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NOTE ON EXISTENCE OF PERIODIC SOLUTIONS TO THE THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATION

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The differential equation under consideration is

$$x''' + e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + g(t,x,x',x'')c(t,x') + h(t,x,x',x'') = 0 , \tag{1}$$

where e,f,g,h,a,b,c are continuous real functions of real variables moreover w-periodic relative to variable t. Hereafter of the functions e,f,g,a,b,c we assume that they satisfy the bounding conditions in this manner: for all $t,x,x',x'' \in (-\infty,+\infty)$ holds

$$|f(t,x,x',x'')| \leqq F , \quad \text{where} \quad F > 0 , \tag{2}$$

and

$$|g(t,x,x',x'')| \leqq G , \quad \text{where} \quad G > 0 , \tag{3}$$

for all $t,x \in (-\infty,+\infty)$ and for all $x',x'' \in (-\infty,+\infty)$:

$$|e(t,x,x',x'')| \leq E_2|x''| + E_1|x'| + E ,$$
$$\text{where } E_2 \geq 0,\ E_1 \geq 0,\ E > 0 , \tag{4}$$

for all $t, x''' \in (-\infty, +\infty)$:

$$|a(t,x''')| \leq A , \quad \text{where } A > 0 , \tag{5}$$

for all $t, x'' \in (-\infty, +\infty)$:

$$|b(t,x'')| \leq B_2|x''| + B, \quad \text{where } B_2 \geq 0,\ B > 0 , \tag{6}$$

and for all $t, x' \in (-\infty, +\infty)$:

$$|c(t,x')| \leq C_1|x'| + C, \quad \text{where } C_1 \geq 0,\ C > 0 , \tag{7}$$

holds so that it may be written

$$|e(t,x,x',x'')a(t,x''')| \leq M_2|x''| + M_1|x'| + M , \tag{8}$$
$$\text{where } M_2 = E_2A \geq 0,\ M_1 = E_1A \geq 0,\ M = EA > 0 ,$$

$$|f(t,x,x',x'')b(t,x'')| \leq N_2|x''| + N, \tag{9}$$
$$\text{where } N_2 = FB_2 \geq 0,\ N = FB > 0 ,$$

$$|g(t,x,x',x'')c(t,x')| \leq P_1|x'| + P, \tag{10}$$
$$\text{where } P_1 = GC_1 \geq 0,\ P = GC > 0 .$$

The function h will acquire several forms, namely $h(x) - q$, $h_1(x') + h(x) - q$, $h_2(x'') + h(x) - q$ and $h_2(x'') + h_1(x') + h(x) - q$, where on $q = q(t,x,x',x'')$ we assume that hereafter for all $t, x \in (-\infty, +\infty)$ and for all $x', x'' \in (-\infty, +\infty)$ satisfy the inequality

$$|q(t,x,x',x'')| \leq Q_2|x''| + Q_1|x'| + Q , \tag{11}$$

where $Q_2 \geq 0$, $Q_1 \geq 0$, $Q > 0$.

The assumptions on $h(x)$, $h_1(x')$ and $h_2(x'')$ or on their some generalized forms will be formulated in the following theorems. However in this paper we don't present all possible modifications of theorems relative to acceptable properties of the last functions. Similar theorems and the same sort of theorems on (1) with additional term linear to x as a special case of h(x) are formulated in [1] or [2]. In return, closing our investigations, we extend the theorem on existence of a periodic solution to (1) with a general form of h.

The sufficient condition of the existence of w-periodic solution x(t) to (1) is - according to Leray-Schauder alternative of the fixed point Theorem - that all solutions x(t) of the one-parametric system

$$x''' + m\Big\{e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + g(t,x,x',x'')c(t,x') + h(t,x,x',x'') - \sum_{j=0}^{2} k_j x^{(2-j)}\Big\} + \sum_{j=0}^{2} k_j x^{(2-j)} = 0 , \qquad (12)$$

where $m \in \langle 0,1 \rangle$ is a parameter, are together with $x'(t)$ and $x''(t)$ bounded by the same constant independent of m and the equation

$$x''' + \sum_{j=0}^{2} k_j x^{(2-j)} = 0 \qquad (13)$$

with $k_j \in R$ $(j = 0,1,2)$ has not any nontrivial w-periodic solution [this condition is satisfied e.g. taking for simplicity $k_0 = k_1 = 0$, $k_2 \neq 0$].

To prove the theorems we note that the integration on the interval $\langle t, t+w \rangle$, $t \in (-\infty, +\infty)$, is restricted hereafter on the interval $\langle 0, w \rangle$ only, the results will be the same; whereby

$$x^{(j)}(0) = x^{(j)}(w) \quad \text{for} \quad j = 0,1,2. \qquad (14)$$

The composed functions, obtained after the substitution $x^{(j)}(t)$ instead of $x^{(j)}$, $j = 0,1,2,3$, into (12), we symbolize e.g. $e[t,x(t),\dots]$ or $e(t,\dots)$ sake to brevity.

Besides the Schwarz inequality we employ also the Wirtinger inequalities

$$\int_t^{t+w} p^{(j)^2}(v)dv \leqq w_0^2 \int_t^{t+w} p^{(j+1)^2}(v)dv, \; j = 1,2, \quad w_0 = \frac{w}{2\pi} \tag{15}$$

applicable for arbitrary continuous w-periodic function $p(t)$ with the square integrable derivatives $p^{(j)2}(t)$, $j = 1,2$, on the interval $\langle t,t + w\rangle$, $t \in (-\infty,+\infty)$, on $\langle 0,w\rangle$ only.

Theorem 1. Let (2) - (7) and (11) hold in the differential equation

$$x''' + e(t,x,x',x")a(t,x''') + f(t,x,x',x")b(t,x") + $$
$$+ g(t,x,x',x")c(t,x') + h(x) = q(t,x,x',x") \; . \tag{1.1}$$

Let there exist constants $k \in R - (0)$ and $\hat{H} \geqq 0$, $H > 0$ such that the inequality

$$|h(x) - kx| \leqq \hat{H}|x| + H \tag{H}$$

is satisfied for all $x \in (-\infty,+\infty)$. If

$$(M_2 + N_2 + Q_2)w_0 + (M_1 + P_1 + Q_1)w_0^2 < 1 \tag{R}$$

and

$$\hat{H} < |k| \; , \tag{R_0}$$

then (1.1) has a w-periodic solution.

P r o o f : Substituting $x^{(j)}(t)$ instead of $x^{(j)}$, $j = 1,2, 3$, into the system

$$x''' + m\{e(t,x,x',x")a(t,x''') + f(t,x,x',x")b(t,x") + $$
$$+ g(t,x,x',x")c(t,x') + h(x) - kx - q(t,x,x',x")\} + $$
$$+ kx = 0 \; , \tag{12_1}$$

where $m \in \langle 0,1\rangle$ is a parameter and $k \in R$, $k \neq 0$, a suitable fixed constant, multiplying the arised identity by $x'(t)$ and integrating, we go to

$$\int_0^w x''^2(t)dt = m\left\{\int_0^w e(t,\ldots)a[t,x'''(t)]x'(t)dt + \right.$$

$$+ \int_0^w f(t,\ldots)b[t,x''(t)]x'(t)dt +$$

$$+ \int_0^w g(t,\ldots)c[t,x'(t)]x'(t)dt -$$

$$\left. - \int_0^w q(t,\ldots)x'(t)dt\right\}, \quad \text{since}$$

$$\int_0^w x'''(t)x'(t)dt = -\int_0^w x''^2(t)dt \text{ and } \int_0^w h[x(t)]x'(t)dt =$$

$$= \int_0^w x(t)x'(t)dt = 0$$

with respect to (14). Regarding (8) - (11) and using (15) we get successively

$$\left|\int_0^w e(t,\ldots)a[t,x'''(t)]x'(t)dt\right| \leqq$$

$$\leqq (M_2 w_0 + M_1 w_0^2)\int_0^w x''^2(t)dt + M\sqrt{w}\, w_0 \sqrt{\int_0^w x''^2(t)dt} \ ,$$

$$\left|\int_0^w f(t,\ldots)b[t,x''(t)]x'(t)dt\right| \leqq N_2 w_0 \int_0^w w''^2(t)dt +$$

$$+ N\sqrt{w}\, w_0 \sqrt{\int_0^w x''^2(t)dt} \ ,$$

$$\left|\int_0^w g(t,\dots)c[t,x'(t)]x'(t)dt\right| \leqq P_1 w_o^2 \int_0^w x''^2(t)dt +$$

$$+ P\sqrt{w}\ w_o \sqrt{\int_0^w x''^2(t)dt} \ ,$$

$$\left|\int_0^w q(t,\dots)x'(t)dt\right| \leqq (Q_2 w_o + Q_1 w_o^2) \int_0^w x''^2(t)dt +$$

$$+ Q\sqrt{w}\ w_o \sqrt{\int_0^w x''^2(t)dt} \ ,$$

so that

$$\int_0^w x''^2(t)dt \leqq [(M_2 + N_2 + Q_2)w_o +$$

$$+ (M_1 + P_1 + Q_1)w_o^2] \int_0^w x''^2(t)dt +$$

$$+ (M + N + P + Q)\sqrt{w}\ w_o \sqrt{\int_0^w x''^2(t)dt} \ .$$

Denoting $K := \{1 - [(M_2 + N_2 + Q_2)w_o + (M_1 + P_1 + Q_1)w_o^2]\}$, then with respect to (R) we have

$$\int_0^w x''^2(t)dt \leqq D_2^2 \ , \ \text{where } D_2 := \frac{1}{K}(M+N+P+Q)\sqrt{w}\ w_o > 0 \qquad (16)$$

and using (15), moreover

$$\int_0^w x'^2(t)dt \leqq D_1^2 \ , \quad \text{where } D_1 := w_o D_2 > 0 \ . \qquad (17)$$

Owing to Rolle Theorem applied on a w-periodic function $x(t)$, $t \in \langle 0,w \rangle$, then there exists a point $t_1 \in (0,w)$ such that

$x'(t_1) = 0$. According to the relation

$$\int_{t_1}^{t} x''(s)ds = x'(t) - x'(t_1) \ ,$$

where t_1, $t \in (0,w)$, holds the inequality

$$|x'(t)| = |\int_{t_1}^{t} x''(s)ds| \leqq |\int_0^w x''(t)dt| \leqq \sqrt{w}\, D_2 := D' > 0 \qquad (18)$$

for any w-periodic solution x(t) to (12_1).

Multiplying (12_1) by x(t)sgn(k) and integrating the arised identity, we have

$$|k|\int_0^w x^2(t)dt = m\ \mathrm{sgn}(k)\Big\{ - \int_0^w e(t,\dots)a[t,x'''(t)]x(t)dt -$$

$$- \int_0^w f(t,\dots)b[t,x''(t)]x(t)dt -$$

$$- \int_0^w g(t,\dots)c[t,x'(t)]x(t)dt -$$

$$- \int_0^w \{h[x(t)] - kx(t)\}\ x(t)dt + \int_0^w q(t,\dots)x(t)dt \Big\} .$$

Now regarding (8) - (11) and (H), using (16), (17) and Schwarz inequality, we get successively

$$|\int_0^w e(t,\dots)a[t,x'''(t)]x(t)dt| \leqq$$

$$\leqq (M_2D_2 + M_1D_1 + M\sqrt{w}) \sqrt{\int_0^w x^2(t)dt} \ ,$$

$$\left|\int_0^w f(t,\dots)b[t,x''(t)]x(t)dt\right| \leq (N_2D_2 + N\sqrt{w}) \sqrt{\int_0^w x^2(t)dt} \ ,$$

$$\left|\int_0^w g(t,\dots)c[t,x'(t)]x(t)dt\right| \leq (P_1D_1 + P\sqrt{w}) \sqrt{\int_0^w x^2(t)dt} \ ,$$

$$\left|\int_0^w \{ h[x(t)] - kx(t)\} \ x(t)dt\right| \leq \hat{H}\int_0^w x^2(t)dt +$$
$$+ H\sqrt{w} \sqrt{\int_0^w x^2(t)dt} \ ,$$

$$\left|\int_0^w q(t,\dots)x(t)dt\right| \leq (Q_2D_2 + Q_1D_1 + Q\sqrt{w}) \sqrt{\int_0^w x^2(t)dt} \ ,$$

so that

$$(|k| - \hat{H})\int_0^w x^2(t)dt \leq [(M_2 + N_2 + Q_2)D_2 +$$
$$+ (M_1 + P_1 + Q_1)D_1 +$$
$$+ (M + N + P + H + Q)\sqrt{w}] \sqrt{\int_0^w x^2(t)dt}$$

and denoting $K_0 = |k| - \hat{H}$, then with regard to (R_0) yields

$$\int_0^w x^2(t)dt \leq D_0^2, \quad \text{where } D_0 := \frac{1}{K_0}[(M_2 + N_2 + Q_2)D_2 +$$
$$+ (M_1 + P_1 + Q_1)D_1 +$$
$$+ (M + N + P + H + Q)\sqrt{w}] > 0 \ . \qquad (19)$$

Consequently such point $t_0 \in \langle 0,w \rangle$ exists that the inequality $|x(t_0)|\sqrt{w} \leq D_0$ holds for any w-periodic solution $x(t)$

to (12_1). Then in keeping with the relation

$$\int_{t_0}^{t} x'(s)ds = x(t) - x(t_0), \quad t, t_0 \in \langle 0, w \rangle ,$$

we have

$$|x(t)| = |x(t_0) + \int_{t_0}^{t} x'(s)ds| \leqq \frac{D_0}{\sqrt{w}} + |\int_0^w x'(t)dt| \leqq$$

$$\leqq \left(\frac{D_0}{\sqrt{w}} + \sqrt{w}\, D_1 \right) := D > 0 . \qquad (20)$$

Multiplying (12_1) by $x'''(t)$ and integrating the arised identity, we get

$$\int_0^w x'''^2(t)dt = m\Big\{ - \int_0^w e(t,\ldots)a[t,x'''(t)]x'''(t)dt -$$

$$- \int_0^w f(t,\ldots)b[t,x''(t)]x'''(t)dt -$$

$$- \int_0^w g(t,\ldots)c[t,x'(t)]x'''(t)dt -$$

$$- \int_0^w \{h[x(t)] - kx(t)\}x'''(t)dt +$$

$$+ \int_0^w q(t,\ldots)x'''(t)dt \Big\} .$$

Regarding (8) - (11) and (H) and using the Schwarz inequality with (16), (17), (19) yields

$$\int_0^w x'''^2(t)dt \leqq D_3^2 ,$$

where $D_3 := [(M_2 + N_2 + Q_2)D_2 + (M_1 + P_1 + Q_1)D_1 + (M + N + P + H + Q)\sqrt{w} + \hat{H}D_o] > 0$. (21)

Owing to Rolle´s Theorem applied now on the function $x''(t)$, $t \in \langle 0,w \rangle$, satisfying (14) then there exists a point $t_2 \in (0,w)$ such that $x''(t_2) = 0$. According to the relation

$$\int_{t_2}^{t} x'''(s)ds = x''(t) - x''(t_2) ,$$

where $t, t_2 \in (0,w)$, the inequality

$$|x''(t)| = |\int_{t_2}^{t} x'''(s)ds| \leqq |\int_0^w x'''(t)dt| \leqq$$

$$\leqq \sqrt{w}\, D_3 := D'' > 0 \qquad (22)$$

holds for any w-periodic solution $x(t)$ to (12_1).

From (18), (20) and (22) implies that any w-periodic solution $x(t)$ to (12_1) satisfy the inequality

$$|x^{(j)}(t)| \leqq \bar{D} , \quad j = 0,1,2, \qquad (23)$$

where the positive constant $\bar{D} = \max(D,D',D'')$ is independent of the parameter $m \in \langle 0,1 \rangle$. This fact together with $k \in R$, $k \neq 0$, proves the theorem.

Note : If we consider $h(t,x)$ instead of $h(x)$ in the equation (1.1) then the corresponding theorem on existence of a w-periodic solution $x(t)$ to referred equation takes $\hat{H} = 0$ in the assumption (H), i.e. (H) will be replaced by the inequality

$$|h(t,x) - kx| \leqq H, \ H > 0,$$

holding for all $t,x \in (-\infty,+\infty)$, while the assumption (R) remains.

Theorem 2. Let (2) - (7) and (11) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + $$
$$+ g(t,x,x',x'')c(t,x') + h_1(x') + h(x) = q(t,x,x',x''). \tag{1.2}$$

Let there exist constants $k \in R - (0)$ and $\hat{H} \geqq 0$, $H > 0$ such that for all $x \in (-\infty,+\infty)$ is satisfied the inequality

$$|h(x) - kx| \leqq \hat{H}|x| + H \tag{H}$$

and let for all $y \in (-\infty,+\infty)$ holds

$$h_1(y)y \leqq H_1^* := \begin{cases} 0 \text{ if } h_1(y)y \leqq 0 \\ H_1 > 0 \text{ if } 0 < h_1(y)y \leqq H_1 \end{cases} . \tag{H_1^*}$$

If

$$(M_2 + N_2 + Q_2)w_0 + (M_1 + P_1 + Q_1)w_0^2 < 1 \tag{R}$$

and

$$\hat{H} < |k| , \tag{R_0}$$

then (1.2) has a w-periodic solution.

Applying the same process of the proof as by Theorem 1 we use the inequality

$$\int_0^w h_1[x'(t)]x'(t)dt \leqq H_1^* w$$

holding with regard to (H_1^*) and in keeping (8) - (11) we go to

$$\int_0^w x''^2(t)dt \leqq [(M_2 + N_2 + Q_2)w_0 + $$
$$+ (M_1 + P_1 + Q_1)w_0^2]\int_0^w x''^2(t)dt + $$
$$+ (M + N + P + Q)\sqrt{w}\, w_0 \sqrt{\int_0^w x''^2(t)dt} + H_1^* w .$$

Denoting $K := \left\{1 - [(M_2 + N_2 + Q_2)w_o + (M_1 + P_1 + Q_1)w_o^2]\right\}$, then with respect to (R) we have

$$\left(\sqrt{\int_0^w x''^2(t)dt} - \frac{M+N+P+Q}{2K}\sqrt{w}\, w_o\right)^2 \leqq \frac{H_1^* w}{K} + \left(\frac{M+N+P+Q}{2K} w_o\right)^2 w ,$$

from whose

$$\int_0^w x''^2(t)dt \leqq D_2^2,$$

$$\text{where } D_2 := \frac{\sqrt{w}}{2K}\left[(M+N+P+Q)w_o + \sqrt{4KH_1^* + (M+N+P+Q)^2 w_o^2}\right] > 0 , \tag{24}$$

$$\int_0^w x'^2(t)dt \leqq D_1^2 , \text{ where } D_1 := w_o D_2 > 0 \tag{25}$$

and consequently [cf.(18)] $|x'(t)| \leqq \sqrt{w}\, D_2 := D' > 0$. (26)

Further, regarding (8) - (11) and (H), using (23), (24) and denoting

$$\overline{H}_1 = \max|h_1(x')| \text{ for } |x'| \leqq D' , \tag{$\overline{H}_1$}$$

we go to

$$|k|\int_0^w x^2(t)dt \leqq \hat{H}\int_0^w x^2(t)dt + [(M_2 + N_2 + Q_2)D_2 + (M_1 + P_1 + Q_1)D_1 + (M + N + P + \overline{H}_1 + H + Q)\sqrt{w}]\sqrt{\int_0^w x^2(t)dt}$$

and denoting $K_o := (|k| - \hat{H})$ then with respect to (R_o) we have

$$\int_0^w x^2(t)dt \leqq D_o^2, \text{ where } D_o := \frac{1}{K_o}\,[M_2 + N_2 + Q_2)D_2 + $$

$$+ (M_1 + P_1 + Q_1)D_1 + (M + N + P + \overline{H}_1 + H + $$

$$+ Q)\sqrt{w}] > 0 \qquad (27)$$

and consequently [cf.(20)] $|x(t)| \leqq (\frac{D_o}{\sqrt{w}} + \sqrt{w}\, D_1) := D > 0.$ (28)

Finally, according with (8) - (11), (H) and taking into account (24), (25), (27) and ($\overline{H}_1$) we get

$$\int_0^w x'''^2(t)dt \leqq D_3^2 , \quad \text{where}$$

$$D_3 := [(M_2 + N_2 + Q_2)D_2 + (M_1 + P_1 + Q_1)D_1 + \hat{H}D_o + $$

$$+ (M + N + P + \overline{H}_1 + H + Q)\sqrt{w}] > 0 \qquad (29)$$

and consequently [cf.(22)] $|x''(t)| \leqq \sqrt{w}\, D_3 := D'' > 0$. (30)

From (26), (28) and (30) implies that the inequality (23) is satisfied which together with $k \neq 0$ means that the sufficient condition of existence of a w-periodic solution x(t) to (1.2) is fulfilled.

Theorem 2.1. Let (2) - (7) and (11) hold in the differential equation (1.2). Let there exist constants $k \in R - (0)$ and $\hat{H} \geqq 0$, $H > 0$ such that for all $x \in (-\infty, +\infty)$ is satisfied the inequality

$$|h(x) - kx| \leqq \hat{H}|x| + H \qquad (H)$$

and let for all $y \in (-\infty, +\infty)$ holds

$$|h_1(y)| \leqq \hat{H}_1|y| + H_1 , \qquad (H_1)$$

where $\hat{H}_1 \geqq 0$, $H_1 > 0$. If

$$(M_2 + N_2 + Q_2)w_o + (M_1 + P_1 + Q_1 + \hat{H}_1)w_o^2 < 1 \qquad (R_1)$$

and

$$\hat{H} < |k| \qquad (R_0)$$

then (1.2) has a w-periodic solution.

Proof is analogous to that of Theorem 1. Now we have

$$\int_0^w x''^2(t)dt \leqq D_2^2 , \quad \text{where } D_2 := \frac{1}{K_1}(M + N + P + Q + H_1)\sqrt{w}\, w_0 > 0$$

with $K_1 := \{1 - [(M_2 + N_2 + Q_2)w_0 + (M_1 + P_1 + Q_1 + \hat{H}_1)w_0^2]\} > 0$ taking account (R_1), so that

$$\int_0^w x'^2(t)dt \leqq D_1^2 , \quad \text{where } D_1 := w_0 D_2 > 0 \text{ and consequently}$$

$$|x'(t)| \leqq \sqrt{w}\, D_2 := D' > 0. \qquad (31)$$

Further we go to $\int_0^w x^2(t)dt \leqq D_0^2$, where $D_0 := \frac{1}{K_0}[(M_2 + N_2 + Q_2)D_2 + (M_1 + P_1 + Q_1 + \hat{H}_1)D_1 + (M + N + P + H + Q + H_1)\sqrt{w}] > 0$ with $K_0 := (|k| - \hat{H}) > 0$ regarding to (R_0) and consequently [cf. (20)] $|x(t)| \leqq \left(\frac{D_0}{\sqrt{w}} + \sqrt{w}\, D_1\right) := D > 0$. (32)

Finally we get $\int_0^w x'''^2(t)dt \leqq D_3^2$, where $D_3 := [(M_2 + N_2 + Q_2)D_2 + (M_1 + P_1 + Q_1 + \hat{H}_1)D_1 + \hat{H}D_0 + (M + N + P + H + H_1 + Q)\sqrt{w}] > 0$ and consequently [cf. (22)]

$$|x''(t)| \leqq \sqrt{w}\, D_3 := D'' > 0 . \qquad (33)$$

From (31), (32) and (33) follows that the inequality (23) is satisfied; this fact together with the assumption $k \neq 0$ proves the theorem.

Theorem 2.2. Let (2) - (7) and (11) hold in the differential equation (1.2). Let there exist constants $k \in R - (0)$ and $H > 0$ such that the inequality

$$|h(x) - kx| \leqq H \qquad (H_0)$$

is satisfied for all $x \in (-\infty,+\infty)$. Let $h_1(y) \in C^1(-\infty,+\infty)$ and let

$$h_1'(y) \leqq H_1^* := \begin{cases} 0 \text{ if } h_1'(y) \leqq 0 \\ H_1' > 0 \text{ if } 0 < h_1'(y) \leqq H_1' \end{cases} \qquad (H')$$

hold for all $y \in (-\infty,+\infty)$. If

$$(M_2 + N_2 + Q_2)w_0 + (M_1 + P_1 + Q_1 + H_1^*)w_0^2 < 1 \qquad (R_2)$$

then (1.2) has a w-periodic solution.

Starting the proof with an estimate of integral $\int_0^w x'''^2(t)dt$ at first, we use besides (8) - (11) and (H_0) the inequality (H'): integrating by parts we have

$$-\int_0^w h_1[x'(t)]x'''(t)dt = \int_0^w h_1'[x'(t)]x''^2(t)dt \leqq$$

$$\leqq H_1^* \int_0^w x''^2(t)dt \leqq H_1^* w_0^2 \int_0^w x'''^2(t)dt.$$

Then

$$\int_0^w x'''^2(t)dt \leqq D_3^2, \text{ where } D_3 := \frac{1}{K_2}(M+N+P+H+Q)\sqrt{w} > 0$$

with $K_2 := \{1 - [(M_2 + N_2 + Q_2)w_0 + (M_1 + P_1 + H_1^* + Q_1)w_0^2]\} > 0$ under (R_2). Following procedure to acquire an estimate of integral $\int_0^w x^2(t)dt$ and all bounding constants [see (18), (20), (22)] needed to prove the validity of (23) is similar to proving process by the foregoing theorems.

Analogicaly to Theorem 2.2 may be proved the following theorem on existence of a periodic solution to (1.2) with more generalized functions h_1 and h.

Theorem 2.3. Let (2) - (7) and (11) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + g(t,x,x',x'')c(t,x') + h_1(t,x') + h(t,x) = q(t,x,x',x'') \quad . \tag{1.2.1}$$

Let there exist constants $k \in R - (0)$ and $H > 0$ such that for all $t \in (-\infty,+\infty)$ and for all $x \in (-\infty,+\infty)$ is satisfied the inequality

$$|h(t,x) - kx| \leq H$$

and for all $t \in (-\infty,+\infty)$ and for all $y \in (-\infty,+\infty)$ holds

$$|h_1(t,y)| \leq \hat{H}_1|y| + H_1 \ ,$$

where $\hat{H}_1 \geq 0$, $H_1 > 0$. If

$$(M_2 + N_2 + Q_2)w_0 + (M_1 + P_1 + \hat{H}_1 + Q_1)w_0^2 < 1$$

then (1.2.1) has a w-periodic solution.

Theorem 3. Let (2) - (7) and (11) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + g(t,x,x',x'')c(t,x') + h_2(x'') + h(x) = q(t,x,x',x'') \ . \tag{1.3}$$

Let there exist constants $k \in R - (0)$ and $H > 0$ such that the inequality

$$|h(x) - kx| \leq H \tag{H_0}$$

is satisfied for all $x \in (-\infty,+\infty)$. If

$$(M_2 + N_2 + Q_2)w_0 + (M_1 + P_1 + Q_1)w^2 < 1 \qquad \text{(R)}$$

then (1.3) has a w-periodic solution.

P r o o f : Substituting $x^{(j)}(t)$ instead of $x^{(j)}$, $j = 0, 1, 2, 3$, into

$$x''' + m\Big\{e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + g(t,x,x',x'')c(t,x') + h_2(x'') + h(x) - kx - q(t,x,x',x'')\Big\} + kx = 0 \,, \qquad (12_2)$$

where $m \in \langle 0,1\rangle$ is a parameter and $k \in R - (0)$ is a suitable fixed constant, multiplying the arised identity by $x'''(t)$ and integrating, we get

$$\int_0^w x'''^2(t)dt = m\Big\{-\int_0^w e(t,\dots)a[t,x'''(t)]x'''(t)dt - \int_0^w f(t,\dots)b[t,x''(t)]x'''(t)dt - \int_0^w g(t,\dots)c[t,x'(t)]x'''(t)dt - \int_0^w \{h[x(t)] - kx(t)\}x'''(t)dt + \int_0^w q(t,\dots)x'''(t)dt\Big\} \,, \text{ since}$$

$\int_0^w h_2[x(t)]x'''(t)dt = \int_0^w x(t)x'''(t)dt = 0$ with respect to (14) and using (8) - (11) and (H_0) we go to

$$\int_0^w x'''^2(t)dt \leq (M_2w_0+M_1w_0^2+N_2w_0+P_1w_0^2+Q_2w_0+Q_1w_0^2)\int_0^w x'''^2(t)dt + (M+N+P+H+Q)\sqrt{w}\sqrt{\int_0^w x'''^2(t)dt} \,.$$

Denoting $K := \{1 - [(M_2+N_2+Q_2)w_o + (M_1+P_1+Q_1)w_o^2]\}$ and taking in account (R), we arrive to

$$\int_0^w x'''^2(t)dt \leqq D_3^2, \text{ where } D_3 := \frac{1}{K}(M + N + P + H + Q)\sqrt{w} > 0,$$

so that

$$\int_0^w x''^2(t)dt \leqq D_2^2 \text{ , where } D_2 := w_o D_3 > 0 \text{ ,} \tag{34}$$

$$\int_0^w x'^2(t)dt \leqq D_1^2 \text{ , where } D_1 := w_o D_2 > 0 \tag{35}$$

and consequently [cf.(22), (18)]

$$|x''(t)| \leqq \sqrt{w}\, D_3 := D'' > 0 \text{ ,} \tag{36}$$

$$|x'(t)| \leqq \sqrt{w}\, D_2 := D' > 0 \text{ .} \tag{37}$$

Multiplying (12_2) by x(t)sgn(k) and integrating, we get

$$|k| \int_0^w x^2(t)dt = m\, \text{sgn}(k) \Big\{ - \int_0^w e(t,\dots)a[t,x'''(t)]x(t)dt - \int_0^w f(t,\dots)b[t,x''(t)]x(t)dt - \int_0^w g(t,\dots)c[t,x'(t)]x(t)dt - \int_0^w h_2[x''(t)]x(t)dt - \int_0^w \{h[x(t)] - kx(t)\}x(t)dt + \int_0^w q(t,\dots)x(t)dt\Big\}$$

and using (8) - (11) and (H_o) with regard to (34), (35) we have

$$|k| \int_0^w x^2(t)dt \leq (M_2D_2+M_1D_1+M\sqrt{w}+N_2D_2+N\sqrt{w}+P_1D_1+P\sqrt{w}+\overline{H}_2\sqrt{w}+$$
$$+ H\sqrt{w}+Q_2D_2+Q_1D_1+Q\sqrt{w}) \sqrt{\int_0^w x^2(t)dt} \ ,$$

where $\overline{H}_2 = \max|h_2(x")|$ for $|x"(t)| \leq D"$, so that

$$\int_0^w x^2(t)dt \leq D_o^2, \text{ where } D_o := \frac{1}{|k|} [(M_2+N_2+Q_2)D_2+(M_1+P_1+Q_1)D_1+$$
$$+(M+N+P+\overline{H}_2+H+Q)\sqrt{w}] > 0$$ and consequently [cf.(20)]

$$|x(t)| \leq (\frac{D_o}{\sqrt{w}} + \sqrt{w}\ D_1) := D > 0 \ . \tag{38}$$

From (36), (37) and (38) implies that the inequality (23) holds for any w-periodic solution x(t) to (12_2) on the interval $(-\infty,+\infty)$ independently of the parameter m; this fact together with the assumption $k \in R$, $k \neq 0$, proves the theorem.

Proceeding as by the foregoing theorem we may prove

Theorem 3.1. Let (2) - (7) and (11) hold in the difeerential equation

$$x''' + e(t,x,x',x")a(t,x''') + f(t,x,x',x")b(t,x") +$$
$$+ g(t,x,x',x")c(t,x') + h_2(t,x") + h(t,x) =$$
$$= q(t,x,x',x") \ . \tag{1.3.1}$$

Let there exist constants $k \in R - (0)$ and $H > 0$ such that the inequality

$$|h(t,x) - kx| \leq H$$

is satisfied for all $t \in (-\infty,+\infty)$ and for all $x \in (-\infty,+\infty)$. Let

$$|h_2(t,z)| \leq H_2|z| + H_1 \ ,$$

where $H_2 \geq 0$, $H_1 > 0$, holds for all $t \in (-\infty,+\infty)$ and for all $z \in (-\infty,+\infty)$. If

$$(M_2 + N_2 + H_2 + Q_2)w_0 + (M_1 + P_1 + Q_1)w_0^2 < 1$$

then (1.3.1) has a w-periodic solution.

It is possible perform the analogical theorems on existence of a periodic solution x(t) to (1.3.1) with the specifications $h = h_2(x") + h(t,x) - q$ or $h = h_2(t,x") + h(x) - q$ in (1), respectively.

<u>Theorem 3.2.</u> Let (2) - (7) and (11) hold in the differential equation (1.3). Let $h(x) \in C^1(-\infty,+\infty)$ whereby for all $x \in (-\infty,+\infty)$ holds

$$|h'(x)| \leqq H' \text{ with } H' > 0 \qquad (H')$$

and let there exist constants $k \in R - (0)$, $H \geqq 0$ and $H_0 > 0$ such that is satisfied the inequality

$$|h(x) - kx| \leqq H|x| + H_0 . \qquad (H)$$

If

$$(M_2 + N_2 + Q_2)w_0 + (M_1 + P_1 + Q_1)w_0^2 + H'w_0^3 < 1 \qquad (R_3)$$

and

$$H < |k| \qquad (R_0)$$

then (1.3) has a w-periodic solution.

The proof proceeds as that of Theorem 3. Now, integrating by parts and taking in account (14), (15) and (H'), we use the inequality

$$|\int_0^w h[x(t)]x'''(t)dt| = |-\int_0^w h'[x(t)]x'(t)x"(t)dt| \leqq$$

$$\leqq H'w_0^3 \int_0^w x'''^2(t)dt ,$$

so that with respect to (8) - (11) we have

$$\int_0^w x'''^2(t)dt \leqq D_3^2 \text{ , where } D_3 := \frac{1}{K_3}(M+N+P+H_o+Q)\sqrt{w} > 0$$

with $K_3 := \{1 - [(M_2+N_2+Q_2)w_o + (M_1+P_1+Q_1)w_o^2 + H' w_o^3]\} > 0$ under (R_3) and also

$$\int_0^w x''^2(t)dt \leqq D_2^2 \text{ , where } D_2 := w_o D_3 > 0$$

$$\int_0^w x'^2(t)dt \leqq D_1^2 \text{ , where } D_1 := w_o D_2 > 0 \text{ ;}$$

consequently [cf.(22), (18)]

$$|x''(t)| \leqq D'' \quad \text{with} \quad D'' := \sqrt{w}\, D_3 > 0 \tag{39}$$

$$|x'(t)| \leqq D' \quad \text{with} \quad D' := \sqrt{w}\, D_2 > 0 \quad . \tag{40}$$

Further we go to

$$\int_0 x^2(t)dt \leqq D_o^2 \text{ , where } D_o := \frac{1}{K_o}\,[(M_2+N_2+Q_2)D_2+(M_1+P_1+Q_1)D_1+$$

$+(M+N+P+H_o+Q)\sqrt{w}] > 0$ with $K_o := (|k| - H) > 0$ under (R_o) so that [cf.(20)]

$$|x(t)| \leqq D \quad \text{with} \quad D := \left(\frac{D_o}{\sqrt{w}} + \sqrt{w}\, D_1\right) > 0 \text{ .} \tag{41}$$

From (39), (40) and (41) implies the validity of (23) which together with $k \in R$, $k \neq 0$, guarantees the fulfilment of the sufficient condition of existence of a w-periodic solution $x(t)$ to (1.3).

Theorem 4. Let (2) - (7) and (11) hold in the differential equation

$$x'''(t) + e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') +$$
$$+ g(t,x, x',x'')c(t,x') + h_2(x'') + h_1(x') + h(x) =$$
$$= q(t,x,x',x'') \text{ .} \tag{1.4}$$

Let there exist constants $k \in R - (0)$ and $H_0 > 0$ such that the inequality

$$|h(x) - kx| \leqq H_0 \qquad (H_0)$$

is satisfied for all $x \in (-\infty, +\infty)$.

Further let there hold one of two following assumptions:

1) the inequality

$$|h_1(y)| \leqq H_1|y| + H \quad \text{with } H_1 \geqq 0,\ H > 0 \qquad (H_1)$$

is satisfied for all $y \in (-\infty,+\infty)$ whereby

$$(M_2 + N_2 + Q_2)w_0 + (M_1 + P_1 + H_1 + Q_1)w_0^2 < 1$$

2) $h_1(y) \in C^1(-\infty,+\infty)$ and

$$h_1'(y) \leqq H_1^* := \begin{cases} 0 & \text{if } h_1'(y) \leqq 0 \\ H_1' > 0 & \text{if } 0 < h_1'(y) \leqq H_1' \end{cases} \qquad (H_1')$$

holds for all $y \in (-\infty,+\infty)$ whereby

$$(M_2 + N_2 + Q_2)w_0 + (M_1 + P_1 + H_1^* + Q_1)w_0^2 < 1 .$$

Then (1.4) has a w-periodic solution.

The proving process of theorem with the assumption 1) is analogical to that of Theorem 3. Starting, we use besides (8) - (11) the inequality

$$\left|\int_0^w \{h[x(t)] - kx(t)\}x'''(t)dt\right| \leqq H_0\sqrt{w}\sqrt{\int_0^w x'''^2(t)dt}$$

with regard to (H_0) and

$$\left|\int_0^w h_1[x'(t)]x'''(t)dt\right| \leqq H_1 w_0^2 \int_0^w x'''^2(t)dt + H\sqrt{w}\sqrt{\int_0^w x'''^2(t)dt}$$

with regard to (H_1).

The proof of theorem ubder assumption 2) we proceed as by Theorem 3 too, but now besides (8) - (11) and (H_0) we use after integration by parts the inequality

$$- \int_0^w h_1[x'(t)]x'''(t)dt = \int_0^w h_1'[x'(t)]x''^2(t)dt \leqq$$

$$\leqq H_1^* \int_0^w x''^2(t)dt \leqq H_1^* w_0^2 \int_0^w x'''^2(t)dt.$$

<u>Theorem 4.1.</u> Let (2) - (7) and (11) hold in the differential equation (1.4). Let there exist constants $k \in R - (0)$ and $H > 0$ such that the inequality

$$|h(x) - k^3x| \leqq H$$

is satisfied for all $x \in (-\infty, +\infty)$ and let the inequality

$$|h_1(y) - 3k^2y| \leqq \hat{H}_1|y| + H_1 ,$$

where $\hat{H}_1 \geqq 0$, $H_1 > 0$, hold for all $y \in (-\infty, +\infty)$. If

$$(M_2 + N_2 + Q_2)w_0 + (M_1 + P_1 + \hat{H}_1 + Q_1 + 3k^2)w_0^2 < 1$$

then (1.4) has a w-periodic solution.

The proving process of this theorem is the same as of Theorem 3. Now the differential equation (1.4) is included into the system

$$\begin{aligned} x''' &+ m\{e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + \\ &+ g(t,x,x',x'')c(t,x') + h_2(x'') - 3kx'' + h_1(x') - \\ &- 3k^2x' + h(x) - k^3x - q(t,x,x',x'')\} + 3kx'' + \\ &+ 3k^2x' + k^3x = 0 , \end{aligned}$$

where $-k$ is a triple root of the characteristic equation to (13).

Following theorems affirm on existence of a periodic solution to (1) with a somewhat generalized form of h. Their proving process is analogous to that of Theorem 3.

<u>Theorem 4.2.</u> Let (2) - (7) and (11) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + g(t,x,x',x'')c(t,x') + h_2(t,x'') + h_1(t,x') + h(t,x) = q(t,x,x',x'') \ . \qquad (1.4.1)$$

Let there exist constants $k \in R - (0)$ and $H > 0$ such that for all $t \in (-\infty,+\infty)$ and for all $x \in (-\infty,+\infty)$ is satisfied the inequality

$$|h(t,x) - kx| \leq H \ .$$

Let for all $t \in (-\infty,+\infty)$ and for all $y \in (-\infty,+\infty)$ hold

$$|h_1(t,y)| \leq \hat{H}_1|y| + H_1 \quad \text{with} \quad \hat{H}_1 \geq 0, \quad H_1 > 0$$

and let for all $t \in (-\infty,+\infty)$ and for all $z \in (-\infty,+\infty)$ hold

$$|h_2(t,z)| \leq \hat{H}_2|z| + H_2 \quad \text{with} \quad \hat{H}_2 \geq 0, \ H_2 > 0 \ .$$

If

$$(M_2 + N_2 + \hat{H}_2 + Q_2)w_0 + (M_1 + P_1 + \hat{H}_1 + Q_1)w_0^2 < 1$$

then (1.4.1) has a w-periodic solution.

Note that similar theorems on existence of a w-periodic solution $x(t)$ to (1) in cases of $h = h_2(x'') + h_1(x') + h(t,x) - q$ or $h = h_2(x'') + h_1(t,x') + h(x) - q$ or $h = h_2(t,x'') + h_1(x') + h(x) - q$ may be performed moreover with modified assumptions relating to $h_1(x')$ or $h(x)$, respectively.

<u>Theorem 4.3.</u> Let (2) - (7) and (11) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + g(t,x,x',x'')c(t,x') + h_2(x',x'') + h_1(x,x'') + h(x,x') = q(t,x,x',x'') \ . \qquad (1.4.2)$$

Let there exist constants $k \in R - (0)$ and $\hat{H} \geq 0$, $H_0 > 0$

such that for all $x \in (-\infty,+\infty)$ and for all $y \in (-\infty,+\infty)$ is satisfied the inequality

$$|h(x,y) - kx| \leqq \hat{H}|y| + H_0 \quad .$$

Let for all $x \in (-\infty,+\infty)$ and for all $z \in (-\infty,+\infty)$ hold

$$|h_1(x,z)| \leqq \hat{H}_1|z| + H \quad \text{with } \hat{H}_1 \geqq 0,\ H > 0$$

and for all $y \in (-\infty,+\infty)$ and for all $z \in (-\infty,+\infty)$ hold

$$|h_2(y,z)| \leqq \hat{H}_2|z| + H_2|y| + H_1$$

with $\hat{H}_2 \geqq 0$, $H_2 \geqq 0$, $H_1 > 0$. If

$$(M_2 + N_2 + \hat{H}_1 + \hat{H}_2 + Q_2)w_0 + (M_1 + P_1 + \hat{H} + H_2 + Q_1)w_0^2 < 1$$

then (1.4.2) has a w-periodic solution.

The following theorem on existence of a periodic solution to (1) represents a generalization of both foregoing theorems in view of the form of h in (1).

Theorem 4.4. Let (2) - (7) and (11) hold in the differential equation

$$\begin{aligned} x''' &+ e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + \\ &+ g(t,x,x',x'')c(t,x') + h_2(t,x',x'') + h_1(t,x,x') + \\ &+ h(t,x,x'') = q(t,x,x',x'') \quad . \end{aligned} \tag{1.4.3}$$

Let there exist constants $k \in R - (0)$ and $\hat{H} \geqq 0$, $H_0 > 0$ such that for all $t,x \in (-\infty,+\infty)$ and for all $z \in (-\infty,+\infty)$ is satisfied the inequality

$$|h(t,x,z) - kx| \leqq \hat{H}|z| + H_0 \quad .$$

Let for all $t,x \in (-\infty,+\infty)$ and for all $y \in (-\infty,+\infty)$ hold

$$|h_1(t,x,y)| \leqq \hat{H}_1|y| + H \quad \text{with } \hat{H}_1 \geqq 0,\ H > 0$$

and let for all $t \in (-\infty,+\infty)$ and for all $y,z \in (-\infty,+\infty)$ hold

$$|h_2(t,y,z)| \leqq \hat{H}_3|z| + \hat{H}_2|y| + H_1$$

with $\hat{H}_3 \geqq 0$, $\hat{H}_2 \geqq 0$, $H_1 > 0$. If

$$(M_2 + N_2 + \hat{H} + \hat{H}_3 + Q_2)w_o + (M_1 + P_1 + \hat{H}_1 + \hat{H}_2 + Q_1)w_o^2 < 1$$

then (1.4.3) has a w-periodic solution.

Note that the assumptions on h_1 and h in Theorem 4.3 and 4.4 may be twofold modified. Moreover, special cases of the last theorem go out if $h = h_2(t,x',x") + h(x) - q$ or $h = h_2(x") + h_1(t,x,x') - q$ or $h = h_1(x') + h(t,x,x") - q$ in (1).

In return a generalization of all foregoing theorems give the next theorem on existence of a periodic solution to (1) with a general form of h in (1):

<u>Theorem 4.5.</u> Let (2) - (7) hold in the differential equation (1). Let there exist constants $k \in R - (0)$ and $H_2 \geqq 0$, $H_1 \geqq 0$, $H > 0$ such that for all $t,x \in (-\infty,+\infty)$ and for all $y,z \in (-\infty,+\infty)$ is satisfied the inequality

$$|h(t,x,y,z) - kx| \leqq H_2|z| + H_1|y| + H \quad . \tag{H}$$

If

$$(M_2 + N_2 + H_2)w_o + (M_1 + P_1 + H_1)w_o^2 < 1 \tag{R}$$

then (1) has a w-periodic solution.

We proceed the proof as that of Theorem 3. Substituting $x^{(j)}(t)$ instead $x^{(j)}$, $j = 0,1,2,3$, into (12) with $k_o = k_1 = 0$ in (13), where $k_2 = k \in R - (0)$ is a suitable constant, multiplying the arised identity by $x'''^2(t)$ and integrating, we go to

$$\int_0^w x'''^2(t)dt = m\Big\{ - \int_0^w e(t,\dots)a[t,x'''(t)]x'''(t)dt -$$

$$- \int_0^w f(t,\dots)b[t,x"(t)]x'''(t)dt -$$

$$- \int_0^w g(t,\dots)c[t,x'(t)]x'''(t)dt -$$

$$- \int_0^w [h(t,\dots) - kx(t)]x'''(t)dt \Big\}$$

from whose with respect to (8) - (10) and (H) we have

$$\int_0^w x'''^2(t)dt \leqq D_3^2 \text{ , where } D_3 := \frac{1}{K} (M + N + P + H)\sqrt{w} > 0$$

with $K := \left\{1 - \left[(M_2 + N_2 + H_2)w_o + (M_1 + P_1 + H_1)w_o^2\right]\right\} > 0$ under (R) and also [see (15)]

$$\int_0^w x''^2(t)dt \leqq D_2^2 \quad \text{with} \quad D_2 := w_o D_3 > 0 \tag{42}$$

$$\int_0^w x'^2(t)dt \leqq D_1^2 \quad \text{with} \quad D_1 := w_o D_2 > 0 \tag{43}$$

whereby [cf.(18), (20)]

$$|x''(t)| \leqq \sqrt{w}\, D_3 := D'' > 0 \tag{44}$$

$$|x'(t)| \leqq \sqrt{w}\, D_2 := D' > 0 \quad . \tag{45}$$

Multiplying (12) by x(t)sgn(k) and integrating, we go to

$$|k|\int_0^w x^2(t)dt = m \text{ sgn}(k) \Big\{ - \int_0^w e(t,\dots)a[t,x'''(t)]x(t)dt -$$

$$- \int_0^w f(t,\dots)b[t,x''(t)]x(t)dt -$$

$$- \int_0^w g(t,\dots)c[t,x'(t)]x(t)dt - \int_0^w [h(t,\dots) -$$

$$- kx(t)]x(t)dt \Big\}$$

so that with respect to (8) - (10) and (H) again and using (42), (43) we have

$$\int_0^w x^2(t)dt \leqq D_o^2 \quad \text{with} \quad D_o := \frac{1}{|k|}\left[(M_2 + N_2 + H_2)D_2 + \right.$$
$$\left. + (M_1 + P_1 + H_1)D_1 + (M + N + P + H)\sqrt{w}\right] > 0 \tag{46}$$

whereby [cf.(22)] $|x(t)| \leqq \left(\frac{D_o}{\sqrt{w}} + \sqrt{w}\, D_1\right) := D > o$. (47)

From (44), (45) and (47 implies that the inequality (23) holds independetly of a parameter m; this fact together with the assumption $k \neq 0$ proves the theorem.

SOUHRN

POZNÁMKA O EXISTENCI PERIODICKÝCH ŘEŠENÍ NELINEÁRNÍ DIFERENCIÁLNÍ ROVNICE TŘETÍHO ŘÁDU

VLADIMÍR VLČEK

Je vyšetřována parametrická nelineární diferenciální rovnice (1), která krom samostatně vystupující 3.derivace obsahuje člen nelineární vzhledem k této derivaci. Přítomnost takového členu značně ovlivňuje restrikční koeficienty obsažené v konstantách omezujících řešení a jeho derivace. Přitom stejnoměrná ohraničenost všech řešení (a jejich derivací) jistého jednoparametrického systému diferenciálních rovnic postačuje - s ohledem na použitou metodu důkazu - k existenci periodických řešení uvažované rovnice. Ukazuje se, že nalezené ohraničující konstanty u rovnice (1) se svým tvarem blíží odpovídajícím konstantám u rovnice vyššího řádu analogického typu bez členu nelineárního vzhledem k nejvyšší derivaci.

РЕЗЮМЕ

ЗАМЕТКА К СУЩЕСТВОВАНИЮ ПЕРИОДИЧЕСКИХ РЕШЕНИЙ НЕЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ 3-ГО ПОРЯДКА

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Изучается параметрическое нелинейное дифференциальное уравнение (1), у которого вместе с самостоятельно выступающей 3-ей производной появляется тоже член взглядом к ней нелинейный. Присутствие этого члена значительно влияет на рестриктивные коэффициенты, которые принимают участие в постоянных ограничивающих решение и его производные. При этом равномерная ограниченность всех решений /и их производных/ определенной однопараметрической системы дифференциальных уравнений - по примененному методу доказательства - достаточна к существованию периодического решения (1). Показывается, что структура постоянных для уравнения (1) приближается той же самой для уравнения высшего порядка, у которого член, взглядом к наивысшей производной нелинейный, отсутствует.

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