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A USE OF THE LAPLACE METHOD IN DIFFERENTIAL EQUATIONS

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This paper is practically the continuation of paper |3|. The Laplace method and the Kummer transformation are applied on the certain class of linear partial differential equations of the n-order, which is greater then that one in the above introduced paper.

Introduction

We will study the equation

$$\sum_{k=0}^{n} a_k(t,y) \frac{\partial^k u}{\partial t^k} = \sum_{p=0}^{m} b_p(y) \frac{\partial^p u}{\partial y^p} + q(t,y) \qquad (1)$$

in interval $(t,y) \in J_1 \times J_2$, where $J_1 = \langle 0,\infty)$ and $J_2 = \langle y_0,\infty)$, y_0 is a real number, with initial and boundary conditions

$$\frac{\partial^k u}{\partial t^k}(0,y) = f_k(y) \qquad (k = 0,1,\dots,n-1) \tag{2}$$

$$\frac{\partial^p u}{\partial y^p}(t,y_0) = g_p(t) \qquad (p = 0,1,\dots,m-1) \tag{3}$$

where m and n are natural numbers.

Notation and assumptions

$C^{(n,m)}(J_1,J_2)$ will be the set of all continues **functions** $f(x,y)$ with their continues derivatives up to the order n regarding x and up to the order m regarding y. Analogically $C^{(n)}(J)$ will be the set of all continues functions with their continues derivatives up to the order n; $C(J) = C^{(0)}(J)$.

Let us suppose $B_i(y) \in C(J_2)$, $B_n(y)\, a_n(t,y) > 0$, $a_n(t,y) \in C^{(n,0)}(J_1,J_2)$, $a_{n-1}(t,y) \in C^{(n-1,0)}(J_1,J_2)$ for every $(t,y) \in J_1 \times J_2$. Denote

$$x(t,y) = \int_0^t \left(\frac{B_n(y)}{a_n(s,y)} \right)^{\frac{1}{n}} ds , \tag{4}$$

and $t(x,y)$ is an inverse function of the function $x(t,y)$ for every $y \in J_2$. Further, let us denote

$$P(x,y) = \int_0^x \left(- \frac{B_{n-1}(y)}{B_n(y)} + \frac{1}{nB_n(y)} \left(\frac{B_n(y)}{a_n(t,y)}\right)^{\frac{n-1}{n}} \cdot \left(- \frac{n-1}{2} \frac{\partial}{\partial t} a_n(t,y) + a_{n-1}(t,y)\right)\right) dx \tag{5}$$

and

$$A_k(x,y) = \frac{1}{k!} \sum_{i=k}^{n} T_{ki}(t(x,y))\, a_i(t(x,y),y)$$

where

$$T_{ki}(t(x,y)) = \frac{\partial^i}{\partial r^i}\left[(x(t(x,y)+r,y) - x(t(x,y)))^k\right]_{r=0}$$

for $k = 0,1,\dots,n$.

Lut us suppose, that

$$\lim_{x\to\infty} t(x,y) = \infty$$

for every $y \in J_2$ and there exist Laplace images of functions

$$e^{P(x,y)}q(t(x,y),y),\ e^{P(x,y)}g_p(t(x,y),y)$$

for every $y \in J_2$, $p = 0,1,\dots,m-1$. Let us denote them

$$Q(\lambda,y),\ G_p(\lambda,y).$$

Denote also

$$S(\lambda,y) = \sum_{k=0}^{n}\sum_{j=0}^{k}\Big[\frac{\partial^j}{\partial x^j}\left[e^{P(x,y)-\lambda x} A_k(x,y)\right] (-1)^{j+1}$$

$$\sum_{i=0}^{k-j-1} f_i(y)\frac{1}{i!}\frac{\partial^{k-j-1}}{\partial r^{k-j-1}}\left[(t(x+r,y)-t(x,y))^i\right]\Big]_{\substack{x=0\\ r=0}} - Q(\lambda,y)$$

We will require that

$$a_k(t,y) = \det H_k \tag{6}$$

where $H_k = (h^k_{ij})^{n-k+1}_{i,j=1}$

$$
h_{ij}^{k} = \begin{cases} \dfrac{\dfrac{\partial^{n-j+1}}{\partial r^{n-j+1}}\left[(x(t+r,y)-x(t,y))^{n-i+1}\right]_{r=0}}{(n-i+1)!\ \left(\dfrac{\partial x(t,y)}{\partial t}\right)^{n-i+1}} & i \geq j,\ j \neq n-k+1 \\ 0 & i < j,\ j \neq n-k+1 \\ \dfrac{\sum\limits_{s=n-i+1}^{n} (n-s+1)B_s (e^{P(x,y)})^{(s-n+i-1)}}{e^{P(x,y)} \left(\dfrac{\partial x(t,y)}{\partial t}\right)^{n-i+1}} & j = n-k+1 \end{cases}
$$

As in paper [3] we can assure, that these assumptions agree with (4) and (5).

Equivalent equation

Let the equation

$$
\left(\sum_{i=0}^{n} B_i(y)\,\lambda^i\right)v(\lambda,y) + S(\lambda,y) = \sum_{p=0}^{m} b_p(y)\,\frac{\partial^p v}{\partial y^p} \qquad (7)
$$

is defined in intervals $\lambda \in J_1$, $y \in J_2$ with boundary conditions

$$
\frac{\partial^p v}{\partial y^p}(\lambda,y_0) = G_p(\lambda,y_0) \qquad (p = 0,1,\ldots,m-1) \qquad (8)
$$

Definition: $u(t,y)$ is a solution of equation (1) if and only if $u(t,y)$ satisfies equation (1) everywhere and $u(t,y) \in C^{(n,m)}(J_1,J_2)$. Analogically $v(\lambda,y)$ is a solution of equation (7) if and only if $v(\lambda,y)$ satisfies equation (7) everywhere and $v(\lambda,y) \in C^{(0,m)}((0,\infty),J_2)$.

Let u(t,y) is the function such that

$u(t,y) \in C^{(n,m)}(J_1,J_2)$.

Let us suppose, that

$$\lim_{x\to\infty} \frac{\partial^j}{\partial x^j} (e^{P(x,y)-\lambda x} A_k(x,y)) \frac{\partial^{k-j-1} u}{\partial x^{k-j-1}} = 0$$

for every $k = 0,1,\dots,n$, $j = 0,1,\dots,n$, $k > j$ and

$$\sum_{k=0}^{m} b_k(y) \frac{\partial^k}{\partial y^k} \int_0^{\infty} e^{P(x,y)-\lambda x} u(t(x),y)dx =$$

$$= \int_0^{\infty} \sum_{k=0}^{m} b_k(y) e^{P(x,y)-\lambda x} \frac{\partial^k}{\partial y^k} u(t(x),y)dx \qquad (9)$$

hold for every $\lambda \in R^+$. Further, let us suppose, that there exist Laplace images of functions

$$A_k(x,y) e^{P(x,y)} \frac{\partial^k u}{\partial x^k}$$

for every $k = 0,1,\dots,n$, $y \in J_2$.

Let the function u(t,y) satisfies condition (3).

Lemma : Let us suppose above introduced assumptions. Then for every $k = 0,1,\dots,n$ hold

$$A_k(x,y) = e^{-P(x,y)} \sum_{i=k}^{n} \binom{i}{k} B_i(y) \frac{\partial^{i-k}}{\partial x^{i-k}} (e^{P(x,y)}).$$

The proof of this lemma is analogous to the proof of the lemma in paper [3]. We leave it out.

Main resultes

To fulfilled condition (9) it is sufficient e.g. that

$P(x,y) = P(x)$ and $t(x,y) = t(x)$. We can carry out it in several ways as we will see in following theorems.

Theorem 1 : We will suppose all introduced assumptions from the chapter "Equivalent equation" and "Notation and assumptions" and let $v(\lambda,y)$ is for every $y \in J_2$ the Laplace image of the function

$$e^{P(x,y)} u(t(x),y)$$

and also the solution of the linear differential equation (7) with boundary conditions (8). Let for functions $a_n(t,y)$ and $a_{n-1}(t,y)$ hold

$$a_n(t,y) = F(t)B_n(y)$$

$$a_{n-1}(t,y) = R(t)B_n(y)$$

where $F(t) \in C^{(n)}(J_1)$, $R(t) \in C^{(n-1)}(J_1)$.

Further let there is positive real number K such that for every $y \in J_2$ is right $B_{n-1}(y) = K.B_n(y)$. Then $u(t,y)$ is a solution of partial differential equation (1) with initial and boundary conditions (2) and (3).

The proof of this theorem is analogous to the proof of the theorem in paper [3]. We leave it out and only prove the independance of functions $P(x,y)$ and $t(x,y)$ on y.

$$x(t,y) = \int_0^t \left(\frac{B_n(y)}{a_n(s,y)}\right)^{\frac{1}{n}} ds = \int_0^t \left(\frac{B_n(y)}{B_n(y)F(s)}\right)^{\frac{1}{n}} ds =$$

$$= \int_0^t \left(\frac{1}{F(s)}\right)^{\frac{1}{n}} ds$$

It implies $t(x,y) = t(x)$.

Further

$$P(x,y) = \int_0^x \left(-\frac{B_{n-1}(y)}{nB_n(y)} + \frac{1}{nB_n(y)} \left(\frac{B_n(y)}{a_n(t,y)}\right)^{\frac{n-1}{n}}\right. .$$

$$\cdot\left(-\frac{n-1}{2}\frac{\partial}{\partial t}a_n(t,y)+a_{n-1}(t,y)\right))\,dx=$$

$$=\int_0^x\left(\frac{-K\,B_n(y)}{nB_n(y)}+\frac{1}{nB_n(y)}\left(\frac{B_n(y)}{F(t)B_n(y)}\right)^{\frac{n-1}{n}}\cdot\right.$$

$$\left.\cdot\left(-\frac{n-1}{2}F'(t)B_n(y)+R(t)B_n(y)\right)\right)dx=$$

$$=\int_0^x\left(-\frac{K}{n}+\frac{1}{n}\left(\frac{1}{F(t)}\right)^{\frac{n-1}{n}}\left(-\frac{n-1}{2}F'(t)+R(t)\right)\right)dx=P(x).$$

Theorem 2 : We will suppose all introduced assumptions from the chapter "Equivalent equation" and "Notation and assumptions" and let $v(\lambda,y)$ is for every $y\in J_2$ the Laplace image of the function

$$e^{P(x,y)}\;u(t(x),y)$$

and also the solution of linear differential equation (7) with boundary conditions (8). Let for the functions $a_n(t,y)$ and $a_{n-1}(t,y)$ hold

$$a_n(t,y)=F(t)\;B_n(y)$$

$$a_{n-1}(t,y)=F(t)^{\frac{n-1}{n}}\;\frac{1}{n}\;B_{n-1}(y)+B_n(y)\;V(t),$$

where $F(t)\in C^{(n)}(J_1)$, $V(t)\in C^{(n-1)}(J_1)$. Then $u(t,y)$ is a solution of linear partial differential equation (1) with initial and boundary conditions (2) and (3).

The proof of this Theorem is analogous to the proof of the Theorem in paper [3]. As in Theorem 1 we leave it out and only prove the independence of functions $P(x,y)$ and $t(x,y)$ on y.

$$x(t,y)=\int_0^t\left(\frac{B_n(y)}{a_n(s,y)}\right)^{\frac{1}{n}}ds=\int_0^t\left(\frac{B_n(y)}{B_n(y)F(s)}\right)^{\frac{1}{n}}ds=\int_0^t\left(\frac{1}{F(s)}\right)^{\frac{1}{n}}ds$$

it implies $t(x,y)=t(x)$.

$$P(x,y) = \int_0^x \left(-\frac{B_{n-1}(y)}{nB_n(y)} + \frac{1}{nB_n(y)}\left(\frac{B_n(y)}{a_n(t,y)}\right)^{\frac{n-1}{n}}\right.$$

$$\left.\left(-\frac{n-1}{2}\frac{\partial}{\partial t}a_n(t,y) + a_{n-1}(t,y)\right)\right) dx =$$

$$= \int_0^x \left(-\frac{B_{n-1}(y)}{nB_n(y)} + \frac{1}{nB_n(y)}\left(\frac{B_n(y)}{B_n(y)F(t)}\right)^{\frac{n-1}{n}}\right.$$

$$\left.\left(-\frac{n-1}{2}B_n(y)F'(t)+B_n(y)V(t)+F(t)^{\frac{n-1}{n}}\frac{1}{n}B_{n-1}(y)\right)\right)dx =$$

$$= \int_0^x \frac{1}{n}\left(-\frac{n-1}{2}F'(t) + V(t)\right)\left(\frac{1}{F(t)}\right)^{\frac{n-1}{n}} dx = P(x)$$

Conclusion

This method is suitable in the cases when coefficients $a_k(t,y)$ $(k = 0,1,\dots,n)$ fulfill (6) and when we know the solution of equation (7) with boundary conditions (8). Say in other way we transfer equation (1), with boundary conditions (2) and (3) to equation (7) with boundary conditions (8) using the Laplace method and Kummer transformation in the form $e^{P(x)} \cdot u(t(x),x)$. We can solved equation (7) by any numerical method and in some cases we can reached the solution in analytical form.

SOUHRN

UŽITÍ LAPLACEOVY METODY V DIFERENCIÁLNÍCH ROVNICÍCH

Článek popisuje možnost řešení lineární parciální diferenciální rovnice n-tého řádu typu

$$\sum_{k=0}^{n} a_k(t,y)\frac{\partial^k n}{\partial t^k} = \sum_{p=0}^{m} b_p(y)\frac{\partial^p n}{\partial y^p} + q(t,y)$$

pomocí kombinace Laplaceovy a Kummerovy transformace. Koeficienty $a_k(t,y)$, $b_p(y)$ ($k = 0,1,\ldots,n$, $p = 0,1,\ldots,m$) a funkce $g(t,y)$ musí splňovat ještě další předpoklady.

Touto kombinací lze lépe řešit některé typy parciálních diferenciálních rovnic než klasickou metodou.

РЕЗЮМЕ

ПРИМЕНЕНИЕ МЕТОДА ЛАПЛАСА В ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЯХ

Я. ГАНЧЛ

В статье показывается возможность решить линейные дифференциальные уравнения в частных производных n-того порядка типа

$$\sum_{k=0}^{n} a_k(t,y)\frac{\partial^k u}{\partial t^k} = \sum_{p=0}^{m} b_p(y)\frac{\partial^p u}{\partial y^p} + q(t,y)$$

при помощи комбинации преобразования Лапласа и Куммера. Коэффициенты $a_k(t,y), b_k(y) (k=0,1,\dots,n,\ p=0,1,\dots,m)$ должны удовлетворять данным условиям.

При помощи этой комбинации возможно решить некоторые типы дифференциальных уравнений в частных производных, которые трудно решить классическим методом.

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