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ON CERTAIN THREE-POINT REGULAR BOUNDARY VALUE PROBLEMS FOR NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS DEPENDING ON THE PARAMETER [1]

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Abstract

Applying a method based on a surjectivity result in $\mathbf{R}^n$, we investigate the existence and uniqueness of solutions of the differential equation

$$x'' = f(t, x, x', \lambda)$$

depending on the parameter λ satisfying for a suitable value of λ the three-point boundary conditions $x'(0) = A$, $x(1) = B$, $x(2) = C$.

Key words: Second-order differential equation depending on the parameter, three-point boundary value problem, surjective mapping.

MS Classification: 34B10

1 Introduction

Consider the differential equation

$$x'' = f(t, x, x', \lambda), \tag{1}$$

where $f \in C^0(\langle 0, 2\rangle \times \mathbf{R}^3)$ depending on the parameter λ. A method based on a surjectivity result in $\mathbf{R}^n$ (see [2], [8]) is developed by means of which it is considered the problem to determine sufficient conditions under which there

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exists a unique value λ_0 of the parameter λ to any numbers A, B, C ($\in \mathbf{R}$) so that equation (1) for $\lambda = \lambda_0$ admits a (and then unique) solution x satisfying the boundary conditions

$$x'(0) = A, \qquad x(1) = B, \qquad x(2) = C. \tag{2}$$

This paper was motivated by an interesting paper of Šeda [8] that is concerned with the correctness of the boundary value problem

$$x'' = f(t, x, x'), \qquad x'(a) = A, \quad x(b) - x(t_0) = B,$$

where $a < t_0 < b$, A, B are real numbers.

We observe that some boundary value problems for differential equations and functional differential equations depending on a parameter have been studied for example in [5]–[7] using the Schauder linearization technique and the Schauder fixed point theorem. In [1] and [3], a suitable implementation of parameters into homogeneous linear differential equations quarantees the existence of a solution x satisfying $x(t_1) = x(t_2) = x(t_3) = 0$ $(-\infty < t_1 < t_2 < t_3 < \infty)$.

2 Definitions, lemmas and results

Lemma 1 *Assume $h : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous mapping. If h is injective, then h is a homeomorphism of $\mathbf{R}^n$ onto itself if and only if it satisfies the condition*

$$\lim_{|x| \to \infty} |h(x)| = \infty. \tag{3}$$

Proof For the proof see [2; p. 23] and [8].

Let $X = \{(x(t), x'(t));\ x \in C^1(\langle 0, 2 \rangle)\}$ be the Banach space with the norm

$$||(x, x')|| = \max \{ \max_{0 \le t \le 2} |x(t)|, \max_{0 \le t \le 2} |x'(t)| \}$$

and let $F : X \to \mathbf{R}^3$ be a continuous operator. Consider the boundary condition

$$F(x(t), x'(t)) = (A, B, C), \tag{4}$$

where $(A, B, C) \in \mathbf{R}^3$. Condition (2) is a special case of (4).

Say that $x \in C^2(\langle 0, 2 \rangle)$ is a *solution of boundary value problem* (1), (4) if there exists a $\lambda_0 \in \mathbf{R}$ such that x is a solution of (1) for $\lambda = \lambda_0$ and x satisfies (4).

The existence of a solution to boundary value problem (1), (4) is given in the following theorem.

Theorem 1 *Let equation* (1) *have the following properties:*

(H_1) *The Cauchy problem* $x(0) = x_0$, $x'(0) = x_1$ *for equation* (1) *has a unique solution* $x(t, x_0, x_1, \lambda)$ *on* $\langle 0, 2\rangle$ *for each* $(x_0, x_1, \lambda) \in \mathbf{R}^3$,

(H_2) *Problem* (1), (4) *has at most one solution for each* $(A, B, C) \in \mathbf{R}^3$,

(H_3) (*Compactness condition*) *If* $\{x(t, x_k, y_k, \lambda_k)\}$ *is an arbitrary sequence of solutions of* (1) *such that*

$$\{F(x(t, x_k, y_k, \lambda_k),\ x'(t, x_k, y_k, \lambda_k))\}$$

is bounded, then the sequences $\{x_k\}$, $\{y_k\}$ *and* $\{\lambda_k\}$ *are bounded.*

Then there exists a unique solution of boundary value problem (1), (4) *for each* $(A, B, C) \in \mathbf{R}^3$.

Proof The mapping $H : \mathbf{R}^3 \to X$ defined by $H(x_0, y_0, \lambda_0) = (x(t, x_0, y_0, \lambda_0), x'(t, x_0, y_0, \lambda_0))$ is continuous (see [4]). In view of continuity of H, the composite mapping $h = F \circ H$ is continuous from $\mathbf{R}^3$ into $\mathbf{R}^3$. Problem (1), (4) has a solution for each $(A, B, C) \in \mathbf{R}^3$ (and then unique by (H_2)) if h is surjective. To show the surjectivity, we shall use Lemma 1.

Condition (3) in Lemma 1 means that the h-preimage of each bounded set in $\mathbf{R}^3$ is bounded in $\mathbf{R}^3$, which is equivalent to condition (H_3).

Corollary 1 *Under assumptions* (H_1)–(H_3) *problem* (1), (4) *has a unique solution for each* $(A, B, C) \in \mathbf{R}^3$ *and this solution as well as its derivative are continuous functions of the variables* (t, A, B, C) *in* $\langle 0, 2\rangle \times \mathbf{R}^3$.

Proof Let h, H be defined as in the proof of Theorem 1. Under assumptions (H_1)–(H_3), h is not only surjective, but even homeomorphic (see Lemma 1). Because H is also homeomorphic, $F|_{H(\mathbf{R}^3)}$ as well as its inverse mapping $\left(F|_{H(\mathbf{R}^3)}\right)^{-1}$ is homeomorphic, too. This implies that the solution x of problem (1), (4) and its derivative continuously depend on (t, A, B, C) in $\langle 0, 2\rangle \times \mathbf{R}^3$.

Lemma 2 *Assume assumptions* (H_1) *and*

(H_4) $f(t, ., y, \lambda)$ *is increasing on* $\mathbf{R}$ *for each fixed* $(t, y, \lambda) \in \langle 0, 2\rangle \times \mathbf{R}^2$,

(H_5) $f(t, x, y, .)$ *is increasing on* $\mathbf{R}$ *for each fixed* $(t, x, y) \in \langle 0, 2\rangle \times \mathbf{R}^2$,

are fulfilled.

Then problem (1), (4) *admits at most one solution for each* $(A, B, C) \in \mathbf{R}^3$.

Proof Suppose there exist two solutions x_i of (1) for $\lambda = \lambda_i$ ($i = 1, 2$) satisfying boundary conditions (2) with $x = x_i$, and suppose $\lambda_2 \geq \lambda_1$. Setting $w = x_2 - x_1$, then $w'(0) = w(1) = w(2) = 0$. If w has a positive local maximum at a $\xi \in (0, 2)$, then $w(\xi) > 0$, $w'(\xi) = 0$, $w''(\xi) \le 0$ which contradicts

$$w''(\xi) = f(\xi, x_2(\xi), x_2'(\xi), \lambda_2) - f(\xi, x_1(\xi), x_2'(\xi), \lambda_1) > 0.$$

Since $w(0) > 0$ implies $w''(0) > 0$, we see $w(t) \le 0$ on $\langle 0, 2\rangle$. Then with regard to $w(1) = 0$ we have $w'(1) = 0$, $w''(1) \le 0$. On the other hand

$$w''(1) = f(1, x_2(1), x_2'(1), \lambda_2) - f(1, x_2(1), x_2'(1), \lambda_1) \ge 0,$$

therefore $w''(1) = 0$ and then necessarily $\lambda_1 = \lambda_2$. Now, if $w(0) = 0$, then (cf. (H_1)) $w = 0$. If $w(0) \ne 0$, then we can assume without loss of generality $w(0) > 0$, hence

$$w''(0) = f(0, x_2(0), x_2'(0), \lambda_2) - f(0, x_1(0), x_2'(0), \lambda_2) > 0,$$

consequently there exists a $\tau \in (0, 1)$ such that w has a positive local maximum at τ which leads to a contradiction.

Lemma 3 *Suppose that f satisfies the condition*

(H_6) *For each numbers $S > 0$, $M > 0$ and $L > 0$ there exists a number $K > 0$ such that*

$$f(t, x, y, \lambda) \ge S \quad \textit{for all} \quad t \in \langle 0, 2\rangle,\ x \ge -M,\ |y| \le L,\ \lambda \ge K, \tag{5}$$

$$f(t, x, y, \lambda) \le -S \quad \textit{for all} \quad t \in \langle 0, 2\rangle,\ x \le M,\ |y| \le L,\ \lambda \le -K. \tag{6}$$

Let $x(t)$ be a solution of (1) for $\lambda = \lambda_0$ on $\langle 0, 2\rangle$ such that

$$|x'(0)| \le Q, \quad |x(1)| \le Q, \quad |x(2)| \le Q \tag{7}$$

for a positive constant Q.

Then $|\lambda_0| < K$, where K corresponds in condition (H_6) to $S = L = 4Q$ and $M = Q$.

Proof Let x be a solution of (1) for $\lambda = \lambda_0$ on $\langle 0, 2\rangle$ satisfying (7) with a $Q > 0$. Assume K corresponds in condition (H_6) to $S = L = 4Q$, $M = Q$ and suppose $\lambda_0 \ge K$.

If $x(0) \ge Q$, then $x''(t) \ge 4Q$ for all $t \in \langle 0, \xi\rangle$ $(\subset \langle 0, 2\rangle)$ where $-Q \le x'(0) \le x'(t) \le 4Q$. Therefore $x'(t) \ge -Q + 4Qt$, $x(t) \ge Q - Qt + 2Qt^2$ for $t \in \langle 0, \xi\rangle$, consequently $x(t) > 0$ on $\langle 0, \xi\rangle$ and if $\xi < 2$ and $x'(\xi) = 4Q$, then necessarily $x'(t) > 4Q$ for $t \in (\xi, 2\rangle$. Hence $x(1) \ge 2Q$ which is a contradiction to $|x(1)| \le Q$.

If $x(0) \le -Q$, then there exists an $\eta \in \langle 0, 1)$ such that $x(\eta) = -Q$, $x'(\eta) \ge 0$ and if $\eta > 0$ we have $x'(\eta) > 0$. Therefore $x(t) \ge -Q$ and $x'(t) \ge \min\{x'(\eta) + 4Q(t-\eta), 4Q\}$ on $\langle \eta, 2\rangle$, hence

$$x(2) - x(1) = \int_1^2 x'(s)\, ds \geq 2Q$$

which is a contradiction to $|x(2) - x(1)| \le 2Q$.

Let $|x(0)| < Q$. If $|x(t)| \le Q$ on $\langle 0, 2\rangle$, then $x'(t) \ge \min\{x'(0) + 4Qt, 4Q\}$ for $t \in \langle 0, 2\rangle$ and therefore there exists a $\xi \in (0, \frac{5}{4}\rangle$ such that $x'(\xi) = 4Q$, $x'(t) < 4Q$ on $\langle 0, \xi)$. Then $x'(t) \ge 4Q$ on $\langle \xi, 2\rangle$ consequently,

$$x(2) \ge x(0) + \int_0^\xi x'(s)ds + 4Q(2-\xi) > -Q + (-Q\xi + 2Q\xi^2) + 4Q(2-\xi) \ge \frac{31}{8}Q$$

and we come to contradiction with $|x(2)| \le Q$. Let $0 < \varepsilon < 2$ and $|x(t)| < Q$ for $t \in \langle 0, \varepsilon)$, $|x(\varepsilon)| = Q$. Then in case of $x(\varepsilon) = Q$ we have $x'(\varepsilon) \ge 0$, $x'(t) > 0$ for $t \in (\varepsilon, 2\rangle$, consequently $x(2) > Q$ which is a contradiction to $|x(2)| \le Q$. In case of $x(\varepsilon) = -Q$ we have $x'(\varepsilon) \le 0$ and therefore it is necessarily $\varepsilon \in (0, 1)$. In the opposite case (that is $\varepsilon \ge 1$) since $x'(t) \ge \min\{x'(0) + 4Qt, 4Q\}$ on $\langle 0, \varepsilon\rangle$, we have

$$x(\varepsilon) \ge x(0) + \int_0^\varepsilon x'(s)ds \ge x(0) + \int_0^1 x'(s)ds \ge 0$$

which is a contradiction to $x(\varepsilon) = -Q$. Then, of course, there exists a $\tau \in \langle \varepsilon, 1\rangle$ such that $x(\tau) = -Q$, $\operatorname{sign} x'(\tau) = \operatorname{sign}(\tau - \varepsilon)$ and as above we get a contradiction.

The case $\lambda_0 \le -K$ can be treated similarly.

Lemma 4 *Assume that f satisfies the condition*

(H_7) *For each numbers $L > 0$ and $K > 0$ there exists a number M ($\ge L$) such that*

$$f(t, x, y, \lambda) > 0 \quad \textit{for all} \quad t \in \langle 0, 1\rangle,\ x \ge M,\ |y| \le L,\ |\lambda| \ge K, \tag{8}$$

$$f(t, x, y, \lambda) < 0 \quad \textit{for all} \quad t \in \langle 0, 1\rangle,\ x \le -M,\ |y| \le L,\ |\lambda| \le K. \tag{9}$$

Let $x(t)$ be a solution of (1) *for $\lambda = \lambda_0$ on $\langle 0, 2\rangle$ such that inequalities* (7) *are satisfied for a positive constant Q. Then*

$$|x(0)| \le M + Q,$$

where M corresponds in condition (H_7) to $L = Q$ and $K \ge |\lambda_0|$.

Proof Let x be a solution of (1) for $\lambda = \lambda_0$ on $\langle 0, 2\rangle$ satisfying (7) for a $Q > 0$. Suppose that M corresponds in condition (H_7) to $L = Q$ and $K \ge |\lambda_0|$.

Assume $x(0) > M + Q$. Then using (8) we obtain $x''(0) > 0$, hence $x'(t)$ $(\geq x'(0) \geq -Q)$ is increasing on each interval $\langle 0, \xi \rangle$ $(\subset \langle 0, 1 \rangle)$ where $x'(t) \leq Q$. If

$$|x'(t)| \leq Q \quad \text{on} \quad \langle 0, 1 \rangle, \tag{10}$$

then by the mean value theorem there exists a $t_0 \in (0,1)$ such that $(Q \leq)$ $M < x(0) - x(1) = -x'(t_0)$, which contradicts (10). Suppose (10) is not fulfilled. If $x'(0) = Q$, then $x''(0) > 0$ and since $|x(1)| \leq Q$, x has a local maximum at a $t_1 \in (0,1)$ and therefore $x(t_1) > x(0)$, $x'(t_1) = 0$, $x''(t_1) \leq 0$. On the other hand $x''(t_1) = f(t_1, x(t_1), 0, \lambda_0) > 0$ (by (H_7)), and we obtain a contradiction. Let $x'(0) < Q$ and let $\langle 0, \xi_1 \rangle$ $(0 < \xi_1 < 1)$ be the maximal interval such that $x'(t) \leq Q$ on $\langle 0, \xi_1 \rangle$ and $x'(\xi_1) = Q$. Then

$$x(\xi_1) = x(0) + \int_0^{\xi_1} x'(s)\, ds > M + Q - Q\xi_1 > M$$

consequently, $x''(\xi_1) > 0$ and $x'(t) \geq Q$ on $\langle \xi_1, 1 \rangle$. Therefore $x(1) > Q$ which is impossible.

Assume $x(0) < -M - Q$. Then using (9) we get $x''(0) < 0$ and $x'(t)$ $(\leq x'(0) \leq Q)$ is decreasing on each interval $\langle 0, \xi \rangle$ $(\subset < 0, 1 >)$ where $x'(t) \geq -Q$. Let (10) be fulfilled. Then by the mean value theorem there exists a $t_0 \in (0,1)$ such that $(-Q \geq)$ $-M > x(0) - x(1) = -x'(t_0)$ which is a contradiction to (10). Suppose (10) is not fulfilled. If $x'(0) = -Q$, then $x''(0) < 0$ and since $|x(1)| \leq Q$, x has a local minimum at a $t_1 \in (0,1)$ and therefore $x(t_1) < x(0)$, $x'(t_1) = 0$, $x''(t_1) \geq 0$. On the other hand $x''(t_1) = f(t_1, x(t_1), 0, \lambda_0) < 0$ by (H_7), a contradiction. Let $x'(0) > -Q$ and let $\langle 0, \varepsilon \rangle$ $(\subset < 0, 1 \rangle)$ be the maximal interval such that $x'(t) \geq -Q$ on $\langle 0, \varepsilon \rangle$ and $x'(\varepsilon) = -Q$. Then

$$x(\varepsilon) = x(0) + \int_0^{\varepsilon} x'(s)\, ds < -M - Q + Q\varepsilon < -M$$

consequently, $x''(\varepsilon) < 0$ and $x'(t) \leq -Q$ for $\varepsilon \leq t \leq 1$. Hence $x(1) < -Q$ which is impossible. This completes the proof.

Lemma 5 *Suppose that f satisfies conditions (H_6) and (H_7). Let $x_k(t)$, $k = 1, 2, \ldots$, be a sequence of solutions of (1) for $\lambda = \lambda_k$ on $\langle 0, 2 \rangle$ such that*

$$|x_k'(0)| \leq Q, \quad |x_k(1)| \leq Q, \quad |x_k(2)| \leq Q, \quad k = 1, 2, \ldots,$$

where Q is a positive constant. Then the sequences

$$\{\, x_k(0) \,\} \quad \text{and} \quad \{\, \lambda_k \,\}$$

are bounded.

Proof By Lemma 3 there exists a positive constant K_1 corresponding in condition (H_6) to $S = L = 4Q$ such that $|\lambda_k| \leq K_1$, $k = 1, 2, \ldots$. Using Lemma 4 we get

$$|x_k(0)| \leq M + Q, \qquad k = 1, 2, \ldots,$$

where M $(\geq Q)$ corresponds in condition (H_7) to $L = Q$ and $K = K_1$.

Theorem 2 *Assume that assumptions* (H_1), (H_4)–(H_7) *are satisfied. Then problem* (1), (2) *has a unique solution for each* $(A, B, C) \in \mathbf{R}^3$ *and this solution as well as its derivative are continuous functions of the variables* (t, A, B, C) *on* $\langle 0, 2\rangle \times \mathbf{R}^3$.

Proof Let assumptions (H_1), (H_4)–(H_7) be satisfied. With respect to Theorem 1 and Corollary 1 where $F(x(t), x'(t)) = (x'(0), x(1), x(2))$, it is sufficient to show that assumptions (H_1)–(H_3) are satisfied. Assumptions (H_4) and (H_5) ((H_6) and (H_7)) imply that assumption (H_2) ((H_3)) is fulfilled (see Lemma 2 and Lemma 5, respectively). Hence theorem is proved.

Example 1 Consider the differential equation

$$x'' = (1 + t^2)(x + \cos(x'^2)) + (1 + |x'|)\lambda. \tag{11}$$

This equation fulfils all assumptions of Theorem 2 and therefore there exists a unique $\lambda_0 \in \mathbf{R}$ to any $(A, B, C) \in \mathbf{R}^3$ such that equation (11) for $\lambda = \lambda_0$ has a (and then unique) solution x satisfying (2).

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