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ON TRANSFORMATIONS OF TWO LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

JITKA LAITOCHOVÁ

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Abstract. We are concerned with the Kummer's transformation [1], [2] of two linear differential equations in the form $(ru'+qu)'-(qu'+pu)=0$ into itself.

Key words: Ordinary differential equation of the second order, Kummer's transformation.

MS Classification: 34A30

Consider the following homogeneous linear differential equations of the second order

$$[r(t)u'+q(t)u]'-[q(t)u'+p(t)u]=0, \qquad \text{(rqp)}$$
$$r,q,p\in C^{(0)}(j), \quad r(t)\neq 0 \text{ in } j,$$

$$[R(T)U'+Q(T)U]'-[Q(T)U'+P(T)U]=0, \qquad \text{(RQP)}$$
$$R,Q,P\in C^{(0)}(J), \quad R(T)\neq 0 \text{ in } J.$$

Definition 1. By a solution of an equation (rqp) [5] we mean every continuous function u with a continuous derivative u' for which there is an associate function v continuously differentiated such that

$$v(t)=r(t)u'(t)+q(t)u(t), \quad v'(t)=q(t)u'(t)+p(t)u(t).$$

Definition 2. If j, J are open intervals and $t \in j$, $T \in J$, then by a transformation of the differential equation (RQP) into the differential equation (rqp) we mean an ordered pair $[f,h]$ of functions $f(t)$, $h(t)$ defined in an open interval i, $i \subset j$, and having such properties that $h(i)=I$, $I \subset J$, $f \in C^{(2)}(i)$, $h \in C^{(3)}(i)$, $f(t)h'(t) \neq 0$ for $t \in i$ [1] and for every solution Y of (RQP) the function

$$y(t)=f(t)Y[h(t)]$$

is a solution of the differential equation (rqp) in the interval i.

The foregoing relation between the solution y and Y is called the transformation equation. The function h is called the parametrization and the function f the multiplier of the transformation $[f,h]$.

Main results.

Theorem 1. *Let* $\frac{R[h(t)]}{r(t)h'(t)} \in C^{(1)}(i)$, $\left\{\frac{r^2(t)h'(t)}{R[h(t)]}\left(\frac{R[h(t)]}{r(t)h'(t)}\right)'\right\}'$ $\left[\frac{R[h(t)]f'(t)}{f^3(t)h'(t)}\right] \in C^{(1)}(i)$. *Then the differential equation (RQP) is transformed into the differential equation (rqp) by the transformation $[f,h]$ if and only if the parametrization h satisfies the nonlinear third order differential equation*

$$\frac{1}{2}\left\{\frac{r^2(t)h'(t)}{R[h(t)]}\left(\frac{R[h(t)]}{r(t)h'(t)}\right)'\right\}' + \frac{1}{4}\, r(t)\left(\frac{r(t)h'(t)}{R[h(t)]}\right)^2\left(\frac{R[h(t)]}{r(t)h'(t)}\right)'^2 + \frac{r(t)h'^2(t)P[h(t)]}{R[h(t)]} - \frac{h'^2(t)r^2(t)Q[h(t)]}{R^2[h(t)]}\left(\frac{R[h(t)]}{r(t)h'(t)}\right)' - p(t) = 0 \qquad (1)$$

and the coefficient q is given by the formula

$$q(t) = \frac{r(t)h'(t)Q[h(t)]}{R[h(t)]} \qquad (2)$$

for the parametrization $h=h(t)$, and the multiplier f is given by the formula

$$f(t) = k\sqrt{\left|\frac{R[h(t)]}{r(t)h'(t)}\right|}, \quad t \in i,\ k \neq 0. \qquad (3)$$

Proof. Let the differential equation (RQP) be transformed into (rqp) by means of $[f,h]$. We will show that h satisfies the equation (1) when f satisfies the equation (3) and the coefficient q is given by (2).

Let us differentiate with respect to t the transformation equation

$$y(t) = f(t)Y[h(t)]. \tag{4}$$

We get

$$y'(t) = f'(t)Y[h(t)]+f(t)h'(t)Y'[h(t)]. \tag{5}$$

Let us multiply both sides of (5) by $R(h)/f^2h'$. Substituting from (4) for Y(h) we get

$$\frac{R[h(t)]}{f^2(t)h'(t)}\, y'(t) = \frac{R[h(t)]f'(t)}{f^3(t)h'(t)}y(t)+ \frac{R[h(t)]}{f(t)}\, Y'[h(t)]. \tag{6}$$

By consequence of (4) we have

$$\frac{Q[h(t)]}{f^2(t)}\, y(t) = \frac{Q[h(t)]}{f(t)}\, Y[h(t)] \tag{7}$$

and adding (6) and (7) we get

$$\frac{R[h(t)]}{f^2(t)h'(t)}\, y'(t) + \frac{Q[h(t)]}{f^2(t)}\, y(t) = \tag{8}$$
$$\frac{R[h(t)]f'(t)}{f^3(t)h'(t)}\, y(t) + \frac{1}{f(t)}\, \{R[h(t)]Y'[h(t)]+Q[h(t)]Y[h(t)]\}.$$

As there is a derivative of the right side of the foregoing equation we obtain by differentiating the following equality

$$\left\{ \frac{R[h(t)]}{f^2(t)h'(t)}\, y'(t) + \frac{Q[h(t)]}{f^2(t)h'(t)}\, y(t) \right\}' =$$
$$\left\{ \frac{R[h(t)]f'(t)}{f^3(t)h'(t)} \right\}' y(t)+ \frac{R[h(t)]f'(t)}{f^3(t)h'(t)}\, y'(t) -$$
$$\frac{f'(t)}{f^2(t)}\, \{ R[h(t)]\, Y'[h(t)] + Q[h(t)]\, Y[h(t)] \} +$$
$$\frac{1}{f(t)}\, \{ R[h(t)]\, Y'[h(t)] + Q[h(t)]\, Y[h(t)] \}'$$

From here and from the identity

$$\{ R[h(t)]Y'[h(t)]+Q[h(t)]Y[h(t)] \}' =$$
$$\{ Q[h(t)]Y'[h(t)]+P[h(t)]Y[h(t)]h'(t) \}$$

we get

$$\left\{ \frac{R[h(t)]}{f^2(t)h'(t)}\, y'(t) + \frac{Q[h(t)]}{f^2(t)}\, y(t)\right\}' =$$

$$\left\{ \frac{R[h(t)]f'(t)}{f^3(t)h'(t)} \right\}' y(t) + \frac{R[h(t)]f'(t)}{f^3(t)h'(t)} y'(t) -$$
$$\frac{f'(t)}{f^2(t)} \{ R[h(t)]Y'[h(t)] + Q[h(t)]Y[h(t)] \} +$$
$$\frac{h'(t)}{f(t)} \{ Q[h(t)]Y'[h(t)] + P[h(t)]Y[h(t)] \}$$

After substitution for Y and Y′ from the equations (4) and (5) we have

$$\left\{ \frac{R[h(t)]}{f^2(t)h'(t)} y'(t) + \frac{Q[h(t)]}{f^2(t)} \right\}' = \frac{Q[h(t)]}{f^2(t)} y'(t) + \tag{9}$$
$$\left\{ \left(\frac{f'(t)R[h(t)]}{f^3(t)h'(t)} \right)' - 2\frac{f'(t)Q[h(t)]}{f^3(t)} + \frac{h'(t)P[h(t)]}{f^2(t)} + \frac{f'^2R[h(t)]}{f^4(t)h'(t)} \right\} y(t)$$

This is a linear second order differential equation for the function y and it has to be identical with the equation (rqp) except a nonzero multiplicative constant. So we have

$$\frac{R[h(t)]}{f^2(t)h'(t)} = cr(t) \tag{10}$$

$$\frac{Q[h(t)]}{f^2(t)} = cq(t) \tag{11}$$

$$\left\{ \frac{f'(t)R[h(t)]}{f^3(t)h'(t)} \right\}' - 2\frac{f'(t)Q[h(t)]}{f^3(t)} + \frac{h'(t)P[h(t)]}{f^2(t)} +$$
$$\frac{f'^2(t)R[h(t)]}{f^4(t)h'(t)} = cp(t), \tag{12}$$

where $c \neq 0$ is an appropriate constant.
After rearrangement of (10) and (11) we get

$$cf^2(t) = \frac{R[h(t)]}{r(t)h'(t)} \tag{10'}$$

$$cf^2(t) = \frac{Q[h(t)]}{q(t)}. \tag{11'}$$

From here we receive the condition (2).
As $f \in C^{(2)}(i)$ we have by differentiating the identity (10′) that

$$2cf(t)f'(t) = \left(\frac{R[h(t)]}{r(t)h'(t)} \right)'$$

and from here and (10′) we get

$$\frac{f'(t)}{f(t)} = \frac{1}{2} \frac{r(t)h'(t)}{R[h(t)]} \left(\frac{R[h(t)]}{r(t)h'(t)} \right)'. \tag{13}$$

If we substitute for f/f' and f^2 in (12) we have

$$\left\{ \frac{1}{2}\, \frac{R[h(t)]}{h'(t)}\, \frac{r(t)h'(t)}{R[h(t)]}\left(\frac{R[h(t)]}{r(t)h'(t)}\right)'\, \frac{cr(t)h'(t)}{R[h(t)]} \right\}' -$$

$$2\frac{1}{2}\, \frac{r(t)h'(t)}{R[h(t)]}\left(\frac{R[h(t)]}{r(t)h'(t)}\right)' \frac{cr(t)h'(t)}{R[h(t)]} Q[h(t)] + \frac{cr(t)h'^2(t)}{R[h(t)]} P[h(t)] +$$

$$\frac{1}{4}\, \frac{r^2(t)h'^2(t)}{R^2[h(t)]}\left(\frac{R[h(t)]}{r(t)h'(t)} \right)'^2 \frac{cr(t)h'(t)}{h'(t)R[h(t)]}\, R[h(t)] = cp(t)$$

and after rearrangement

$$\frac{1}{2}\left\{ \frac{r^2(t)h'(t)}{R[h(t)]}\left(\frac{R[h(t)]}{r(t)h'(t)}\right)' \right\}' + \frac{1}{4}\, r(t)\left(\frac{r(t)h'(t)}{R[h(t)]}\right)^2\left(\frac{R[h(t)]}{r(t)h'(t)}\right)'^2 +$$

$$\frac{r(t)h'^2(t)P[h(t)]}{R[h(t)]} - \frac{r^2(t)h'^2(t)Q[h(t)]}{R^2[h(t)]}\left(\frac{R[h(t)}{r(t)h'(t)} \right)' = p(t)$$

And this is the equation (1) for the parametrization h.
From (10) yields immidiately that the multiplier f is given by the formula (3).

Conversely, let functions f,h satisfy the equations (1),(3). We will show that the transformation [f,h] transforms the equation (RQP) into the equation (rqp) if the coefficient q is given by the formula (2).
The validity of the formulas (11), (10) and consequently (13) follows from (2) and (3). From (1) we get (12) by the help of (10).
Let Y be a solution of (RQP). We will show that the function y given by (4) is a solution of (rqp). From the transformation equation (4) and its derivative (5) we obtain

$$Y[h(t)] = \frac{y(t)}{f(t)} \tag{4'}$$

$$Y'[h(t)] = \frac{1}{f(t)h'(t)}\, y'(t) - \frac{f'(t)}{f^2(t)h'(t)}\, y(t) \tag{5'}$$

If we multiply both sides of (5') by R(h)/f we have the equality (6). If we add (7) after rearrangement we get

$$\frac{R[h(t)]}{f^2(t)h'(t)}\, y'(t) + \frac{Q[h(t)]}{f^2(t)}\, y(t) =$$

$$\frac{R[h(t)]f'(t)}{f^3(t)h'(t)}\, y(t) + \frac{1}{f(t)}\, \{R[h(t)]Y'[h(t)] + Q[h(t)]Y[h(t)]\},$$

which is the equation (8) and from it we get (9). From here by help of (10), (11) and (12) we have

$\{ r(t)y'(t) + q(t)y(t)\}' - \{q(t)y'(t) + p(t)y(t)\} = 0$

or $y(t)$ is a solution of the equation (rqp).

Remark 1. In assuming that $r \in C^{(2)}(i)$, $R \in C^{(2)}(I)$ then the parametrization h satisfies in i the nonlinear differential equation of the third order

$$-\{h,t\} + \left(-\frac{1}{2}\frac{R''[h(t)]}{R[h(t)]} - \frac{1}{4}\frac{R'^2[h(t)]}{R^2[h(t)]} + \frac{P[h(t)]}{R[h(t)]} + \frac{Q[h(t)]R'[h(t)]}{R^2[h(t)]}\right) h'^2(t) + \frac{1}{2}\frac{Q[h(t)]}{R[h(t)]} = \tag{14}$$

$$-\frac{1}{2}\frac{r''(t)}{r(t)} - \frac{1}{4}\frac{r'^2(t)}{r^2(t)} + \frac{p(t)}{r(t)} + \frac{q(t)r'(t)}{r^2(t)},$$

where the symbol $\{h,t\} = \frac{1}{2}\frac{h'''(t)}{h'(t)} - \frac{3}{4}\frac{h''^2}{h'^2}$ and denotes the Schwarzian derivative of function h.

Indeed, if we set $X = R(h)/r$ we get from (10) that $cf^2=X/h'$. By differentiating this equation we get

$$2cf(t)f'(t) = \frac{X'[h(t)]}{h'(t)} - X[h(t)]\frac{h''(t)}{h'^2(t)}$$

and from the foregoing relations we obtain

$$2\frac{f'(t)}{f(t)} = \frac{X'[h(t)]}{X[h(t)]} - \frac{h''(t)}{h'(t)}.$$

On substituting for f'/f and f^2 into (12) we have

$$\frac{1}{2}\left\{\left(\frac{X'[h(t)]}{X[h(t)]} - \frac{h''(t)}{h'(t)}\right) c\,\frac{h'(t)}{X[h(t)]}\frac{R[h(t)]}{h'(t)}\right\}' -$$

$$2\,\frac{1}{2}\left(\frac{X'[h(t)]}{X[h(t)]} - \frac{h''(t)}{h'(t)}\right) c\,\frac{h'(t)}{X[h(t)]}\,Q[h(t)] + c\,\frac{h'^2(t)P[h(t)]}{X[h(t)]} +$$

$$\frac{1}{4}\left(\frac{X'[h(t)]}{X[h(t)]} - \frac{h''(t)}{h'(t)}\right)^2 c\,\frac{h'(t)}{X[h(t)]}\frac{R[h(t)]}{h'(t)} = cp(t)$$

On substituting the expression Xr for $R(h)$ we get

$$-\frac{1}{2}\frac{h'''(t)}{h'(t)} + \frac{3}{4}\frac{h''^2(t)}{h'^2(t)} + \frac{P[h(t)]}{r(t)X[h(t)]}h'^2(t) -$$

$$\frac{1}{2}\left(\frac{r'(t)}{r(t)}+\frac{X'[h(t)]}{X[h(t)]}-\frac{h'(t)Q[h(t)]}{r(t)X[h(t)]}\right)\frac{h''(t)}{h'(t)}+\frac{1}{2}\left(\frac{X'[h(t)]}{X[h(t)]}\right)'+$$

$$\frac{1}{2}\frac{r'(t)}{r(t)}\frac{X'[h(t)]}{X[h(t)]}+\frac{h'(t)Q[h(t)]}{r(t)X[h(t)]}\frac{X'[h(t)]}{X[h(t)]}+\frac{1}{4}\frac{X'^2[h(t)]}{X^2[h(t)]}=\frac{p(t)}{r(t)},$$

where $X=R(h)/r$.

From here

$$X'[h(t)]=\frac{R'[h(t)]h'(t)}{r(t)}-\frac{r'(t)R[h(t)]}{r^2(t)},$$

$$\frac{X'[h(t)]}{X[h(t)]}=\frac{R'[h(t)]}{R[h(t)]}h'(t)-\frac{r'(t)}{r(t)}.$$

Inserting for X and X' into the foregoing equation we get after rearragement (14), which is the assartion of our theorem.

Remark 2. Setting $Q=0$ we have $q=0$ and the equations (RQP) resp. (rqp) go over into linear differential equations of the second order of Sturm form [3] and we get this theorem:

Let

$$(r(t)u')'-p(t)u=0 \tag{rp}$$

$$(R(T)U')'-P(T)U=0 \tag{RP}$$

be linear second order differential equations of Sturm form, where $r,p\in C^{(0)}(j)$, $r\neq 0$, $R,P\in C^{(0)}(J)$, $R\neq 0$.
The differential equation (RP) is transformed into the differential equation (rp) by the transformation $[f,h]$ if and only if the parametrization h satisfies in i, $i\subset j$, the nonlinear differential equation of the third order

$$\frac{1}{2}\left\{\frac{r^2(t)h'(t)}{R[h(t)]}\left(\frac{R[h(t)]}{r(t)h'(t)}\right)'\right\}'+\frac{1}{4}r(t)\left(\frac{r(t)h'(t)}{R[h(t)]}\right)^2\left(\frac{R[h(t)]}{r(t)h'(t)}\right)'^2+$$

$$\frac{r(t)h'^2(t)P[h(t)]}{R[h(t)]}-p(t)=0,$$

where $\frac{R[h(t)]}{r(t)h'(t)}\in C^{(1)}(i)$, $\frac{R[h(t)]f'(t)}{f^3(t)h'(t)}\in C^{(1)}(i)$ and the multiplier f is given by the formula

$$f(t)=k\sqrt{\left|\frac{R[h(t)]}{r(t)h'(t)}\right|},\ k\neq 0,\ t\in i.$$

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