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NOTE TO THE EXISTENCE OF PERIODIC SOLUTIONS FOR HIGHER-ORDER DIFFERENTIAL EQUATIONS WITH NONLINEAR RESTORING TERM AND TIME-VARIABLE COEFFICIENTS

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Abstract: The existence of ω-periodic solutions to equation (1) is proved for n = 2,...,8.

Key words: Periodic solution, Leray-Schauder continuation theorem.

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Introduction

In order to generalize the results from [2], [3] the first author has found in [1] the sufficient conditions for the existence of an ω-periodic solution of the equation

$$x^{(n)} + \sum_{j=1}^{n-1} a_j(t)x^{(n-j)}(t) + h(x) = p(t) , \quad n > 1 \qquad (1)$$

where $a_j(t) \in C^j(R)$, $j = 1,\dots,n-1$, $p(t) \in C(R)$ are ω-periodic functions, $h(x) \in C^1(R)$.

The purpose of this paper is a further improvement of the mentioned results, including [1], for a concrete $n = 2,\dots,8$. The proving technique consists as well as in [1] in the application of the following standard Leray-Schauder alternative.

Proposition. Equation (1) admits an ω-periodic solution if all ω-periodic solutions $x(t)$ of the one-parametric system

$$x^{(n)} + \mu\left\{\sum_{j=1}^{n-1} a_j(t)x^{(n-j)}(t) + h(x) - p(t)\right\} + (1-\mu)cx = 0 , \qquad (2)$$

where $\mu \in (0,1\rangle$ is a parameter and $c \neq 0$ is a suitable real constant, are uniformly a priori bounded together with their derivatives up to the (n-1)-th order including, independently of the parameter μ, and the equation

$$x^{(n)} + cx = 0 ,$$

originated from (2) for $\mu = 0$, has no nontrivial ω-periodic solution.

Let us note that the last requirement can be always satisfied for a sufficiently small $c \neq 0$.

For the L_2-estimates desired we use the well-known Schwarz inequality jointly with the Wirtinger inequality (see [3])

$$\int_0^{\omega} [x^{(j)}(t)]^2 dt \leq \omega_0^2 \int_0^{\omega} [x^{(j+1)}(t)]^2 dt , \quad j = 1,\dots,n-1, \qquad (W)$$

$$\omega_0 = \frac{\omega}{2\pi} ,$$

holding for a continuously up to the (j+1)th-order differentiable ω-periodic function $x(t)$.

The further basic tool we employ in the proof of the main statement are the following integral identities obtained after the integration by parts of the single terms of the sum in (1).

Lemma. The following integral identities are satisfied for every ω-periodic solution $x(t)$ of (2):

$$\int_0^{\omega} a_1(t)x^{(n-1)}(t)x^{(n)}(t)dt = -\frac{1}{2}\int_0^{\omega} a_1'(t)[x^{(n-1)}(t)]^2 dt , \qquad (3)$$

$$\int_0^{\omega} a_2(t)x^{(n-2)}(t)x^{(n)}(t)dt = \frac{1}{2}\int_0^{\omega} a_2''(t)[x^{(n-2)}(t)]^2dt -$$

$$- \int_0^{\omega} a_2(t)[x^{(n-1)}(t)]^2dt , \qquad (4)$$

$$\int_0^{\omega} a_3(t)x^{(n-3)}(t)x^{(n)}(t)dt = -\frac{1}{2}\int_0^{\omega} a_3'''(t)[x^{(n-3)}(t)]^2dt +$$

$$+ \frac{3}{2}\int_0^{\omega} a_3'(t)[x^{(n-2)}(t)]^2dt , \qquad (5)$$

$$\int_0^{\omega} a_4(t)x^{(n-4)}(t)x^{(n)}(t)dt = \frac{1}{2}\int_0^{\omega} a_4^{(4)}(t)[x^{(n-4)}(t)]^2dt -$$

$$- 2\int_0^{\omega} a_4''(t)[x^{(n-3)}(t)]^2dt +$$

$$+ \int_0^{\omega} a_4(t)[x^{(n-2)}(t)]^2dt , \qquad (6)$$

$$\int_0^{\omega} a_5(t)x^{(n-5)}(t)x^{(n)}(t)dt = -\frac{1}{2}\int_0^{\omega} a_5^{(5)}(t)[x^{(n-5)}(t)]^2dt +$$

$$+ \frac{5}{2}\int_0^{\omega} a_5'''(t)[x^{(n-4)}(t)]^2dt -$$

$$- \frac{5}{2}\int_0^{\omega} a_5'(t)[x^{(n-3)}(t)]^2dt , \qquad (7)$$

$$\int_0^{\omega} a_6(t)x^{(n-6)}(t)x^{(n)}(t)dt = \frac{1}{2}\int_0^{\omega} a_6^{(6)}(t)[x^{(n-6)}(t)]^2dt -$$

$$- 3\int_0^{\omega} a_6^{(4)}(t)[x^{(n-5)}(t)]^2dt +$$

$$+ \frac{5}{2}\int_0^{\omega} a_6''(t)[x^{(n-4)}(t)]^2dt -$$

$$- \int_0^{\omega} a_6(t)[x^{(n-3)}(t)]^2 dt \,, \tag{8}$$

$$\int_0^{\omega} a_7(t)x^{(n-7)}(t)x^{(n)}(t)dt = -\frac{1}{2}\int_0^{\omega} a_7^{(7)}(t)[x^{(n-7)}(t)]^2 dt + \frac{7}{2}\int_0^{\omega} a_7^{(5)}(t)[x^{(n-6)}(t)]^2 dt - 7\int_0^{\omega} a_7'''(t)[x^{(n-5)}(t)]^2 dt + \frac{7}{2}\int_0^{\omega} a_7'(t)[x^{(n-4)}(t)]^2 dt \,. \tag{9}$$

Proof - is trivial.

Notation:

$$\left.\begin{aligned}
A_{n1} &= \max_{t\in\langle 0,\omega\rangle} \left\{-\frac{1}{2} a_1'(t) - a_2(t)\right\} && \text{for } n = 3,\dots,8 \\
A_{42} &= \max_{t\in\langle 0,\omega\rangle} \left\{-\frac{1}{2} a_2''(t) + \frac{3}{2} a_3'(t)\right\} \\
A_{n2} &= \max_{t\in\langle 0,\omega\rangle} \left\{A_{42} + a_4(t)\right\} && \text{for } n = 5,\dots,8 \\
A_{53} &= \max_{t\in\langle 0,\omega\rangle} \left\{-\frac{1}{2} a_3'''(t) - 2a_4''(t)\right\} \\
A_{63} &= \max_{t\in\langle 0,\omega\rangle} \left\{A_{53} - \frac{5}{2} a_5'(t)\right\} \\
A_{64} &= \max_{t\in\langle 0,\omega\rangle} \left\{\frac{1}{2} a_4^{(4)}(t) + \frac{5}{2} a_5^{(3)}(t)\right\} \\
A_{n3} &= \max_{t\in\langle 0,\omega\rangle} \left\{A_{63} - a_6(t)\right\} && \text{for } n = 7,8 \\
A_{74} &= \max_{t\in\langle 0,\omega\rangle} \left\{A_{64} + \frac{5}{2} a_6''(t)\right\}
\end{aligned}\right\} \tag{10}$$

$$A_{84} = \max_{t\in\langle 0,\omega\rangle} \left\{A_{74} + \frac{7}{2}\, a_7'(t)\right\}$$

$$A_{75} = \max_{t\in\langle 0,\omega\rangle} \left\{-\frac{1}{2}\, a_5^{(5)}(t) - 3\, a_6^{(4)}(t)\right\}$$

$$A_{85} = \max_{t\in\langle 0,\omega\rangle} \left\{A_{75} - 7\, a_7^{(3)}(t)\right\}$$

$$A_{86} = \max_{t\in\langle 0,\omega\rangle} \left\{\frac{1}{2}\, a_6^{(6)}(t) + \frac{7}{2}\, a_7^{(5)}(t)\right\}$$

$$A_{n,n-1} = \max_{t\in\langle 0,\omega\rangle} \left\{\frac{1}{2}(-1)^{n-1}\, a_{n-1}^{(n-1)}(t)\right\} \quad \text{for } n = 2,\dots,8$$

and for $n\in\{2,\dots,8\}$, $1 \le j \le n-1$:

$$A_{nj}^{+} = \begin{cases} A_{nj} & \text{if } A_{nj} > 0 \\ 0 & \text{if } A_{nj} \le 0 . \end{cases}$$

Theorem. Let

(i) $\int_0^{\omega} p(t)dt = 0$,

(ii) $\exists (H' \ge 0,\ H'\text{-const.}): |h'(x)| \le H'$ for all $x \in R$,

(iii) $\Omega_n := \left\{1 - \sum_{j=1}^{n-1} A_{nj}^{+}\, \omega_0^{2j} - H'\omega_0^{n}\right\} > 0$, $n\in\{2,\dots,8\}$,

(iv) $\exists (R > 0,\ R\text{-const.}) : h(x)\operatorname{sgn} x > h$ or $h(x)\operatorname{sgn} x < -h$

for $|x| > R$, where $h := \sum_{j=1}^{n-1} A_j D_{n,n-j}^{(n-j)} > 0$

with $|a_j(t)| \le A_j$, $A_j \ge 0$ for $j = 1,\dots,n-1$; $n\in\{2,\dots,8\}$,

$D_{nj}^{(j)} := \sqrt{\omega}\, D_{n,j+1}$, $D_{nj} := \omega_0 D_{n,j+1} > 0$, $D_{nn} = \dfrac{P\sqrt{\omega}}{\Omega_n}$,

$P := \max_{t\in\langle 0,\omega\rangle} |p(t)|$.

Then equation (1) for $n = 2,\dots,8$ admits an ω-periodic solution.

P r o o f . Let $x(t)$ be a solution of (2), where $x^{(j)}(0) = x^{(j)}(\omega)$, $j = 0,1,2,\dots,n-1$. Substituting $x(t)$ into (2), multiplying it by $x^{(n)}(t)$ and integrating the obtained identity from 0 to ω, we come to

$$\int_0^{\omega}[x^{(n)}(t)]^2dt = -\mu\Big\{\sum_{j=1}^{n-1}\int_0^{\omega}a_j(t)x^{(n-j)}(t)x^{(n)}(t)dt\Big\} - \mu\int_0^{\omega}h(x(t))x^{(n)}(t)dt + \mu\int_0^{\omega}p(t)x^{(n)}(t)dt - (1-\mu)c\int_0^{\omega}x(t)x^{(n)}(t)dt .$$

After integrating by parts we get by means of Lemma, (W), the Schwarz inequality and (ii), (iv) [cf. notation (10)] that

$$\int_0^{\omega}[x^{(n)}(t)]^2dt \le \Big(\sum_{j=1}^{n-1}A_{nj}^{+}\,\omega_0^{2j} + H'\omega_0^{n} + \frac{1+(-1)^n}{2}|c|\Big)\int_0^{\omega}[x^{(n)}(t)]^2dt + P\sqrt{\omega}\sqrt{\int_0^{\omega}[x^{(n)}(t)]^2dt} ,$$

i.e. [see (iii)]

$$\int_0^{\omega}[x^{(n)}(t)]^2dt < \frac{P^2\omega}{\Omega_n^2} := D_{nn}^2 , \quad D_{nn} := \frac{P\sqrt{\omega}}{\Omega_n} \ (>0) .$$

Applying (W) again, we come to

$$\int_0^{\omega}[x^{(j)}(t)]^2dt \le D_{nj}^2 , \quad D_{nj} := \omega_0 D_{n,j+1} \ (>0)$$

for $j = 1,\dots,n-1$; $n\in\{2,\dots,8\}$.

According to Rolle's theorem, the points $t_j \in (0,\omega)$,

$j = 1,\dots,n-1$, exist such that $x^{(j)}(t_j) = 0$ for $j = 1,\dots,n-1$, and thus we arrive at the inequalities

$$|x^{(j)}(t)| \le \int_0^\omega |x^{(j+1)}(t)|\,dt \le \sqrt{\omega}\, D_{n,j+1} := D_{nj}^{(j)} \ (>0) \tag{11}$$

$j = 1,\dots,n-1$.

Now substituting $x(t)$ into (2) and integrating from 0 to ω, we obtain [cf. (i)]

$$\int_0^\omega [\mu h(x(t)) + (1-\mu)cx(t)]\,dt = -\mu \sum_{j=1}^{n-1} \int_0^\omega a_j(t) x^{(n-j)}(t)\,dt.$$

If $\min_{t\in\langle 0,\omega\rangle} |x(t)| > R$, $R > 0$ const., then choosing $c \ne 0$ in order to be [see (iv)]

$$ch(x)x > 0$$

we get by means of (iv) and (11) that

$$\int_0^\omega |h(x(t))|\,dt \le \sum_{j=1}^{n-1} \int_0^\omega |a_j(t) x^{(n-j)}(t)|\,dt \le \omega \sum_{j=1}^{n-1} A_j D_{n,n-j}^{(n-j)} := \omega h,$$

when multiplying the foregoing identity by $\operatorname{sgn}(cx)$. But this leads to the contradiction with assumption (iv). Therefore it must be

$$\min_{t\in\langle 0,\omega\rangle} |x(t)| \le R$$

and consequently

$$|x(t)| \le R + \int_0^\omega |x'(t)|\,dt \le R + \sqrt{\omega}\, D_{n2} := D_{n1} \ (>0) \tag{12}$$

for $t \in \langle 0,\omega\rangle$.

It follows from (11) and (12) that

$$\sum_{j=0}^{n-1} |x^{(j)}(t)| \le \sum_{j=1}^{n-1} D_{nj}^{(j)} + D_{n1} := D > 0$$

holds for every ω-periodic solution x(t) of (2), independently of $\mu \in (0,1\rangle$.

Hence, in view of Proposition, the proof of our Theorem is completed.

Concluding remark.

It could be expected, in view of the obtained result, that Theorem may be analogously extended for the nth-order equation, where the integral identities from Lemma obey in the case of the odd order (j = 2k-1, k $\in$ N) the relation

$$\int_0^{\omega} a_{2k-1}(t)x^{(n-2k+1)}(t)x^{(n)}(t)dt = -\frac{1}{2}\int_0^{\omega} a_{2k-1}^{(2k-1)}(t)[x^{(n-2k+1)}(t)]^2 dt + \\ + \dots + (-1)^k \frac{2k-1}{2}\int_0^{\omega} a_{2k-1}'(t)[x^{(n-k)}(t)]^2 dt$$

(k terms),

while for the even order (j = 2k, k $\in$ N) it is

$$\int_0^{\omega} a_{2k}(t)x^{(n-2k)}(t)x^{(n)}(t)dt = \frac{1}{2}\int_0^{\omega} a_{2k}^{(2k)}(t)[x^{(n-2k)}(t)]^2 dt - \\ - k\int_0^{\omega} a_{2k}^{(n-2k-2)}(t)[x^{(n-2k+1)}(t)]^2 dt + \\ + \dots + (-1)^k \int_0^{\omega} a_{2k}(t)[x^{(n-k)}(t)]^2 dt$$

(k+1 terms)

in general.

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