_0^1(v)s^{01}(x,y_j)+ \\ +l_j\varphi_1^1s^{01}(x,y_{j+1})+\tfrac{1}{2}l_j^2\varphi_0^2(v)s^{02}(x,y_j). \qquad (FV2DS)$$

Remark 2 The values computed in the steps 1°–4° are shown on Fig. 1b.

Remark 3 Using the mapping matrix $\mathbf{M}_{ij}$, we can write

$$s(x,y)=[\varphi_0(v),\ \varphi_1(v),\ l_j\varphi_0^1(v),\ l_j\varphi_1^1(v),\ \tfrac{1}{2}l_j^2\varphi_0^2(v)]\cdot\mathbf{M}_{ij}\cdot\begin{bmatrix}\varphi_0(u)\\ \varphi_1(u)\\ h_i\varphi_0^1(u)\\ h_i\varphi_1^1(u)\\ \tfrac{1}{2}h_i^2\varphi_0^2(u)\end{bmatrix} \qquad (9)$$

as a symbolic description of the algorithm presented in a matrix form.

2.3 Boundary conditions

According to the construction of $S_{44}(\Delta)$ we have

$$\dim S_{44}(\Delta) = (n+5)(m+5) = (n+2)(m+2) + 3(n+2) + 3(m+2) + 9. \quad (10)$$

The conditions of interpolation

$$s(x_i, y_j) = s_{ij}, \quad i = 0(1)n+1,\ j = 0(1)m+1 \qquad (CI)$$

represent altogether $(n+2)(m+2)$ given parameters.
The remaining conditions for the unique determination of $s(x, y)$ we have to prescribe e.g. as boundary conditions on D in such a way to enable us to compute all local parameters for $s(x, y)$ on every subrectangle D_{ij}.

Theorem 2 *Given the conditions of interpolation (CI) on the grid (Δ) in D, the following boundary conditions given at the boundary gridpoints determine uniquely the biquartic spline $s(x, y)$ on D:*

1°	*the values* s^{10}, s^{20}	*in the*	(x_0, y_j)		
	s^{10}	*boundary*	(x_{n+1}, y_j),	$j = 0(1)m+1$	
2°	s^{01}, s^{02}	*points*	(x_i, y_0)		
	s^{01}		(x_i, y_{m+1}),	$i = 0(1)n+1$	$(BC2)$
3°	$s^{11}, s^{21}, s^{12}, s^{22}$	*in the*	(x_0, y_0)		
	s^{11}, s^{12}	*corners*	(x_{n+1}, y_0)		
	s^{11}, s^{21}		(x_0, y_{m+1})		
	s^{11}		(x_{n+1}, y_{m+1})		

(see Fig 2. for the location of boundary values prescribed).

Proof will be given in the next section as an algorithm for the computing all local parameters on D_{ij} from (CI) and $(BC2)$.

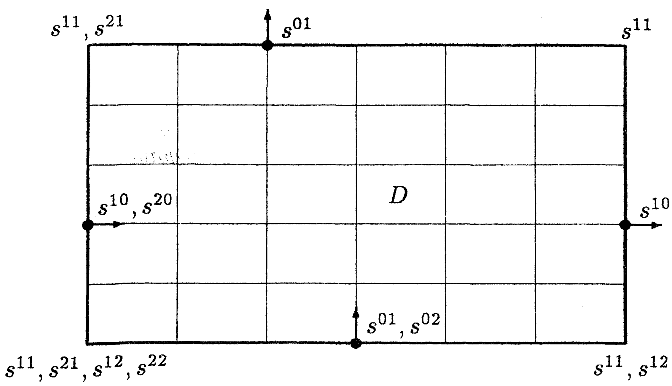

Fig. 2

2.4 Algorithm for computing 2D local parameters

We will repeatedly use the one-dimensional **Algorithm (m)** in x and y directions applied to $s^{kl}(x,y)$ on grid lines for computing all local parameters of $s(x,y)$ in every subrectangle D_{ij} from given (CI) and $(BC2)$. We split the whole algorithm into two stages.

2.4.1 Computing local parameters on the boundary lines (BLP)

	on lines	function values	boundary values	computed local parameters
1°	$x = x_0, x_{n+1}$	s	$s^{01}, s^{02} \mid s^{01}$	s^{01}, s^{02}
	$y = y_0, y_{m+1}$	s	$s^{10}, s^{20} \mid s^{10}$	s^{10}, s^{20}
2°	$x = x_0, x_{n+1}$	s^{10}	$s^{11}, s^{12} \mid s^{11}$	s^{11}, s^{12}
	$y = y_0, y_{m+1}$	s^{01}	$s^{11}, s^{21} \mid s^{11}$	s^{11}, s^{21}
3°	$x = x_0, x_{n+1}$	s^{20}	$s^{21}, s^{22} \mid s^{21}$	s^{21}, s^{22}
	$y = y_0, y_{m+1}$	s^{02}	$s^{12}, s^{22} \mid s^{12}$	s^{12}, s^{22}

2.4.2 Computing local parameters inside the rectangle (ILP)

When we apply the **Algorithm (m)** now to the horizontal and vertical inner lines of our mesh grid, we can compute

	on lines	function values	boundary values	computed parameters	
1°	$y = y_j$	s	$s^{10}, s^{20} \mid s^{10}$	s^{10}, s^{20}	
	$x = x_i$	s	$s^{01}, s^{02} \mid s^{01}$	s^{01}, s^{02}	
2°	$y = y_j$	s^{01}	$s^{11}, s^{21} \mid s^{11}$	s^{11}, s^{21}	
	$x = x_i$	s^{10}	$s^{11}, s^{12} \mid s^{11}$	$(s^{11}), s^{12}$	$\star$
3°	$y = y_j$	s^{02}	$s^{12}, s^{22} \mid s^{12}$	$(s^{12}), s^{22}$	$\star$

($\star$ —only recurrences for the second derivate in Algorithm(m) can be used, because the first derivatives are known yet from foregoing step.)

Together with the known values s we have thus (uniquely) computed all local parameters in each subrectangle D_{ij}.

Remark 4 We can obtain variants of the Algorithm (ILP) by changing the directions in steps 1°–3° in 2.4.2.

Remark 5 The one-dimensional **Algorithm (m,M)** can be used in formally a quite similar way.

3 Examples

Example 1 Let us have the function

$$f(x) = 3x^2 \cdot e^{-x}.$$

On the Fig 3.a we can see plots of $f(x)$ and $s_4(x)$ on the knotset $(\Delta x) = \{x_i = i,\ i = 0(1)10\}$ and with (BC) computed from the function $f(x)$. On the Fig 3.b the plot of $s_4(x)$ is given with m_0, m_{n+1}, M_0 approximated from given data s_i as $m_0 = (s_1 - s_0)/h,\ m_{n+1} = (s_{n+1} - s_n)/h,\ M_0 = (s_0 - 2s_1 + s_2)/h^2$.

On the Fig 3.c we see $s_4(x)$ for the knot set $(\Delta x) = \{x_i = 0.5 \cdot i,\ i = 0(1)20\}$ and (BC) approximated as in 1.b.

On the Fig 3.d the foregoing knot set is used and (BC) are computed exactly from the formula for $f'(x)$.

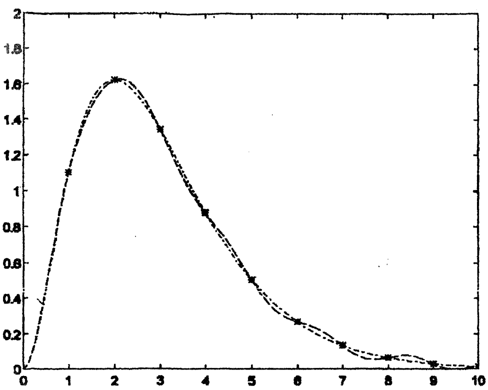

Fig. 3a

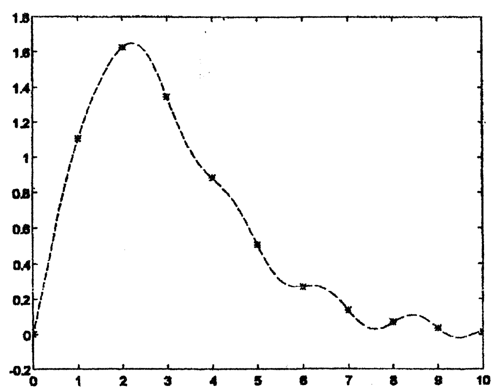

Fig. 3b

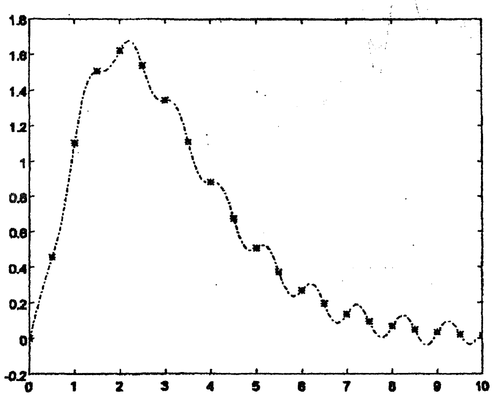

Fig. 3c

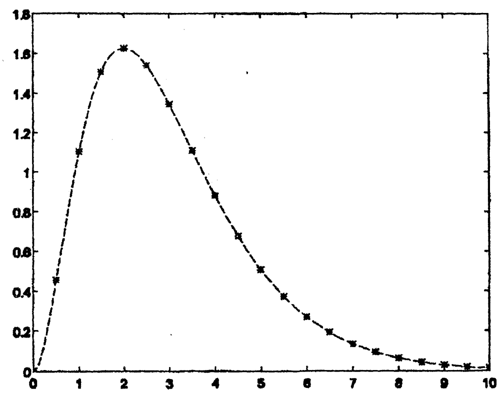

Fig. 3d

Example 2 Let us have the knot set $\{x_i = i,\ i = 0(1)20\}$ with the prescribed function values

$$s = [s_i] = [15, 11, 3, 5, 0, -2, -7, -1, 6, 10, 12, 16, 19, 17, 13, 12, 8, 6, 4, 1, 0].$$

The unstable error propagation is demonstrated on the following figures:

Fig 4.a shows $s_4(x)$ with m_0, m_{n+1}, M_0 approximated from the values s_i,

Fig 4.b—$s_4(x)$ corresponding to small changes in s_i; m_i are computed from s_i on the boundary and $M_0 = 0$. The changes of M_0 seemed to have a small influence on the plots.

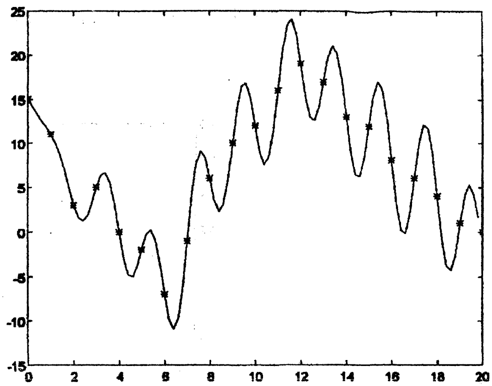

Fig. 4a

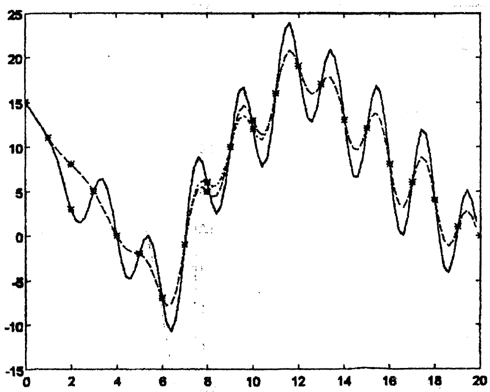

Fig. 4b

Example 3 *Shape preserving properties*

On the Fig 5.a we can see that for the monotone data

$$s = [1, 2, 5, 6, 7, 12, 15, 22, 24, 25, 35, 36, 37, 45, 47, 48, 55, 56, 58, 59, 60]$$

the spline $S_4(x)$ needn't be monotone.

On the Fig 5.b the interpolating spline $S_4(x)$ preserves convexity of the data

$$\begin{aligned} s \;=\; & [0, 7, 13.9, 20.5, 26.9, 32.9, 38.4, 43.2, 47, 49.3, 49.6, 48.7, \\ & 46.8, 44, 40.6, 36.8, 32.3, 27, 20.7, 12.4, 0] \end{aligned}$$

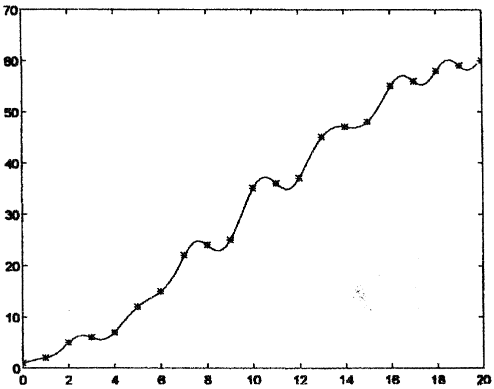

Fig. 5a

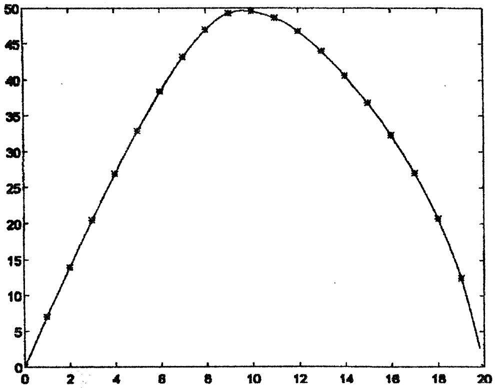

Fig. 5b

Example 4 Let us have the $2D$ data shown in the Fig 6.a ($n = 4, m = 7$). When we compute s^{10}, s^{01} on the boundary from differences and put s^{20}, s^{11}, s^{02}, s^{12}, s^{21}, s^{22} equal to zero, then the corresponding $s_{44}(x, y)$ is plotted on Fig 6.b.

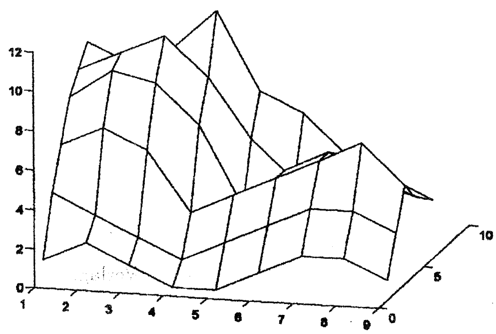

Fig. 6a

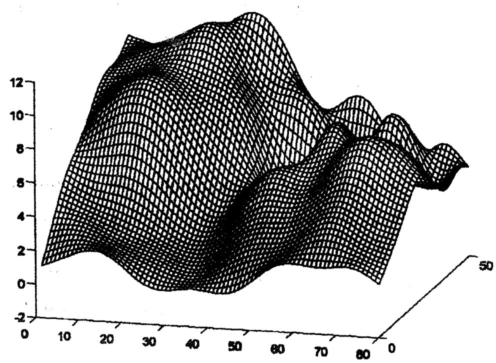

Fig. 6b

Example 5 For the function $f(x,y) = (x^3+3y^3)^{\frac{1}{2}}$ and meshgrid $(\Delta) = \{(x_i,y_j);\ x_i = 0(5)30,\ y_j = 0(3)12\}$ there is no significant difference between plots of $s_{44}(x,y)$ for exact (Fig 7.a) and rounded (Fig 7.b) boundary values of s^{10}, s^{01}.

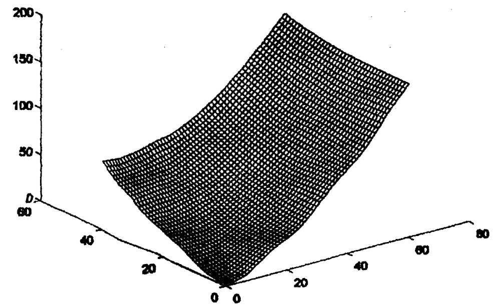

Fig. 7a

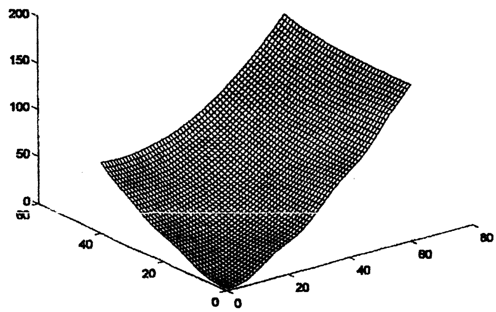

Fig. 7b

Remark 6 The above examples were worked out with MATLAB. The corresponding M-files can be obtained from the author.

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