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Sylvester Theorem for Certain Free Modules *

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Abstract

It was shown in [2] that the inertial law of quadratic forms can be suitably generalized when the vector space is replaced by a free finite-dimensional module over certain linear algebra on $\mathbb{R}$ *(real plural algebra)* introduced in [1]. In the present paper an analogy of the Sylvester theorem is founded.

Key words: Linear algebra, free module, quadratic form, polar basis.

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1 Preface

Definition 1.1 The *real plural algebra of order m* is every linear algebra $\mathbf{A}$ on $\mathbb{R}$ having as a vector space over $\mathbb{R}$ a basis

$$\{1, \eta, \eta^2, \ldots, \eta^{m-1}\}, \quad \text{with } \eta^m = 0.$$

Definition 1.2 The *system of projections* $\mathbf{A} \to \mathbb{R}$ is a system of mappings $p_k : \mathbf{A}$ onto $\mathbb{R}$, defined for $k = 0, \ldots, m-1$ as follows:

$$\forall \beta \in \mathbf{A} \;\; \beta = \sum_{i=0}^{m-1} b_i \eta^i; \quad p_k(\beta) \overset{def}{=} b_k\,.$$

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Now we present a survey of several results from [1] and [2] which we will need in Part II.

Proposition 1.3 $\mathbf{A}$ *is a local ring with the maximal ideal* $\eta\mathbf{A}$. *The ideals* $\eta^j\mathbf{A}$, $1 \le j \le m$, *are all ideals of* $\mathbf{A}$.

Notation 1.4 In the sequel we will always denote by $\mathbf{A}$ the $\mathbb{R}$-algebra from Definition 1.1 and by $\mathbf{M}$ the free finite-dimensional module over the algebra $\mathbf{A}$.

Proposition 1.5 *Let* $\Phi : \mathbf{M}^2 \to \mathbf{A}$ *be a bilinear form. Then there exists exactly one system of bilinear forms* $\Phi_0, \dots, \Phi_{m-1} : \mathbf{M}^2$ *into* $\mathbb{R}$ *such that*

$$\Phi = \sum_{j=0}^{m-1} \Phi_j \eta^j$$

Definition 1.6 The bilinear forms $\Phi_0, \dots, \Phi_{m-1} : \mathbf{M}^2 \to \mathbb{R}$ from Proposition 1.5 will be called *projections of* Φ (Φ_j is the j-th *projection*).

Proposition 1.7 *If* $\Phi_0, \dots \Phi_{m-1} : \mathbf{M}^2 \to \mathbb{R}$ *are bilinear forms then the mapping*

$$\Phi = \sum_{j=0}^{m-1} \Phi_j \eta^j$$

is a bilinear form $\mathbf{M}^2 \to \mathbf{A}$ *if and only if* $\forall \underline{X}, \underline{Y} \in \mathbf{M}$:

(1) $\Phi_0(\eta\underline{X}, \underline{Y}) = 0,$

(2) $\Phi_k(\eta\underline{X}, \underline{Y}) = \Phi_{k-1}(\underline{X}, \underline{Y}), \ 1 \le k \le m-1,$

(3) $\Phi_0(\underline{X}, \eta\underline{Y}) = 0,$

(4) $\Phi_k(\underline{X}, \eta\underline{Y}) = \Phi_{k-1}(\underline{X}, \underline{Y}), \ 1 \le k \le m-1.$

Definition 1.8 A polar basis $\{\underline{U}_1, \dots, \underline{U}_n\}$ of $\mathbf{M}$ with respect to given quadratic form ${}_2\Phi$ is called the *normal polar basis* if for every i, $1 \le i \le n$, there exists k, $0 \le k \le m$, such that

$${}_2\Phi(\underline{U}_i) = \mp\eta^k.$$

Theorem 1.9 *Let a quadratic form* ${}_2\Phi$ *on the* $\mathbf{A}$*-module* $\mathbf{M}$ *be given. Then there exists a normal polar basis of* $\mathbf{M}$ *with respect to* ${}_2\Phi$.

Definition 1.10 Let ${}_2\Phi$ be a quadratic form on $\mathbf{M}$ and let $\mathcal{U} = \{\underline{U}_1, \dots, \underline{U}_n\}$ be its normal polar basis. Putting $\gamma_i = {}_2\Phi(\underline{U}_i)$, $1 \le i \le n$, define a system of sets $\mathcal{I}_k = \{\ 1 \le i \le n;\ \gamma_i = \mp\eta^k\}$, $0 \le k \le m$ and denote $\pi_k = \mathrm{card}(\mathcal{I}_k)$, $0 \le k \le m$.

Then $\mathfrak{Ch}({}_2\Phi, \mathcal{U}) = (\pi_0, \dots, \pi_m)$ is called the *characteristic of the quadratic form* ${}_2\Phi$ *with respect to* $\mathcal{U}$.

Theorem 1.11 *Let a quadratic form ${}_2\Phi$ on* **M** *be given. If $\mathcal{U},\mathcal{V}$ are arbitrary normal polar bases of the form ${}_2\Phi$, then*

$$\mathfrak{Ch}({}_2\Phi,\mathcal{U}) = \mathfrak{Ch}({}_2\Phi,\mathcal{V}).$$

Definition 1.12 Let ${}_2\Phi$ be a quadratic form on **M** and let $\mathcal{U} = \{\underline{U}_1,\ldots,\underline{U}_n\}$ be its normal polar basis. Putting $\gamma_i = {}_2\Phi(\underline{U}_i)$, $1 \le i \le n$, define a system of sets $P_k = \{1 \le i \le n;\ \gamma_i = \eta^k\}$, $N_k = \{1 \le i \le n;\ \gamma_i = -\eta^k\}$, $0 \le k \le m-1$, and denote $\mathfrak{p}_k = \operatorname{card} P_k$, $\mathfrak{n}_k = \operatorname{card} N_k$, $0 \le k \le m-1$.

Then $\mathfrak{S}({}_2\Phi,\mathcal{U}) = (\mathfrak{p}_0,\ldots,\mathfrak{p}_{m-1},\mathfrak{n}_0,\ldots,\mathfrak{n}_{m-1})$ is called *the plural signature of the quadratic form ${}_2\Phi$ with respect to $\mathcal{U}$.*

Theorem 1.13 *Let a quadratic form ${}_2\Phi$ on* **M** *be given. If $\mathcal{U},\mathcal{V}$ are arbitrary normal polar bases of the form ${}_2\Phi$ then $\mathfrak{S}({}_2\Phi,\mathcal{U}) = \mathfrak{S}({}_2\Phi,\mathcal{V})$.*

2 Sylvester theorem for free modules over plural algebras

Notation 2.1 With respect to Theorem 1.11 and Theorem 1.13 the chararacteristic, respectively the plural signature of the quadratic form ${}_2\Phi$ will be denoted only by $\mathfrak{Ch}({}_2\Phi)$ resp. by $\mathfrak{S}({}_2\Phi)$.

Lemma 2.2 *Let $\mathcal{B}$ be a basis of an* **A***-module* **M**. *Then $\mathcal{B}$ forms a basis* mod $\eta\mathbf{M}$ *of* $\mathbf{M}/\eta\mathbf{M}$ *as a vector space over $\mathbb{R}$.*

Proof $\eta\mathbf{A}$ is the maximal ideal of **A** (Proposition 1.3). $\mathbf{M}/\eta\mathbf{M}$ is obviously a real vector space. Let $\mathcal{B} = \{\underline{E}_1,\ldots,\underline{E}_n\}$.

(1) We prove that $\mathcal{B}$ is a set of generators of $\mathbf{M}/\eta\mathbf{M}$ mod $\eta\mathbf{M}$.

Let $\underline{X} \in \mathbf{M}$, $\underline{X} = \sum_{i=1}^n \xi_i \underline{E}_i$, where $\xi_i = \sum_{j=0}^{m-1} x_{ij}\eta^j$, $1 \le i \le n$. Then

$$\underline{X} = \sum_{i=1}^n x_{i0}\underline{E}_i + \eta\left(\sum_{i=1}^n\sum_{j=1}^{m-1} x_{ij}\eta^{j-1}\underline{E}_i\right),$$

the second summand belonging to $\eta\mathbf{M}$.

(2) Now we prove the linear independence of $\mathcal{B}$ mod $\eta\mathbf{M}$ over $\mathbb{R}$.

Let there exist $c_1,\ldots,c_n \in \mathbb{R}$ such that $\sum_{i=1}^n c_i\underline{E}_i \in \eta\mathbf{M}$. Then

$$\eta^{m-1}\sum_{i=1}^n c_i\underline{E}_i = \sum_{i=1}^n \left(\eta^{m-1}c_i\right)\underline{E}_i = \underline{o}$$

and consequently $\eta^{m-1}c_i = 0$ for any $i = 1,\ldots,n$ so that $c_1,\ldots,c_n = 0$.

Notation 2.3 In what follows we will denote by $\mathbb{M}$ the real vector space $\mathbf{M}/\eta\mathbf{M}$ and the coset represented by $\underline{X} \in \mathbf{M}$ we will denote by $[\underline{X}]$.

Lemma 2.4 *Let Φ be a bilinear form $\mathbf{M} \times \mathbf{M} \to \mathbf{A}$ and Φ_0 its 0-th projection. Then the mapping $\mathscr{F} : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$, $([\underline{X}], [\underline{Y}]) \mapsto \Phi_0(\underline{X}, \underline{Y})$ is well-defined and is a bilinear form $\mathbb{M} \times \mathbb{M} \to \mathbb{R}$.*

Proof We will verify the correctness of the definition of $\mathscr{F}$. Let $[\underline{X}] = [\underline{X}']$, so that there is a $\underline{Z} \in \mathbf{M}$ such that $\underline{X} - \underline{X}' = \eta \underline{Z}$. Then $\mathscr{F}([\underline{X}], [\underline{Y}]) - \mathscr{F}([\underline{X}'], [\underline{Y}]) = \Phi_0(\underline{X}, \underline{Y}) - \Phi_0(\underline{X}', \underline{Y}) = \Phi_0(\underline{X} - \underline{X}', \underline{Y}) = \Phi_0(\eta \underline{Z}, \underline{Y}) = 0$ [due to Proposition 1.7] for every $\underline{Y} \in \mathbf{M}$.

Analogously, if $[\underline{Y}] = [\underline{Y}']$ then $\mathscr{F}([\underline{X}], [\underline{Y}]) = \mathscr{F}([\underline{X}], [\underline{Y}'])$. So the definition of $\mathscr{F}$ is correct. Since Φ_0 is a bilinear form, $\mathscr{F}$ is a bilinear form as well.

Lemma 2.5 *Let a quadratic form Φ on $\mathbf{M}$ and simultaneously a basis $\mathcal{B} = \{\underline{E}_1, \ldots, \underline{E}_n\}$ of $\mathbf{M}$ be given. Let $F = (\phi_{ij})$ be the matrix of ${}_2\Phi$ with respect to $\mathcal{B}$. Then $F^* = (p_0(\phi_{ij}))$ is the matrix of the quadratic form ${}_2\mathscr{F} : \mathbb{M} \to \mathbb{R}$[1] with respect to the basis $\mathcal{B}^* = \{[\underline{E}_1], \ldots, [\underline{E}_n]\}$ of $\mathbb{M}$.*

Proof Putting $f_{ijs} = p_s(\phi_{ij})$, $1 \le i,\ j \le n$, for $s = 0, \ldots, m-1$, we obtain for every $[\underline{X}] \in \mathbb{M}$: Let $\underline{X} = \sum_{i=1}^{n} \xi_i \underline{E}_i$ and $\xi_i = \sum_{j=0}^{m-1} x_{ij} \eta^j$, $1 \le i \le n$ $([\underline{X}] = \sum_{i=1}^{n} x_{i0}[\underline{E}_i])$. Then

$$
{}_2\mathscr{F}([\underline{X}]) = \mathscr{F}([\underline{X}], [\underline{X}]) = \Phi_0(\underline{X}, \underline{X}) = p_0(\Phi(\underline{X}, \underline{X})) \quad \text{[see Proposition 1.5.]} =
$$

$$
= p_0\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \phi_{ij}\xi_i\xi_j\right) = p_0\left(\sum_{k+l+s=0}^{m-1}\ \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ijs}x_{ik}x_{jl}\eta^{k+l+s}\right) =
$$

$$
= \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij0}x_{i0}x_{j0} = \sum_{i=1}^{n}\sum_{j=1}^{n} p_0(\phi_{ij})x_{i0}x_{j0}\,.
$$

Theorem 2.6 *Let ${}_2\Phi$ be a quadratic form on $\mathbf{M}$ and $F = (\phi_{ij})$ an arbitrary matrix of ${}_2\Phi$. Then the plural signature $\mathfrak{S}({}_2\Phi)$ is equal to $(n, 0, \ldots, 0)$ if and only if*

$$
p_0\left(\begin{vmatrix} \phi_{11} & \ldots & \phi_{1k} \\ \ldots & \ldots & \ldots \\ \phi_{k1} & \ldots & \phi_{kk} \end{vmatrix}\right) > 0 \quad \textit{for all } k,\ 1 \le k \le n.
$$

Proof Clearly, p_0 is a homomorphism of $\mathbb{R}$-algebra $\mathbf{A}$ into $\mathbb{R}$ and therefore

$$
p_0\left(\begin{vmatrix} \phi_{11} & \ldots & \phi_{1k} \\ \ldots & \ldots & \ldots \\ \phi_{k1} & \ldots & \phi_{kk} \end{vmatrix}\right) = \begin{vmatrix} p_0(\phi_{11}) & \ldots & p_0(\phi_{1k}) \\ \ldots & \ldots & \ldots \\ p_0(\phi_{k1}) & \ldots & p_0(\phi_{kk}) \end{vmatrix}.
$$

Now we must show (with respect to Lemma 2.5) that the form ${}_2\mathscr{F}$ is positive definite just if $\mathfrak{S}({}_2\Phi) = (n, 0, \ldots, 0)$. Let $\{\underline{E}_1, \ldots, \underline{E}_n\}$ be a normal polar basis of Φ. Then[2]

$$
{}_2\mathscr{F}([\underline{X}]) = \Phi_0(\underline{X}, \underline{X}) = \sum_{\substack{j+k+h=0 \\ 0 \le h \le 0,\, i \in P_h}} x_{ij}x_{ik} - \sum_{\substack{j+k+h=0 \\ 0 \le h \le 0,\, i \in N_h}} x_{ij}x_{ik} = \sum_{i \in P_0} x_{i0}^2 - \sum_{i \in N_0} x_{i0}^2, \quad (*)
$$

[1]Determined by the bilinear form $\mathscr{F}$.

[2]This expression of Φ_0 is derived in the proof of III.7 in [2].

for every $\underline{X} = \sum_{i=1}^{n} \xi_i \underline{E}_i$ with $\xi_i = \sum_{j=0}^{m-1} x_{ij}\eta^j$, $1 \le i \le n$ $([\underline{X}] = \sum_{i=1}^{n} x_{i0}[\underline{E}_i])$.

(1) Let $\mathfrak{S}({}_2\Phi) = (n, 0, \ldots, 0)$, i.e. $P_0 = \{1, \ldots, n\}$, $N_0 = \mathcal{I}_1 = \ldots = \mathcal{I}_m = \emptyset$. Then we get from $(*)$: ${}_2\mathscr{F}([\underline{X}]) = \sum_{i=1}^{n} x_{i0}^2$. Thus the form ${}_2\mathscr{F}$ is positive definite.

(2) Let ${}_2\mathscr{F}$ be positive definite. Then $N_0 = \emptyset$ [from $(*)$]. Let us prove that $P_0 = \{1, \ldots, n\}$: Let there exist an m, $1 \le m \le n$; $m \notin P_0$. Then ${}_2\mathscr{F}([\underline{E}_m]) = 0$ [by $(*)$], a contradiction, since the form ${}_2\mathscr{F}$ is positive definite. Therefore $P_0 = \{1, \ldots, n\}$ and consequently $\mathcal{I}_1 = \ldots = \mathcal{I}_m = \emptyset$. Thus $\mathfrak{S}({}_2\Phi) = (n, 0, \ldots, 0)$.

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