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CONCEPT LATTICES OF QUASIORDERED SETS

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Abstract

In this paper there are investigated the concept lattices of quasiordered sets. There are given necessary and sufficient conditions under which the concept lattice of of quasiordered set is distributive or modular or complemented or boolean, respectively.

Key words: concept lattice, distributive or modular or complemented or boolean quasiordered set.

MS Classification: 06A15.

The concept of concept lattice was introduced by R. Wille in [1] and investigated by many other authors.

M. Ernè gave in [2] necessary and sufficient conditions under which the concept lattice satisfies some kinds of distributive identity.

The aim of my paper is to investigate concept lattices in case of quasiordered sets and to show the connections with concepts of distributive and modular ordered sets introduced by J. Rachůnek (see [3]).

First, let us recall some notions concerning concept lattices and ordered and quasiordered sets.

A *context* is a triple (G, M, I), where G and M are sets and I is a binary relation between G and M (i.e. $I \subseteq G \times M$).
For $A \subseteq G$ and $B \subseteq M$, we define:

$$U(A) := \{m \in M;\ gIm \text{ for all } g \in A\}, \quad L(B) := \{g \in G;\ gIm \text{ for all } m \in B\}.$$

We call the set $U(A)$ an *upper cone* of the set A and the set $L(B)$ the *lower cone* of the set B. If $A = \{a\}$, we write instead of $U(\{a\})$ or $L(\{a\})$ briefly $U(a)$ or $L(a)$.

It is easy to see, that the mappings

$$C: \quad \forall X \subseteq G: \ C(X) := L(U(X)),$$

$$D: \quad \forall Y \subseteq M: \ D(Y) := U(L(Y))$$

are the closure operators on the set G or M, respectively. By a *concept lattice* of the triple *(context)* (G, M, I) is understand the set of all closed sets under the closure operator C (i.e. the sets $X \subseteq G$ with $C(X) = X$) ordered by set inclusion.

It is well-known that the lattice of all closed sets under the operator D is a lattice dually isomorphic to the concept lattice.

Further we shall work only with triples, where $G = M$.

Remark 1 If there is not danger of misunderstanding, we write for $A, B \subseteq G$ $LU(A)$ or $UL(A)$ instead of $L(U(A))$ or $U(L(A))$ and $L(A, B)$ or $U(A, B)$ instead of $L(A \cup B)$ or $U(A \cup B)$.

Proposition 1 *Let (G,G,I) be a context. Let us consider the conditions:*

(i) I is reflexive and transitive binary relation

(ii) $\forall a \in G: \ LU(a) = L(a)$

(iii) $\forall A \subseteq G: \ LUL(A) = L(A)$

Then (i)$\Leftrightarrow$(ii) and (iii) holds.

Proof

(i) $\Rightarrow$ (ii) Let $x \in L(a)$ and $y \in U(a)$. Then $(x, a) \in I$, $(a, y) \in I$ and by transitivity $(x, y) \in I$, so $x \in LU(a)$. Conversely, let $x \in LU(a)$, $y \in U(a)$. Thus $(x, y) \in I$ and $a \in U(a)$ by reflexivity, so $(x, a) \in I$ and $x \in L(a)$.

(ii) $\Rightarrow$ (i)

$$U(a) = \{x : (a, x) \in I\}, \quad LU(a) = \{y : (y, x) \in I \text{ for all } x \in U(a)\}.$$

But $a \in LU(a)$ and thus $a \in L(a)$, so $(a, a) \in I$ and I is reflexive.
Let $(x, y), (y, z) \in I$. Then $x \in L(y) = LU(y)$, $z \in U(y)$ and therefore $(x, z) \in I$ and I is transitive.

(iii) Let $x \in L(A)$, $y \in UL(A)$. Then $(x, y) \in I$ and so $x \in LUL(A)$. Let $x \in LUL(A)$, $y \in UL(A)$, $z \in L(A)$, $a \in A$. Then $(z, a), (x, y) \in I$, so $a \in UL(A)$ and $(x, a) \in I$, thus $x \in L(A)$.

By Proposition 1, if I is a quasiorder, we can use (ii) and (iii). Further we shall work only with the concepts (G, G, I), where I is a quasiorder on G.

In the papers [3] and [4], there were introduced concepts of distributive, modular and complemented ordered sets. We shall give a similar definition for case of quasiordered sets: an quasiordered set $(G, \leq)$ is

distributive	if $\forall a, b, c \in G: \ L(U(a,b),c) = LU(L(a,c), L(b,c))$;
modular	if $\forall a, b, c \in G: \ LU(a, L(b, U(a,c))) = L(U(a,b), U(a,c))$;
complemented	if $\forall a \in G \ \ \exists b \in G: \ LU(a,b) = G \ \& \ UL(a,b) = G$;
boolean	if it is distributive and complemented.

Theorem 1 *A concept lattice of the quasiordered set* $(G, \leq)$ *is*

(i) distributive iff $\forall A, B, C \subseteq G$ *it holds:*

$$L(U(A,B), U(C)) = LU(L(U(A), U(C)), L(U(B), U(C)));$$

(ii) modular iff $\forall A, B, C \subseteq G$ *it holds:*

$$L(U(B), U(A, L(U(A,C), U(B)))) = L(U(A,B), U(A,C));$$

(iii) complemented iff $\forall A \subseteq G \ \ \exists B \subseteq G$:

$$LU(A,B) = UL(U(A), U(B)) = G.$$

Proof (i) By the definition, all elements of the concept lattice are of the form $LU(X)$, where X is an arbitrary subset of G. Now, the concept lattice is distributive iff $\forall A, B, C \subseteq G$ it holds:

$$[LU(A) \vee LU(B)] \wedge LU(C) = [LU(A) \wedge LU(C)] \vee [LU(B) \vee LU(C)], \qquad (1)$$

where the operations $\vee$ or $\wedge$ are the join or meet in the concept lattice. It is clear, that the meet coincide with the set intersection and the join is $LU(A) \vee LU(B) = LU(LU(A), LU(B))$. Then the left side of equation (1) is equal to

$$LU(LU(A), LU(B)) \cap LU(C) = L(ULU(A) \cap ULU(B)) \cap LU(C) = \\ = L(U(A) \cap U(B)) \cap LU(C) = L(U(A,B)) \cap LU(C) = L(U(A,B), U(C)).$$

The right side of the same equation is equal to

$$LU((LU(A) \cap LU(C)), (LU(B) \cap LU(C))) = \\ = LU(L(U(A), U(C)), L(U(B), U(C))).$$

(ii) The proof is analogous to that of the claim (i), if we use the modular identity in the form $a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$.

(iii) The greatest element of the concept lattice is the set $G = LU(G)$ (if G has not a unit element, then $U(G) = \emptyset$ and we consider $L(\emptyset) = G$) and the least element is the set $LU(\emptyset)$. Then the concept lattice is complemented iff $\forall A \subseteq G \quad \exists B \subseteq G$:

$$LU(A) \cap LU(B) = LU(\emptyset). \text{and } LU(LU(A), LU(B)) = G.$$

These conditions are equivalent with

$$L(U(A), U(B)) = LU(\emptyset) \text{ and } L(U(A) \cap U(B)) = G,$$

which are equivalent with

$$UL(U(A), U(B)) = U(\emptyset) = G \text{ and } LU(A, B) = G.$$

Remark 2 From (i), (ii) and (iii) of Theorem 1 it is evident, that these conditions are stronger than distributivity, modularity and complementarity of quasiordered set (we can substitute the sets A, B, C by one-element sets and by using of the properties of the closure operator LU we obtain desired identities).

Corollary 1 *If the concept lattice of quasiordered set $(G, \leq)$ is distributive or modular or complemented or Boolean, then $(G, \leq)$ is distributive or modular or complemented or Boolean quasiordered set, respectively.*

Let $(G, \leq)$ be an quasiordred set and $H \subseteq G$. Let us consider the induced context $(H, H \times H \cap \leq)$. We give necessary and sufficient conditions under which the concept lattice of the induced context is a sublattice of the concept $(G, \leq)$.

Theorem 2 *Let $(G, \leq)$ be a quasiordered set, $H \subseteq G$ and $(H, \leq)$ be an induced context. Then the concept lattice of $(H, \leq)$ is a sublattice of the concept latice of $(G, \leq)$ iff*

$$\forall\ A \subseteq H:\ L(U(A) \cap H) \cap H = LU(A).$$

Proof Let A be an arbitrary subset of H. Then the upper cone of A in the induced context is equal to $U'(A) = U(A) \cap H$ and the lower cone is equal to $L'(A) = L(A) \cap H$.
Then the closed sets of the induced context are $L(U(A) \cap H) \cap H$ for arbitrary $A \subseteq H$. If the concept lattice of the induced context is a sublattice, then the suprema and infima of all the pairs of elements must be the same as in the concept lattice of $(G, \leq)$.
Hence, $L(U(A) \cap H) \cap H \cap L(U(B) \cap H) \cap H = LU(A) \cap LU(B)$ for all $A, B \subseteq H$. If we take $A = B$, we obtain the desired condition. The second implication is clear.

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