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Even Order Differential Equations with Measures as Coefficients

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Abstract

The note deals with differential equations with Borel measures as coefficients. The problem of existence and uniqueness of solutions is discussed. The Ritz–Galerkin method is used for determining approximate solutions.

Key words: Differential equation, Borel measure, Ritz–Galerkin method.

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1. Let us consider the boundary value problem

$$\begin{gathered}\sum_{i=1}^{k}(-1)^i a_i u^{(2i)} + \mu_1 u = \mu_2 \\ u^{(i)}(a) = u^{(i)}(b) = 0 \quad \text{for } i = 0, \dots, k-1,\ k \geq 1,\end{gathered} \tag{1}$$

where μ_1 and μ_2 are real Borel measures, $\mu_1 \geq 0$ and $a_i \geq 0$ for $i = 1, \dots, k$.

It was shown in [1] that for $k = 1$ Problem 1 has exactly one solution, which can be approximated by the Ritz–Galerkin method. Here we shall show that Problem 1 can be solved similarly.

Every continuous function u fulfilling the boundary conditions and the differential equation (1) in the distributional sense will be called a solution of Problem 1. This means that

$$\sum_{i=1}^{k}(-1)^i a_i \int_a^b u\varphi^{(2i)} dx + \int_a^b u\varphi d\mu_1(x) = \int_a^b \varphi d\mu_2(x) \quad \text{for } \varphi \in D(a,b), \tag{2}$$

where $D(a,b)$ denotes the Schwartz space of test functions. Therefore $u^{(2k-1)} \in BV(a,b) \subset L^2(a,b)$. Further it implies that $u \in W^{2k-1,2}(a,b)$ ($W^{n,2}(a,b)$, $n = 1,2,\ldots$ is the Sobolev space of functions $u \in L^2(a,b)$ such that the distributional derivative $u^{(n)}$ belongs to $L^2(a,b)$; the space $W^{n,2}(a,b)$ is equipped with the norm $||u|| := (\sum_{i=0}^{n} ||u^{(i)}||_{L^2}^2)^{\frac{1}{2}}$).

Let us define now the space

$$W_0^{n,2}(a,b) := \left\{ u \in W^{n,2}(a,b) : \; u^{(i)}(a) = u^{(i)}(b) = 0 \quad \text{for } i = 1,\ldots,n-1 \right\}.$$

It is easy to show that $W_0^{n,2}(a,b)$ is the closure of $D(a,b)$ in $W^{n,2}(a,b)$ with respect to its norm $||\cdot||$.

An easy computation using integration by parts shows that equation (2) can be rewritten in the form

$$\sum_{i=1}^{k} a_i \int_a^b u^{(i)}\varphi^{(i)}dx + \int_a^b u\varphi d\mu_1(x) = \int_a^b \varphi d\mu_2(x) \quad \text{for } \varphi \in D(a,b). \tag{3}$$

For simplicity of notation we put for $u, \varphi \in W_0^{k,2}(a,b)$

$$\alpha(u,\varphi) := \sum_{i=1}^{k} a_i \int_a^b u^{(i)}\varphi^{(i)}dx + \int_a^b u\varphi d\mu_1(x), \qquad \beta(\varphi) := \int_a^b \varphi d\mu_2(x).$$

Thus the original Problem 1 can be written as

$$\alpha(u,\varphi) = \beta(\varphi) \quad \text{for } \varphi \in W_0^{k,2}(a,b). \tag{4}$$

It results from the discussion given above that if equation (4) has a unique solution $u \in W_0^{k,2}(a,b)$, then u is the unique solution of Problem 1 and $u \in W^{2k-1,2}(a,b)$. Hence we can consider the problem of existence and uniqueness of solutions of equation (4). For this purpose we will define the following two norms in the space $W_0^{k,2}(a,b)$:

$$||u||_\alpha := \sqrt{\alpha(u,u)}$$

$$|||u||| := ||u^{(k)}||_{L^2}.$$

Theorem 1 *The norms* $||\cdot||$, $|||\cdot|||$ *and* $||\cdot||_\alpha$ *are equivalent on* $W_0^{k,2}(a,b)$.

Proof One can show that $|u^{(i)}(x)| \le \sqrt{b-a}||u^{(i+1)}||_{L^2}$ for $x \in (a,b)$, $u \in W_0^{k,2}(a,b)$, $i = 0,\ldots,k-1$. From this we get

$$||u^{(i)}||_{L^2} \le (b-a)^{k-i}|||u||| \tag{5}$$

and

$$\sup_{x\in(a,b)} u^2(x) \le (b-a)||u'||_{L^2}^2 \le (b-a)^{2k-1}|||u|||^2. \tag{6}$$

By (5) we obtain

$$|||u|||^2 \le ||u||^2 = \sum_{i=0}^{k} ||u^{(i)}||_{L^2}^2 \le \sum_{i=0}^{k} (b-a)^{2(k-i)} |||u|||^2 = A|||u|||^2 \tag{7}$$

where $A = \frac{(b-a)^{2k+2}-1}{(b-a)^2-1}$. Therefore the norms $||\cdot||$ and $|||\cdot|||$ are equivalent. Moreover, by virtue of inequalities (6) and (7) there is

$$\begin{aligned} a_k |||u|||^2 &\le ||u||_\alpha^2 = \sum_{i=1}^{k} a_i ||u^{(i)}||_{L^2}^2 + \int_a^b u^2 d\mu_1(x) \le \\ &\le \left(A \max_{i=1,\ldots,k} a_i + (b-a)^{2k-1} \mu_1([a,b]) \right) |||u|||^2. \end{aligned}$$

This finishes the proof. □

In the space $W_0^{k,2}(a,b)$ besides the usual inner product

$$(u,\varphi) := \sum_{i=0}^{k} \left(u^{(i)}, \varphi^{(i)} \right)_{L^2}$$

one can consider two other products, namely $\alpha(u,\varphi)$ and $[u,\varphi] := (u^{(k)}, \varphi^{(k)})_{L^2}$. It follows from Theorem 1 that $W_0^{k,2}(a,b)$ is a Hilbert space with any of these products.

Theorem 2 *Problem 1 has exactly one solution in* $W^{2k-1,2}(a,b)$.

Proof From the previous remarks it follows that it is sufficient to show that equation (4) has exactly one solution in $W_0^{k,2}(a,b)$. β is a continuous linear form on $W_0^{k,2}(a,b)$, $(W_0^{k,2}(a,b), \alpha(\cdot,\cdot))$ is a Hilbert space, therefore the Riesz Representation Theorem of a continuous linear form shows that a solution of (4) exists and is unique. □

2. In this section we shall show how approximate solutions of Problem 1 can be determined in the space $W_0^{k,2}(a,b)$, if a countable complete system of linear independent functions $\{\varphi_n\}_{n=1}^{\infty}$ is given. It will be noted that $\mathrm{cl}(\mathrm{lin}\{\varphi_n;\, n=1,2,\ldots\}) = W_0^{k,2}(a,b)$. 'Lin' denotes the lineal hull and 'cl' the closure of the set $\{\varphi_n;\, n=1,2,\ldots\}$. Let us define the quadratic form

$$F(u) := \frac{1}{2}\alpha(u,u) - \beta(u) \quad \text{for } u \in W_0^{k,2}(a,b).$$

From the Ritz Theorem ([1], p. 21, see also [2]) one can conclude, that u is a solution of Problem 1 if and only if

$$F(u) = \inf_{\varphi \in W_0^{k,2}(a,b)} F(\varphi). \tag{8}$$

Assume that E_n is a subspace of $W_0^{k,2}(a,b)$ spanned by elements $\varphi_1,\ldots,\varphi_n$. Let $u_n := \inf_{\varphi\in E_n} F(\varphi)$. It is known that u_n tends to u with respect to any of the norms considered above. Therefore the problem of determining the approximate solutions of Problem 1 reduces to the possibility of determining functions u_n. The quadratic form F on the space E_n reads as

$$F(\varphi) = G(\lambda_1,\ldots,\lambda_n) =$$
$$= \frac{1}{2}\left[\sum_{i,j=1}^{n}\sum_{s=1}^{k}\lambda_i\lambda_j a_s\left(\varphi_i^{(s)}\varphi_j^{(s)}\right)_{L^2} + \sum_{i,j=1}^{n}\lambda_i\lambda_j\int_a^b \varphi_i\varphi_j\,d\mu_1(x)\right] -$$
$$-\sum_{j=1}^{n}\lambda_j\int_a^b \varphi_j\,d\mu_2(x), \tag{9}$$

where $\varphi = \lambda_1\varphi_1 + \ldots + \lambda_n\varphi_n$. The expression (9) may be rewritten in matrix form

$$G(\Lambda) = \frac{1}{2}\Lambda^T\left(\sum_{s=1}^{k} a_s\Gamma_s + \Delta\right)\Lambda - \Lambda^T P,$$

where

$$\Lambda = [\lambda_1\ldots\lambda_n]^T, \qquad \Gamma_s = \left[\left(\varphi_i^{(s)},\varphi_k^{(s)}\right)_{L^2}\right]_{i,k=1}^{n},$$

$$\Delta = \left[\int_a^b \varphi_i\varphi_k\,d\mu_1(x)\right]_{i,k=1}^{n}, \qquad P = \left[\int_a^b \varphi_1\,d\mu_2(x)\ldots\int_a^b \varphi_n\,d\mu_2(x)\right]^T.$$

It is easy to check that

$$G(\Lambda^*) = \inf_{\Lambda\in\mathbb{R}^n} G(\Lambda)$$

when

$$\left(\sum_{i=1}^{k} a_s\Gamma_s + \Delta\right)\Lambda^* = 0. \tag{10}$$

Equation (10) may be used for determining approximate solutions of Problem 1.

3. To solve Problem 1 we can apply any complete system of linearly independent functions in the space $W_0^{k,2}(a,b)$, but for calculational reasons systems connected with Haar functions in $L^2(a,b)$ are convenient. Let us remind that

$$\xi_n(t) = \begin{cases} 2^{\frac{m}{2}} & \text{for } t\in\left[\frac{2l-2}{2^{m+1}},\frac{2l-1}{2^{m+1}}\right] \\ 2^{-\frac{m}{2}} & \text{for } t\in\left(\frac{2l-1}{2^{m+1}},\frac{2l}{2^{m+1}}\right] \\ 0 & \text{for other } t\in[0,1], \end{cases}$$

where $n = 2^m + l$, $l = 1\ldots 2^m$, $m = 0,1,\ldots$ and $\xi_1 :\equiv 1$ on $[0,1]$ are called Haar functions, they constitute a complete orthonormal system in $L^2(0,1)$ ([3],

p. 132). A complete orthonormal system in $(W_0^{1,2}(0,1), [.,.])$ was constructed in paper [2], which after differentiation gives the Haar functions ξ_n, $n = 2, 3, \ldots$ For various reasons this cannot be repeated for $k > 1$. Otherwise it another construction is possible in the space $W_0^{k,2}(0,1)$. Because of calculational difficulties we will discuss this for the case $k = 2$ only.

To solve this problem we have to find functions in $W_0^{2,2}(0,1)$ which are parts of second order polynomials and are linearly independent. They determine a complete system. After double differentiation they become a linear combination of some Haar functions. Let us begin with the following two functions

$$y_3(t) := \begin{cases} \frac{3t^2}{2} & \text{for } t \in [0, \frac{1}{4}] \\ -\frac{5}{2}\left(t - \frac{4+\sqrt{6}}{10}\right)\left(t - \frac{4-\sqrt{6}}{10}\right) & \text{for } t \in (\frac{1}{4}, \frac{1}{2}] \\ \frac{(t-1)^2}{2} & \text{for } t \in (\frac{1}{2}, 1] \end{cases}$$

and $y_4(t) := -y_3(1-t)$, $t \in [0,1]$.

One can show that $y_\alpha'' = -\xi_2 + 2\sqrt{2}\xi_\alpha$ for $\alpha = 3, 4$. Now, we put for $n = 2^{m+1} + s$, $m = 0, 1, \ldots$, $s = 1, \ldots 2^{m+1}$

$$y_{2^{m+1}+s} := \begin{cases} 2^{-\frac{3m}{2}} y_\alpha \left[2^m \left(t - \frac{4\lfloor\frac{s+1}{2}\rfloor - 4}{2^{m+2}}\right)\right] & \text{for } t \in \left[\frac{4\lfloor\frac{s+1}{2}\rfloor-4}{2^{m+2}}, \frac{4\lfloor\frac{s+1}{2}\rfloor}{2^{m+2}}\right] \\ 0 & \text{for other } t \in [0,1] \end{cases} \tag{11}$$

where $\alpha = 3$ if s is odd and $\alpha = 4$ if s is even. By $\lfloor \cdot \rfloor$ the entire part function is denoted. For $t \in [0,1]$ the following formula holds

$$\begin{aligned} y''_{2^{m+1}+s}(t) &= 2^{\frac{m}{2}} y''_\alpha \left[2^m \left(t - \frac{4\lfloor\frac{s+1}{2}\rfloor - 4}{2^{m+2}}\right)\right] \\ = 2^{\frac{m}{2}} &\left\{-\xi_2 \left[2^m \left(t - \frac{4\lfloor\frac{s+1}{2}\rfloor - 4}{2^{m+2}}\right)\right] + 2\sqrt{2}\xi_\alpha \left[2^m \left(t - \frac{4\lfloor\frac{s+1}{2}\rfloor - 4}{2^{m+2}}\right)\right]\right\} \\ &= -\xi_{2^m + \lfloor\frac{s+1}{2}\rfloor} + 2\sqrt{2}\xi_{2^{m+1}+s}. \end{aligned} \tag{12}$$

Theorem 3 $\mathrm{cl}(\mathrm{lin}\{y_n''; \ n = 1, 2, \ldots\}) = L^2(0,1)$, *where* $y_1'' := \xi_1$ *and* $y_2'' := \xi_2$.

Proof We shall show that for every m, $m = 0, 1, \ldots$ the functions $y_1'', \ldots y''_{2^{m+2}}$ are linearly independent. Let us assume that $\sum_{i=1}^{2^{m+2}} a_i y_i'' = 0$ for some a_i, $i = 1 \ldots 2^{m+2}$. By (12) we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^{2^{m+1}} a_i y_i'' + \sum_{s=1}^{2^{m+1}} a_{2^{m+1}+s} y''_{2^{m+1}+s} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{2^i} a_{2^i+j} y''_{2^i+j} - \sum_{s=1}^{2^{m+1}} a_{2^{m+1}+s}\xi_{2^m+\lfloor\frac{s+1}{2}\rfloor} + 2\sqrt{2}\sum_{s=1}^{2^{m+1}} a_{2^{m+1}+s}\xi_{2^{m+1}+s}. \end{aligned}$$

The above expression is a linear combination of the Haar functions and therefore we have $a_{2^{m+1}+s} = 0$ for $s = 1, \ldots 2^{m+1}$. This implies that $a_{2^m+s} = 0$ for

$s = 1, \ldots 2^m$. Continuing the above consideration we can conclude that $a_i = 0$ for $i = 1, \ldots 2^{m+2}$. By induction it can be shown that for $m = 0, 1, \ldots$, $s = 1, \ldots 2^{m+1}$ we have

$$\xi_{2^{m+1}+s} = \frac{1}{2\sqrt{2}} y''_{2^{m+1}+s} + \sum_{l=0}^{m-1} \left(\frac{1}{2\sqrt{2}}\right)^{l+2} y''_{2^{m-l}+\lfloor \frac{s+2^{l+1}-1}{2^{l+1}} \rfloor} + \left(\frac{1}{2\sqrt{2}}\right)^{m+1} \xi_2. \tag{13}$$

Therefore every Haar function may be represented as a combination of some functions y''_n. Let $f \in L^2(0,1)$. The function f has the Fourier representation

$$f = \sum_{n=1}^{\infty} c_n \xi_n = c_1 \xi_1 + c_2 \xi_2 + \lim_{n \to \infty} \sum_{m=0}^{2^{n+2}} \sum_{s=1}^{2^{m+1}} c_{2^{m+1}+s} \xi_{2^{m+1}+s}. \tag{14}$$

By (13) we obtain

$$f = c_1 \xi_1 + \lim_{n \to \infty} \left[\sum_{m=3}^{2^{n+2}} \tilde{c}^n_m y''_m + A_n \xi_2 \right]. \tag{15}$$

where $\tilde{c}^m_n$ may be determined by formulas (13), (14) and

$$A_n := c_2 + \sum_{m=0}^{2^{n+2}} \frac{1}{(2\sqrt{2})^{m+1}} \sum_{s=1}^{2^{m+1}} c_{2^{m+1}+s}, \quad n = 0, 1, \ldots \tag{16}$$

This completes the proof of the theorem. □

Theorem 4 $\mathrm{cl}(\mathrm{lin}\{y_n;\, n = 3, 4 \ldots\}) = W_0^{2,2}(0,1)$.

Proof In virtue of Theorem 1 the space $W_0^{2,2}(0,1)$ with the norm $|||\cdot|||$ may be considered. Assume that a function $f \in W_0^{2,2}(0,1)$ is given. Then $f'' \in L^2(0,1)$ and the function f'' may be represented by formulas (14)–(16). Clearly $c_1 = \int_0^1 f'' \xi_1 dx = 0$. We shall show that $A := \lim_{n \to \infty} A_n = 0$. It is easy to check that $\int_0^1 \{\int_0^t \xi_{2^n+l}(s) ds\} dt = \frac{1}{2^{\frac{3n}{2}+2}}$ for $n = 0, 1 \ldots$, $l = 1, \ldots 2^n$. By (14) integrating two times we get

$$0 = \int_0^1 \left(\int_0^t f''(s) ds \right) dt = \sum_{n=2}^{\infty} c_n \int_0^1 \left\{ \int_0^t \xi_n(s) ds \right\} dt$$

$$= \sum_{n=0}^{\infty} \sum_{l=1}^{2^n} \frac{c_{2^n+l}}{2^{\frac{3n}{2}+2}} = \frac{c_2}{4} + \frac{1}{4} \sum_{m=0}^{\infty} \sum_{l=1}^{2^{m+1}} \frac{c_{2^{m+1}+l}}{2^{\frac{3(m+1)}{2}}} = \frac{A}{4}.$$

Now, we can put

$$\tilde{f} := \lim_{m \to \infty} \left[\sum_{i=3}^{2^{m+2}} \tilde{c}^m_i y_i \right], \qquad \tilde{f} \in W_0^{2,2}(0,1).$$

This means that

$$\tilde{f''} = \lim_{m\to\infty}\left[\sum_{i=3}^{2^{m+2}} \tilde{c}_i^m y''_i\right] = f''$$

a.e. on $[0,1]$. Hence $f(t) = \tilde{f}(t)$ for each $t \in [0,1]$. This finishes the proof. □

Remark 1 Let us mention that using y_n, $n = 3,4$ one can define a complete orthonormal system in $(W_0^{2,2}(0,1),[.,.])$ by putting

$$\Psi_{2^{m+1}+2s-1} := \frac{1}{2\sqrt{5}}\left(y_{2^{m+1}+2s-1} + y_{2^{m+1}+2s}\right)$$
$$\Psi_{2^{m+1}+2s} : = \frac{1}{4}\left(y_{2^{m+1}+2s-1} - y_{2^{m+1}+2s}\right)$$

for $s = 1,\dots 2^m$, $m = 0,1,\dots$.

Remark 2 Similarly one can construct a countable complete system of linearly independent functions in the space $W_0^{m,2}(0,1)$ consisting of functions which are parts of polynomials of order m and in some joint points $t_1,\dots,t_m$ are at least of class C^{m-1}. One can start with function

$$y(t) = \begin{cases} \frac{a_0 t^m}{m!} & \text{for } t \in [0,t_1] \\ \frac{a_{i,1}t^m}{m!} + \dots + a_{i,m+1} & \text{for } t \in (t_i,t_{i+1}],\ i = 1,\dots m-1 \\ \frac{a_m(t-1)^m}{m!} & \text{for } t \in (t_m,1] \end{cases} \tag{17}$$

which can be always determined (homogeneous system of m^2 equations with $(m-1)(m+1)+2 = m^2+1$ unknowns). We assume that $m = 2^n + l$, $n = 0,1,\dots$, $l = 0,\dots(2^n-1)$. Taking t_i, $i = 1,\dots m$ from among points $\frac{s}{2^{n+1}}$, $s = 1,\dots(2^{n+1}-1)$ it can be defined $2^{n+1} - m = 2^n - l$ linear independent functions according to formula (17). Their m-th derivative is a constant in the intervals determined by points t_i, $i = 1,\dots m$. Thus, they are a linear combination of Haar functions $\xi_1,\dots\xi_{2^{n+1}}$ (see [3], p. 133). Applying linear transformation appearing in formula (11) one can formulate a complete system of linear independent functions in $W_0^{m,2}(0,1)$ analogously as for $m = 2$.

Example 1 For differential equation

$$y^{(4)} + 16\delta_{\frac{1}{2}} y = 1$$
$$y(0) = y(1) = y'(0) = y'(1) = 0,$$

where $\delta_{\frac{1}{2}}$ is the Dirac measure concentrated at point $t = \frac{1}{2}$ we obtain the exact solution

$$y(t) = \begin{cases} \frac{1}{24}t^4 - \frac{49}{600}t^3 + \frac{97}{2400}t^2 & \text{for } t \in [0,\frac{1}{2}] \\ \frac{1}{24}t^4 - \frac{49}{600}t^3 + \frac{97}{2400}t^2 - \frac{1}{300}\left(t-\frac{1}{2}\right)^3 & \text{for } t \in (\frac{1}{2},1]. \end{cases}$$

That is $y\left(\frac{1}{2}\right) = \frac{1}{400} = 2.5\cdot 10^{-4}$. Approximate solutions in spaces E_2, E_6 and E_{14} calculated numerically by the method described in this paper have the following values $1.83\cdot 10^{-4}$, $2.23\cdot 10^{-4}$ and $2.36\cdot 10^{-4}$ at point $t = \frac{1}{2}$.

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