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TO EXISTENCE OF THE PERIODIC SOLUTION OF A THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATION

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Let us consider the equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + g(t,x,x',x'') = 0, \qquad (1)$$

where e,f,g are continuous real functions of real variables and w-periodic ($w > 0$) in the variable t. Furthermore we assume that both function e,f are bounded with respect to all their variables, i.e. there exist positive constants E,F such that

$$|e(t,x,x',x'')| \leq E \qquad (2)$$

and

$$|f(t,x,x',x'')| \leq F \ . \qquad (3)$$

The existence of w-periodic solutions x(t) of (1) will be discussed successively under certain restrictions imposed on the function g with regard to their variables.

In [1], the authors investigate, by means of the Leray-

-Schauder alternative, the special case $g = h(x) - q$, where - besides the corresponding assumptions related to $h(x)$ - the function $q = q(t,x,x',x'')$ is bounded, i.e.

$$|q(t,x,x',x'')| \leqq Q \ , \quad \text{where } Q > 0 \ . \tag{4}$$

Using the same methode, we distinguish the following forms of the function g:

I. $h(t,x) - q$, II. $h_1(x') + h(x) - q$, III. $h_2(x'') + h(x) - q$, IV. $h_2(x'') + h_1(x') + h(x) - q$, together with the condition related to q:

$$|q(t,x,x',x'')| \leqq Q_2|x''| + Q_1|x'| + Q \ , \tag{4_0}$$

where $Q_2 \geqq 0$, $Q_1 \geqq 0$, $Q > 0$ are the arbitrary given constants, instead (4) [Note, that (4) is a special case of (4_0) if $Q_2 = Q_1 = 0$]. Let us remind that the technique of the Leray-Schauder fixed point Theorem consists of the investigation of the one-parametric system

$$x''' + m\Big\{e(t,x,x',x'')x'' + f(t,x,x',x'')x' + g(t,x,x',x'') - \sum_{j=0}^{2} a_j x^{(2-j)}\Big\} + \sum_{j=0}^{2} a_j x^{(2-j)} = 0 \tag{S}$$

with a homotopical parameter $m \in \langle 0,1 \rangle$, where the constants $a_j \in R$ $(j - 0,1,2)$ are chosen in order the linear homogeneous differential equation

$$x''' + \sum_{j=0}^{2} a_j x^{(2-j)} = 0, \tag{5}$$

obtained from (S) for $m = 0$, not to have any nontrivial w-periodic solution. Thus, the sufficient condition of the existence of a w-periodic solution $x(t)$ to (1) - belonging to (S) for $m = 1$ - is, that all w-periodic solutions $x(t)$ related to (S), together with their derivatives $x'(t)$ and $x''(t)$, are bounded by the same constant, independent of the parameter m.

Taking into account the following procedure, we note: concerning the notation of composed functions e,f,g, obtained after the substitution of a w-periodic solution x(t) in the (S) and depending thus of the variable t only, we use the following symbols, e.g. $e(t,x,x',x'') = e[t,x(t),x'(t),x''(t)]$ or $e(t,\dots)$ only, etc.

Similarly, integrating the identity, obtained by the substitution $x(t)$, $x'(t)$, $x''(t)$ into (S), we restrict ourselves (for the brevity) to the interval $\langle 0,w \rangle$, in view of the fact that the obtained results are valid on any interval $\langle t,t+w \rangle$, where $t \in (-\infty,+\infty)$.

At the same time we assume that

$$x^{(j)}(0) = x^{(j)}(w) \quad \text{for } j = 0,1,2. \tag{6}$$

For this purpose we use, besides the well-known Schwarz inequality, the inequalities of the Wirtinger type (see [2])

$$\int_t^{t+w} p^{(j)^2}(s)ds \leqq w_0^2 \int_t^{t+w} p^{(j+1)^2}(s)ds, \; j=1,2, \; w_0=\frac{w}{2\pi} \tag{7}$$

holding for arbitrary continuous w-periodic function p(t) with the square integrable derivatives $p^{(j)^2}(t)$, $j = 1,2$, on the interval $\langle t,t+w \rangle$ for all $t \in (-\infty,+\infty)$, on the interval $\langle 0,w \rangle$.

PART I.

Theorem 1. Let (2), (3) and (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h(t,x) = \\ = q(t,x,x',x'') . \tag{1.1}$$

Let there exist a constant $a \in R - (0)$ and a constant $H > 0$ such that the inequality

$$|h(t,x) - ax| \leqq H \tag{A}$$

is satisfied for all $t \in (-\infty, +\infty)$ and all $x \in (-\infty, +\infty)$. If

$$(E + Q_2)w_0 + (F + Q_1)w_0^2 < 1 , \qquad (R)$$

then the equation (1.1) has a w-periodic solution.

P r o o f : Substituting $x^{(j)}(t)$ on behalf of $x^{(j)}$, j = = 0,1,2,3, into

$$x''' + m \left\{ e(t,x,x'x'')x'' + f(t,x,x',x'')x' + h(t,x) - \right.$$
$$\left. - ax - q(t,x,x',x'') \right\} + ax = 0 , \qquad (S_1$$

where $m \in \langle 0,1 \rangle$ is a parameter and $a \in R$, $a \neq 0$, a suitable fixed constant, multiplying the obtained identity by the function $x'''(t)$ and integrating, we get

$$\int_0^w x'''^2(t)dt = m \left\{ - \int_0^w e(t,\dots)x''(t)x'''(t)dt - \right.$$
$$- \int_0^w f(t,\dots)x'(t)x'''(t)dt - \int_0^w \{ h[t,x(t)] - $$
$$\left. - ax(t) \} x'''(t)dt + \int_0^w q(t,\dots)x'''(t)dt \right\} ,$$

because of $\int_0^w x(t)x'''(t)dt = 0$.

Using (2), (3), (4_0) together with (A), the Schwarz inequality and (7), we receive successively

$$\left| \int_0^w e(t,\dots)x''(t)x'''(t)dt \right| \leqq Ew_0 \int_0^w x'''^2(t)dt,$$

$$\left| \int_0^w f(t,\dots)x'(t)x'''(t)dt \right| \leqq Fw_0^2 \int_0^w x'''^2(t)dt,$$

$$\left| \int_0^w \{h[t,x(t)] - ax(t)\} x'''(t)dt \right| \leqq H\sqrt{w} \sqrt{\int_0^w x'''^2(t)dt},$$

$$\left|\int_0^w q(t,\dots)x'''(t)dt\right| \leq (Q_2 + Q_1 w_o)w_o \int_0^w x'''^2(t)dt +$$

$$+ Q\sqrt{w}\sqrt{\int_0^w x'''^2(t)dt};$$

then

$$\int_0^w x'''^2(t)dt \leq \left[(E + Q_2)w_o + (F + Q_1)w_o^2\right]\int_0^w x'''^2(t)dt +$$

$$+ (H + Q)\sqrt{w}\,.\sqrt{\int_0^w x'''^2(t)dt}\,.$$

Denoting

$$K = 1 - \left[(E + Q_2)w_o + (F + Q_1)w_o^2\right]$$

and taking into account (R), we arrive at

$$\sqrt{\int_0^w x'''^2(t)dt} \leq \frac{1}{K}(H + Q)\sqrt{w} := D_3 > 0, \tag{8}$$

from which

$$\int_0^w x'''^2(t)dt \leq D_3^2$$

and with respect to (7) also

$$\int_0^w x''^2(t)dt \leq w_o^2 D_3^2 = D_2^2\ ,\ D_2 := w_o D_3 > 0 \tag{9}$$

and

$$\int_0^w x'^2(t)dt \leq w_o^2 D_2^2 = D_1^2\ ,\ D_1 := w_o D_2 (= w_o^2 D_3) > 0\,. \tag{10}$$

According to the Rolle Theorem, applied on the w-periodic function $x(t)$, $t \in \langle 0,w \rangle$, differentiable and satisfying (6), such points $t_j \in (0,w)$, $j = 1,?$, exist that $x^{(j)}(t_j) = 0$. Then, in view of the relations

$$\int_{t_j}^{t} x^{(j+1)}(s)ds = x^{(j)}(t) - x^{(j)}(t_j), \quad j = 1,2,$$

where $t_j, t \in (0,w)$, the following inequalities

$$|x'(t)| = |\int_{t_j}^{t} x''(s)ds| \leqq \int_0^w |x''(t)|dt \leqq \sqrt{w \int_0^w x''^2(t)dt} =$$

$$= \sqrt{w}\, D_2 := D' > 0 \tag{11}$$

and

$$|x''(t)| = |\int_{t_j}^{t} x'''(s)ds| \leqq \int_0^w |x'''(t)|dt \leqq \sqrt{w \int_0^w x'''^2(t)dt} =$$

$$= \sqrt{w}\, D_3 := D'' > 0 \tag{12}$$

hold for any w-periodic solution $x(t)$ of (S_1).

Multiplying (S_1) by the function $x(t)\mathrm{sgn}(a)$ and integrating the obtained identity, we go to

$$|a| \int_0^w x^2(t)dt = m\, \mathrm{sgn}(a) \Big\{ - \int_0^w e(t,\dots)x''(t)x(t)dt -$$

$$- \int_0^w f(t,\dots)x'(t)x(t)dt - \int_0^w \{h[t,x(t)] -$$

$$- ax(t)\} x(t)dt + \int_0^w q(t,\dots)x(t)dt \Big\},$$

because of $\int_0^w x'''(t)x(t)dt = 0$ again.

Emploing (2), (3), (4_0), (A), the Schwarz inequality and (7) and with respect to (9), (10), we have now

$$\left|\int_0^w e(t,\dots)x''(t)x(t)dt\right| \leqq ED_2\sqrt{\int_0^w x^2(t)dt},$$

$$\left|\int_0^w f(t,\dots)x'(t)x(t)dt\right| \leqq FD_1\sqrt{\int_0^w x^2(t)dt},$$

$$\left|\int_0^w \{h[t,x(t)] - ax(t)\}x(t)dt\right| \leqq H\sqrt{w}\sqrt{\int_0^w x^2(t)dt},$$

$$\left|\int_0^w q(t,\dots)x(t)dt\right| \leqq (Q_2D_2 + Q_1D_1 + Q\sqrt{w})\sqrt{\int_0^w x^2(t)dt},$$

sothat

$$|a|\int_0^w x^2(t)dt \leqq [ED_2 + FD_1 + Q_2D_2 + Q_1D_1 +$$

$$+ (H + Q)\sqrt{w}]\sqrt{\int_0^w x^2(t)dt)},$$

from which

$$\sqrt{\int_0^w x^2(t)dt} \leqq \frac{1}{|a|}|(E + Q_2)D_2 + (F + Q_1)D_1 + (H + Q)\sqrt{w}] :=$$

$$:= D_0 > 0 \qquad (13)$$

and hence

$$\int_0^w x^2(t)dt \leqq D_0^2 .$$

Consequently, the point $t_0 \in \langle 0,w\rangle$ exists at which the inequality

$$|x(t_0)| \leqq \frac{D_0}{\sqrt{w}}$$

holds for any w-periodic solution x(t) of the system (S_1). If, on the contrary, the opposite inequality

$$|x(t_o)| > \frac{D_o}{\sqrt{w}}$$

be true for the same solution x(t) of (S_1), then from the corresponding integral inequality

$$\int_0^w x^2(t_o)dt > \int_0^w \frac{D_o^2}{w}\,dt = D_o^2$$

we go to a contradiction with the inequality (13), which holds for all solutions $x(t), t \in \langle 0,w\rangle$, of (S_1).

Hence, according to relation

$$\int_{t_o}^{t} x'(s)ds = x(t) - x(t_o)$$

we have for $t_o, t \in \langle 0,w\rangle$

$$|x(t)| = |x(t_o) + \int_{t_o}^{t} x'(s)ds| \leq \frac{D_o}{\sqrt{w}} + \int_0^w |x'(t)|dt \leq \frac{D_o}{\sqrt{w}} +$$

$$+ \sqrt{w}\sqrt{\int_0^w x'^2(t)dt} = (\frac{D_o}{\sqrt{w}} + \sqrt{w}\,D_1) := D > 0\ . \qquad (14)$$

From (11), (12) and (14) follows that for any w-periodic solution x(t) of (S_1) is satisfied the inequality

$$|x^{(j)}(t)| \leq M\ ,\ j = 0,1,2, \qquad (15)$$

on the interval $(-\infty,+\infty)$, where the positive constant M = = max(D,D',D") is independent of the parameter $m \in \langle 0,1\rangle$. This fact together with the assumption $a \in R$, $a \neq 0$, prove our theorem.

In the following theorem the function h(t,x), belonging to the equation (1.1), has a certain form. This concretization makes possible to alter somewhat the process of the proof.

Theorem 1.1. Let (2), (3), (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + g(t)h_0(x) + h(x) = q(t,x,x',x'') . \quad (1.2)$$

Let there exist a constant $a \in R - (0)$ such that for all $x \in (-\infty,+\infty)$ holds the inequality

$$|h(x) - ax| \leqq \hat{H}|x| + H , \quad (A_1)$$

where $\hat{H} \geqq 0, H > 0$. Let there exist a constant $H_0 > 0$ such that

$$|g(t)h_0(x)| \leqq H_0 \quad (A_2)$$

holds for all $t \in (-\infty,+\infty)$ and for all $x \in (-\infty,+\infty)$.

If

$$(E + Q_2)w_0 + (F + Q_1)w_0^2 < 1 \quad (R)$$

and

$$\hat{H} < |a| \quad , \quad (R_0)$$

then the equation (1.2) has a w-periodic solution.

P r o o f : Substituting $x^{(j)}(t)$ on behalf of $x^{(j)}$, $j = 0, 1,2,3$, into

$$x''' + m\left\{e(t,x,x',x'')x'' + f(t,x,x',x'')x' + g(t)h_0(x) + h(x) - ax - q(t,x,x',x'')\right\} + ax = 0 , \quad (S_2)$$

where $m \in \langle 0,1 \rangle$ is a parameter and $a \in R$, $a \neq 0$, a suitable fixed constant, multiplying the obtained identity by the function $x'(t)$ and integrating, we get

$$\int_0^w x''^2(t)dt = m\left\{\int_0^w e(t,\dots)x''(t)x'(t)dt + \int_0^w f(t,\dots)x'^2(t)dt + \int_0^w g(t)h_0[x(t)]x'(t)dt - \int_0^w q(t,\dots)x'(t)dt\right\}$$

because of $\int_0^w x'''(t)x'(t)dt = -\int_0^w x''^2(t)dt$ and $\int_0^w x(t)x\ (t)dt =$

$= \int_0^w h[x(t)]x'(t)dt = 0.$

Using (2), (3), (4_0) and relative to (A_2) is

$$\int_0^w x''^2(t)dt \leqq (Ew_0 + Fw_0^2)\int_0^w x''^2(t)dt + \\ + H_0\sqrt{w}\ w_0\sqrt{\int_0^w x''^2(t)dt} + \\ + (Q_2w_0 + Q_1w_0^2)\int_0^w x''^2(t)dt + \\ + Q\sqrt{w}\ w_0\sqrt{\int_0^w x''^2(t)dt}\ ,$$

i.e.

$$\{1 - [(E + Q_2)w_0 + (F + Q_1)w_0^2]\}\int_0^w x''^2(t)dt \leqq \\ \leqq (H_0 + Q)\sqrt{w}\ w_0\sqrt{\int_0^w x''^2(t)dt}$$

If we denote

$$K = 1 - [(E + Q_2)w_0 + (F + Q_1)w_0^2]\ ,$$

then regarding to (R) yields

$$\sqrt{\int_0^w x''^2(t)dt} \leqq \frac{1}{K}(H_0 + Q)\sqrt{w}\ w_0 := D_2 > 0 \qquad (D_2)$$

from whence

$$\int_0^w x''^2(t)dt \leqq D_2^2$$

and

$$\int_0^w x'^2(t)dt \leq D_1^2 \text{ , where } D_1 := w_0D_2 > 0 \text{ too.} \qquad (D_1)$$

Consequently [cf.(11)] holds

$$|x(t)| \leq \sqrt{w}\, D_2 := D' > 0 \qquad (D')$$

for any w-periodic solution x(t) of (S_2).

Multiplying (S_2) by the function x(t)sgn(a) and integrating. the obtained identity, we go to

$$|a|\int_0^w x^2(t)dt = m\, \mathrm{sgn}(a)\Big\{ -\int_0^w e(t,\dots)x''(t)x(t)dt - $$
$$-\int_0^w f(t,\dots)x'(t)x(t)dt -$$
$$-\int_0^w g(t)h_0[x(t)]x(t)dt - \int_0^w \{h[x(t)] -$$
$$- ax(t)\}x(t)dt + \int_0^w q(t,\dots)x(t)dt\Big\}$$

According to (2), (3), (4_0), with regard to (A_1), (A_2) and using (D_2), (D_1) we get now

$$|a|\int_0^w x^2(t)dt \leq (ED_2 + FD_1)\sqrt{\int_0^w x^2(t)dt} +$$
$$+ H_0\sqrt{w}\sqrt{\int_0^w x^2(t)dt} + \hat{H}\int_0^w x^2(t)dt + H\sqrt{w}\sqrt{\int_0^w x^2(t)dt} +$$
$$+ (Q_2D_2 + Q_1D_1 + Q\sqrt{w})\sqrt{\int_0^w x^2(t)dt}$$

i.e.

$$(|a| - \hat{H}) \int_0^w x^2(t)dt \leq [(E + Q_2)D_2 + (F + Q_1)D_1 + (Q + H_0 + H)\sqrt{w}] \sqrt{\int_0^w x^2(t)dt} .$$

If we denote $K_0 = |a| - \hat{H}$, then regarding to (R_0) yields

$$\sqrt{\int_0^w x^2(t)dt} \leq \frac{1}{K_0} [(E + Q_2)D_2 + (F + Q_1)D_1 + (Q + H_0 + H)\sqrt{w}] := D_0 > 0, \qquad (D_0)$$

from whence

$$\int_0^w x^2(t)dt \leq D_0^2$$

and consequently [cf.(14)] the inequality

$$|x(t)| \leq (\frac{D_0}{w} + D_1\sqrt{w}) := D > 0 \qquad (D)$$

holds for any w-periodic solution x(t) of (S_2).

Now, multiplying (S_2) by the function $x'''(t)$ and integrating the obtained identity, we have

$$\int_0^w x'''(t)dt = m\Big\{ - \int_0^w e(t,\dots)x''(t)x'''(t)dt - \int_0^w f(t,\dots)x'(t)x'''(t)dt - \int_0^w g(t)h_0[x(t)]x'''(t)dt - \int_0^w \{h[x(t)] - ax(t)\}x'''(t)dt + \int_0^w q(t,\dots)x'''(t)dt \Big\} ,$$

from whence, using (2), (3), (4_0), (A_1), (A_2), (D_2), (D_1) and (D_0), we get

$$\int_0^w x'''^2(t)dt \leq (ED_2 + FD_1)\sqrt{\int_0^w x'''^2(t)dt} +$$

$$+ H_0\sqrt{w}\sqrt{\int_0^w x'''^2(t)dt} + (\hat{H}D_0 + H\sqrt{w})\sqrt{\int_0^w x'''^2(t)dt} +$$

$$+ (Q_2D_2 + Q_1D_1 + Q\sqrt{w})\sqrt{\int_0^w x'''^2(t)dt} ,$$

i.e.

$$\sqrt{\int_0^w x'''^2(t)dt} \leq \left[(E + Q_2)D_2 + (F + Q_1)D_1 + \hat{H}D_0 + \right.$$
$$\left. + (H_0 + H + Q)\sqrt{w}\right] := D_3 > 0 , \qquad (D_3)$$

sothat

$$\int_0^w x'''^2(t)dt \leq D_3^2$$

and consequently [cf.(12)] the inequality

$$|x''(t)| \leq \sqrt{w}\, D_3 := D'' > 0 \qquad (D'')$$

for any w-periodic solution x(t) of (S_2) holds.

From (D'), (D) and (D'') follows that for any w-periodic solution x(t) of (S_2) is satisfied the inequality (15) on the interval $(-\infty,+\infty)$, what - together with the assumption $a \in R$, $a \neq 0$ - prove this theorem.

Modification of Theorem 1.1 is

Theorem 1.2. Let (2), (3) and (4_0) hold in the differential equation (1.2). Let there exist constants $a \in R - (0)$ and $H_0 > 0$ such that the inequality

$$|g(t)h_0(x) + h(x) - ax| \leq H_0$$

is satisfied for all $t \in (-\infty,+\infty)$ and for all $x \in (-\infty,+\infty)$.

If

$$(E + Q)w_0 + (F + Q_1)w_0^2 < 1,$$

then the equation (1.2) has a w-periodic solution.

The proof is quite analogical to that of the Theorem 1, since in (1.2) is possible to note $g(t)h_0(x) + h(x) = H(t,x)$.

Note: In case $g(t) = k$, where $k \in R$ is a constant, we may denote $kh_0(x) + h(x) = H(x)$ and we obtain the form of the differential equation investigated in [1].

Closing the part I. we present two more theorems concerning the existence of a w-periodic solution to (1.1) with the special form of the function g.

<u>Theorem 1.3.</u> Let (2), (3), (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h(x) + ax = q(t,x,x',x'') \quad (1.3)$$

where $a \in R - (0)$ is an arbitrary given constant. Let

$$|h(x)| \leqq H|x| + H_0 , \quad (A_0)$$

where $H \geqq 0, H_0 > 0$, hold for all $x \in (-\infty,+\infty)$.

If

$$(E + Q_2)w_0 + (F + Q_1)w_0^2 < 1 \quad (R)$$

and

$$H < |a| , \quad (R_0)$$

then the equation (1.3) has a w-periodic solution.

Now the differential equation (1.3) is contained in the system

$$x''' + m\Big\{e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h(x) -$$
$$- q(t,x,x',x'')\Big\} + ax = 0$$

with parameter $m \in \langle 0,1 \rangle$ again, but the process of the proof is the same as the Theorem 1.1 only with the exception that for the estimate of integral

$$\int_0^w x^2(t)dt$$

wu use the inequality

$$\left|\int_0^w h[x(t)]x(t)dt\right| \leqq H\int_0^w x^2(t)dt + H_0\sqrt{w}\sqrt{\int_0^w x^2(t)dt}$$

holding with regard to (A_0) and for the estimate of integral

$$\int_0^w x'''^2(t)dt$$

we use the inequality or

$$\left|\int_0^w h[x(t)]x'''(t)dt\right| \leqq (HD_0 + H_0\sqrt{w})\sqrt{\int_0^w x'''^2(t)dt} ,$$

where

$$\sqrt{\int_0^w x^2(t)dt} \leqq D_0 , \quad D_0 > 0$$

holds or

$$\left|\int_0^w h[x(t)]x'''(t)dt\right| \leqq \overline{H}\sqrt{w}\sqrt{\int_0^w x'''^2(t)dt} ,$$

where $\overline{H} = \max|h(x)|$ for $|x| \leqq D$, $D > 0$.

Modification of the foregoing theorem is

<u>Theorem 1.3.1.</u> Let (2), (3), (4_0) hold in the differential equation (1.3), where $a \in R - (0)$ is an arbitrary given constant. Let for all $x \in (-\infty, +\infty)$ hold

$$- h(x)x \leq H^* = \begin{cases} 0 \text{ if } 0 \leq h(x)x \\ H > 0 \text{ if } -H \leq h(x)x < 0. \end{cases} \tag{A_1}$$

If

$$(E + Q_2)w_0 + (F + Q_1)w_0^2 < 1 \quad , \tag{R}$$

then the equation (1.3) has a w-periodic solution.

The proof is equal to the proof of Theorem 1.2; at the same time, according to (A_1), we use the inequality

$$- \int_0^w h[x(t)]x(t)dt \leq H^* w$$

for the estimate of integral

$$\int_0^w x^2(t)dt.$$

Remark: Similar theorems on the existence of a w-periodic solution may be presented of the differential equation (1) with $g = h(t,x) + ax - q(t,x,x',x'')$ or $g = h_0(t)h(x) + ax - q(t,x,x',x'')$ etc., where $a \in R - (0)$.

PART II.

<u>Theorem 2</u>. Let (2), (3), (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_1(x') + h(x) = q(t,x,x',x'') . \tag{2.1}$$

Let there exists a constant $a \in R - (0)$ such that the inequality

$$|h(x) - ax| \leqq \hat{H}|x| + H , \tag{A_1}$$

where $\hat{H} \geqq 0, H > 0$, is satisfied for all $x \in (-\infty, +\infty)$.

Let for all $y \in (-\infty, +\infty)$ holds

$$h_1(y)y \leqq H_1^* := \begin{cases} 0 \text{ if } h_1(y)y \leqq 0 \\ H_1 > 0 \text{ if } 0 < h_1(y)y \leqq H_1 . \end{cases} \tag{A_2}$$

If

$$(E + Q_2)w_o + (F + Q_1)w_o^2 < 1 \tag{R}$$

and

$$\hat{H} < |a| , \tag{R_o}$$

the the equation (2.1) has a w-periodic solution.

The process of the proof is the same as the Theorem 1.1. Using, accordingly to (A_2), the inequality

$$\int_0^w h_1[x'(t)]x'(t)dt \leqq H_1^* w$$

and denoting

$$K = 1 - [(E + Q_2)w_o + (F + Q_1)w_o^2]$$

we go - in accord with (R) - to the estimate

$$\int_0^w x''^2(t)dt - \frac{Q\sqrt{w}\; w_o}{K} \sqrt{\int_0^w x''^2(t)dt} \leqq \frac{H_1^* w}{K}$$

i.e.

$$\left(\int_0^w x''^2(t)dt - \frac{Q\sqrt{w}\; w_o}{2K}\right)^2 \leqq \left(\frac{4H_1^*K + Q^2 w_o^2}{4K^2}\right) w$$

from where

$$\int_0^w x''^2(t)dt \leqq D_2^2$$

with $\frac{\sqrt{w}}{2K}(Qw_o + \sqrt{4H_1^{*}K + Q^2w_o^2}) := D_2 > 0$ and consequently

$$\int_0^w x'^2(t)dt \leqq D_1^2 \text{ , where } D_1 := w_oD_2 > 0.$$

On account of the estimate of integral

$$\int_0^w x^2(t)dt$$

we use the inequality

$$\left|\int_0^w \{h[x(t)] - ax(t)\} x(t)dt\right| \leqq \hat{H}\int_0^w x^2(t)dt + H\sqrt{w}\sqrt{\int_0^w x^2(t)dt}$$

holding with regard to (A_1) and

$$\left|\int_0^w h_1[x'(t)]x(t)dt\right| \leqq \overline{H}_1\sqrt{w}\sqrt{\int_0^w x^2(t)dt} \text{ ,}$$

where $\overline{H}_1 = \max|h_1(x')|$ for $|x'| \leqq D'$, $D' := \sqrt{w}\, D_2 > 0$. Then

$$\int_0^w x^2(t)dt \leqq D_o^2 \text{ ,}$$

where $D_o := \frac{1}{K_o}[(E + Q_2)D_2 + (F + Q_1)D_1 + (\overline{H}_1 + H + Q)\sqrt{w}] > 0$ and where $K_o := |a| - \hat{H} > 0$ under the assumption (R_o).

Some modifications of Theorem 2.

Theorem 2.1. Let (2), (3), (4_0) hold in the differential equation (2.1). Let there exist a constant $a \in R - (0)$ and constants $\hat{H} \geqq 0, H > 0$ such that the inequality

$$|h(x) - ax| \leqq \hat{H}|x| + H \qquad (A_1)$$

is satisfied for all $x \in (-\infty, +\infty)$.

Let there exist the constants $H_1 \geqq 0, H_0 > 0$ such that the inequality

$$|h_1(y)| \leqq H_1|y| + H_0 \qquad (A_3)$$

is satisfied for all $y \in (-\infty, +\infty)$.

If

$$(E + Q_2)w_0 + (F + H_1 + Q_1)w_0^2 < 1 \qquad (R_1)$$

and

$$\hat{H} < |a| \ , \qquad (R_0)$$

then the equation (2.1) has a w-periodic solution.

The proof may be performed analogicaly as the same of the Theorem 2, whereby for the estimate of integral

$$\int_0^w x''^2(t)dt$$

we use - with respect to (A_3) - the inequality

$$|\int_0^w h_1[x'(t)]x'(t)dt| \leqq H_1 \int_0^w x''^2(t)dt +$$

$$+ H_0\sqrt{w}\ w_0 \sqrt{\int_0^w x''^2(t)dt} \ .$$

Note: In the case $\hat{H} = 0$ in (A_1) we may start the proof with the estimate of integral

$$\int_0^w x'''^2(t)dt$$

as first.

Theorem 2.2. Let (2), (3), (4_0) hold in the differential equation (2.1). Let there exist a constant $a \in R - (0)$ such that the inequality

$$|h(x) - ax| \leq \hat{H}|x| + H , \qquad (A_1)$$

where $\hat{H} \geq 0, H > 0$, is satisfied for all $x \in (-\infty, +\infty)$ and the inequality

$$|h_1(y) - 3\sqrt[3]{a^2}y| \leq H_1|y| + H_0 , \qquad (A_4)$$

where $H_1 \geq 0, H_0 > 0$, is satisfied for all $y \in (-\infty, +\infty)$.

If

$$(E + Q_2)w_0 + (F + Q_1 + H_1 + 3\sqrt[3]{a^2})w_0^2 < 1 \qquad (R_2)$$

and

$$\hat{H} < |a| , \qquad (R_0)$$

then the equation (2.1) has a w-periodic solution.

The process of the proof is the same as the Theorem 1.1 or - if $\hat{H} = 0$ in (A_1) - as the Theorem 1. Now the differential equation (2.1) is contained in the system

$$\begin{aligned} x''' &+ m\Big\{e(t,x,x',x'')x'' + f(t,x,x',x'')x' - 3\sqrt[3]{a}x'' + \\ &+ h_1(x') - 3\sqrt[3]{a^2}x' + h(x) - ax - q(t,x,x',x'')\Big\} + \\ &+ 3\sqrt[3]{a}x'' + 3\sqrt[3]{a^2}x' + ax = 0 , \end{aligned} \qquad (S_3)$$

where the corresponding linear homogeneous differential equation obtained from (S_3) for $m = 0$ [cf.(5)] has a characteristic equation with the triple root $-\sqrt[3]{a}$.

For the estimate of integral

$$\int_0^w x''^2(t)dt$$

we use - besides (2), (3) and (4_0) - the inequality

$$\left|\int_0^w \{h_1[x'(t)] - 3\sqrt[3]{a^2}x'(t)\} x'(t)dt\right| \leqq$$

$$\leqq H_1 w_0^2 \int_0^w x''^2(t)dt + H_0\sqrt{w}\, w_0 \sqrt{\int_0^w x''^2(t)dt}$$

with regard to (A_4) and for the estimate of integral

$$\int_0^w x^2(t)dt$$

we use - besides (2), (3) and (4_0) - the inequality

$$\left|\int_0^w \{h[x(t)] - ax(t)\} x(t)dt \leqq \hat{H}\int_0^w x^2(t)dt +\right.$$

$$+ H\sqrt{w}\sqrt{\int_0^w x^2(t)dt}$$

with regard to (A_1).

<u>Theorem 2.3.</u> Let (2), (3), (4_0) hold in the differential equation (2.1). Let there exist a constant $a \in R - (0)$ such that the inequality

$$|h(x) - ax| \leqq H \;, \qquad (A_0)$$

where $H > 0$, is satisfied for all $x \in (-\infty, +\infty)$.

Let $h_1(y) \in C^1(-\infty, +\infty)$ and let

$$h_1'(y) \leqq H_1^* := \begin{cases} 0 \text{ if } h_1'(y) \leqq 0 \\ H_1' > 0 \text{ if } 0 < h_1'(y) \leqq H_1' \end{cases} \qquad (A_5)$$

hold for all $y \in (-\infty, +\infty)$.

If

$$(E + Q_2)w_0 + (F + H_1^* + Q_1)w_0^2 < 1 , \qquad (R_3)$$

then the equation (2.1) has a w-periodic solution.

The process of the proof is the same as the Theorem 1. But now, integrating by parts, we have

$$\int_0^w h_1[x'(t)]x'''(t)dt = - \int_0^w h_1'[x'(t)]x''^2(t)dt$$

and with regard to (A_5) we use for the estimate of integral

$$\int_0^w x'''^2(t)dt$$

the inequality

$$- \int_0^w h_1[x'(t)]x'''(t)dt = \int_0^w h_1'[x'(t)]x''^2(t)dt \leqq$$

$$\leqq H_1^* \int_0^w x''^2(t)dt \leqq$$

$$\leqq H_1^* w_0^2 \int_0^w x'''^2(t)dt$$

together with (2), (3), (4_0), (A_0), etc.

Proceeding as in the proof of Theorem 2 it is possible analogicaly to prove

Theorem 2.4. Let (2), (3), (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_1(t,x') + $$
$$+ h(t,x) = q(t,x,x',x'') . \qquad (2.2)$$

Let there exist a constant $a \in R - (0)$ and a constant $H > 0$ such that for all $t \in (-\infty,+\infty)$ and for all $x \in (-\infty,+\infty)$ holds

$$|h(t,x) - ax| \leqq H$$

and for all $t \in (-\infty,+\infty)$ and for all $y \in (-\infty,+\infty)$ is satisfied the inequality

$$|h_1(t,y)| \leqq H_1|y| + H_0 ,$$

where $H_1 \geqq 0$, $H_0 > 0$. If

$$(E + Q_2)w_0 + (F + H_1 + Q_1 w_0^2) < 1 ,$$

then the equation (2.2) has a w-periodic solution.

Remark: In analogy to Theorem 2 and their modifications we may to express the corresponding theorems on the existence of a w-periodic solution to the differential equations

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_1(x') + h(t,x) =$$
$$= q(t,x,x',x'')$$

or

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_1(t,x') +$$
$$+ h(x) = q(t,x,x',x'')$$

with the assumptions on the functions h and h_1 analogical to themselves in the Theorems 2 - 2.3.

Closing the part II. we present the theorems as a special case of Theorem 2.

Theorem 2.5.1. Let (2), (3), (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_1(x') + ax = \\ = q(t,x,x',x''), \qquad (2.1.1)$$

where $a \in R - (0)$ is an arbitrary given constant.

Let there hold one of the following four assumptions:

1) for all $y \in (-\infty,+\infty)$ holds

$$h_1(y)y \leqq H_0^* := \begin{cases} 0 \text{ if } h_1(y)y \leqq 0 \\ H_0 > 0 \text{ if } 0 < h_1(y)y \leqq H_0 \end{cases}$$

as well as

$$(E + Q_2)w_0 + (F + Q_1)w_0^2 < 1$$

2) for all $y \in (-\infty,+\infty)$ holds

$|h_1(y)| \leqq H_2|y| + H_1$, where $H_2 \geqq 0$, $H_1 > 0$,

as well as

$$(E + Q_2)w_0 + (F + H_2 + Q_1)w_0^2 < 1$$

3) for all $y \in (-\infty,+\infty)$ holds

$|h_1(y) - 3\sqrt[3]{a^2}y| \leqq H_2|y| + H_1$, where $H_2 \geqq 0$, $H_1 > 0$,

as well as

$$(E + Q_2)w_0 + (F + H_2 + Q_1 + 3\sqrt[3]{a^2})w_0^2 < 1$$

4) $h_1(y) \in C^1(-\infty,+\infty)$ and for all $y \in (-\infty,+\infty)$ holds

$$h_1'(y) \leqq H_1^* := \begin{cases} 0 \text{ if } h_1'(y) \leqq 0 \\ H_1' > 0 \text{ if } 0 < h_1'(y) \leqq H_1' \end{cases}$$

as well as

$$(E + Q_2)w_0 + (F + H_1^* + Q_1)w_0^2 < 1 .$$

Then the equation (2.1.1) has a w-periodic solution.

Proving this theorem with the assumptions 1) or 2), it is convenient to proceed equal as in the proof of Theorem 2. The theorem with the assumptions 3) or 4) may be proved analogicaly to Theorem 2.2.

<u>Theorem 2.5.2.</u> Let (2), (3), (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_1(x') + h(x) + \\ + ax = q(t,x,x',x'') \, , \qquad (2.1.2)$$

where $a \in R - (0)$ is arbitrary given constant.

Let there hold one of the following four assumptions:

1) for all $x \in (-\infty,+\infty)$ holds

$$-h(x)x \leqq H^* := \begin{cases} 0 & \text{if } 0 \leqq h(x)x \\ H > 0 & \text{if } -H \leqq h(x)x < 0 \, , \end{cases}$$

for all $y \in (-\infty,+\infty)$ holds

$|h_1(y)| \leqq H_1|y| + H_0$, where $H_1 \geqq 0$, $H_0 > 0$,

as well as

$(E + Q_2)w_0 + (F + H_1 + Q_1)w_0^2 < 1$

2) for all $x \in (-\infty,+\infty)$ holds

$$-h(x)x \leqq H^* := \begin{cases} 0 & \text{if } 0 \leqq h(x)x \\ H > 0 & \text{if } -H \leqq h(x)x < 0 \, , \end{cases}$$

for all $y \in (-\infty,+\infty)$ holds

$$h_1(y)y \leqq H_1^* := \begin{cases} 0 & \text{if } h_1(y)y \leqq 0 \\ H_1 > 0 & \text{if } 0 < h_1(y)y \leqq H_1 \end{cases}$$

as well as

$(E + Q_2)w_0 + (F + Q_1)w_0^2 < 1$

3) for all $x \in (-\infty,+\infty)$ holds

$|h(x)| \leqq H|x| + H_0$, where $H \geqq 0$, $H_0 > 0$,

for all $y \in (-\infty,+\infty)$ holds

$$h_1(y)y \leqq H_1^* := \begin{cases} 0 & \text{if } h_1(y)y \leqq 0 \\ H_1 > 0 & \text{if } 0 < h_1(y)y \leqq H_1 \end{cases}$$

as well as

$(E + H + Q_2)w_0 + (F + Q_1)w_0^2 < 1$

4) for all $x \in (-\infty,+\infty)$ holds
$|h(x)| \leq H|x| + H_0$, where $H \geq 0$, $H_0 > 0$,
for all $y \in (-\infty,+\infty)$ holds
$|h_1(y)| \leq \hat{H}_1|y| + H_1$, where $\hat{H}_1 \geq 0$, $H_1 > 0$
as well as
$(E + H + Q_2)w_0 + (F + \hat{H}_1 + Q_1)w_0^2 < 1$.

Then the equation (2.1.2) has a w-periodic solution.

Theorem in any case of the assumptions 1) - 4) may be proved quite analogicaly as Theorem 2, i.e. with the proving process of Theorem 1.1.

PART III.

Theorem 3. Let (2), (3), (4_0) hold in the differential equation

$$x''' + e\,(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_2(x'') + \\ + h(x) = q(t,x,x',x'') . \qquad (3.1)$$

Let there exist a constant $a \in R - (0)$ and a constant $H > 0$ such that the inequality

$$|h(x) - ax| \leq H \qquad (A_0)$$

is satisfied for all $x \in (-\infty,+\infty)$. If

$$(E + Q_2)w_0 + (F + Q_1)w_0^2 < 1 ,$$

then the equation (3.1) has a w-periodic solution.

The process of the proof is the same of Theorem 1. Estimating the integral

$$\int_0^w x'''^2(t)dt$$

we take account of

$$\int_0^w h[x''(t)]x'''(t)dt = \int_0^w x(t)x'''(t)dt = 0.$$

For the estimate of integral

$$\int_0^w x^2(t)dt$$

we use - besides (2), (3), (4_0) and (A_0) - the inequality

$$\left|\int_0^w h_2[x''(t)]x(t)dt\right| \leqq \overline{H}_2\sqrt{w}\sqrt{\int_0^w x^2(t)dt},$$

where $\overline{H}_2 = \max|h_2(x'')|$ for $|x''| \leqq D'' > 0$ [cf.(12)].

Modification of the foregoing theorem is

Theorem 3.1. Let (2), (3), (4_0) hold in the differential equation (3.1).

Let there exist a constant $a \in R - (0)$ and a constant $H > 0$ such that the inequality

$$|h(x) - ax| \leqq H \tag{A_0}$$

holds for all $x \in (-\infty,+\infty)$.

Let there exist constants $H_2 \geqq 0$ and $H_0 > 0$ such that the inequality

$$|h_2(z)| \leqq H_2|z| + H_0 \tag{A_1}$$

is satisfied for all $z \in (-\infty,+\infty)$. If

$$(E + H_2 + Q_2)w_0 + (F + Q_1)w_0^2 < 1 ,$$

then the equation (3.1) has a w-periodic solution.

We may to proceed the proof with an estimate of integral

$$\int_0^w x''^2(t)dt$$

at first, using - besides (2), (3) and (4_0) - the inequality

$$\left|\int_0^w h_2[x''(t)]x'(t)dt\right| \leq \left(H_2 \int_0^w x''^2(t)dt + H_0\sqrt{w}\sqrt{\int_0^w x''^2(t)dt}\right)w_0$$

with regard to (A_1). Or, starting the proof immediately with an estimate of integral

$$\int_0^w x'''^2(t)dt,$$

we use - besides (2), (3), (4_0), (A_0) and regarding to (A_1) - the inequality or

$$\left|\int_0^w h_2[x''(t)]x(t)dt\right| \leq (H_2 D_2 + H_0\sqrt{w})\sqrt{\int_0^w x^2(t)dt} ,$$

where

$$\int_0^w x''^2(t)dt \leq D_2^2 , \ D_2 > 0$$

holds [cf.(9)] or /as in the foregoing theorem/

$$\left|\int_0^w h_2[x''(t)]x(t)dt\right| \leq \bar{H}_2\sqrt{w}\sqrt{\int_0^w x^2(t)dt} ,$$

where $\bar{H}_2 = \max|h_2(x'')|$ for $|x''| \leq D'' > 0$ [cf.(12)], for an estimate of integral

$$\int_0^w x^2(t)dt .$$

Proceeding as in the proof of Theorem 3 it is possible analogicaly to prove

Theorem 3.2. Let (2), (3), (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_2(t,x'') + h(t,x) = q(t,x,x',x'') . \qquad (3.2)$$

Let there exist a constant $a \in R - (0)$ and a constant $H > 0$ such that

$$|h(t,x) - ax| \leq H$$

holds for all $t \in (-\infty,+\infty)$ and for all $x \in (-\infty,+\infty)$.
Let the inequality

$$|h_2(t,z)| \leq H_2|z| + H_0 ,$$

where $H_2 \geq 0$, $H_0 > 0$, is satisfied for all $t \in (-\infty,+\infty)$ and for all $z \in (-\infty,+\infty)$. If

$$(E + Q_2 + H_2)w_0 + (F + Q_1)w_0^2 < 1 ,$$

then the equation (3.2) has a w-periodic solution.

Remark: Analogical theorems on the existence of w-periodic solution to the differential equation (1) with $g = h_2(x'') + h(t,x) - q$ or $g = h_2(t,x'') + h(x) - q$, where $q = q(t,x,x',x'')$, may be given as a special cases of the foregoing theorem.

Closing the part III. we present the theorems as a special case of Theorem 3.

Theorem 3.3.1. Let (2), (3), (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_2(x'') + ax = q(t,x,x',x'') \qquad (3.1.1)$$

where $a \in R - (0)$ is an arbitrary given constant. If

$$(E + Q_2)w_0 + (F + Q_1)w_0^2 < 1 ,$$

then the equation (3.1.1) has a w-periodic solution.

Theorem 3.3.2. Let (2), (3), (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_2(x'') + h(x) + ax = q(t,x,x',x'') \;, \qquad (3.1.2)$$

where $a \in R - (0)$ is an arbitrary given constant.

Let there hold one of the following two assumptions:

1) for all $x \in (-\infty,+\infty)$ holds

$|h(x)| \leq H$, where $H > 0$

as well as

$(E + Q_2)w_0 + (F + Q_1)w_0^2 < 1$

2) $h(x) \in C^1(-\infty,+\infty)$ is such that

$|h'(x)| \leq H'$, where $H' > 0$

and for all $x \in (-\infty,+\infty)$ holds

$$- h(x)x \leq H^* := \begin{cases} 0 \text{ if } 0 \leq h(x)x \\ H > 0 \text{ if } -H \leq h(x)x < 0 \end{cases}$$

as well as

$(E + Q_2)w_0 + (F + Q_1)w_0^2 + H'w_0^3 < 1$.

Then the equation (3.1.2) has a w-periodic solution.

Process of the proof of both theorems is the same of Theorem 3, i.e. we start with an estimate of integral

$$\int_0^w x'''^2(t)dt \;.$$

To the proof of the last theorem with the assumption 2): integrating by parts we get

$$\int_0^w h[x(t)]x'''(t)dt = - \int_0^w h'[x(t)]x'(t)x''(t)dt$$

sothat

$$\left|\int_0^w h[x(t)]x'''(t)dt\right| \leqq H'w_0^3 \int_0^w x'''^2(t)dt.$$

PART IV.

Theorem 4. Let (2), (3), (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_2(x'') + h_1(x') + \\ + h(x) = q(t,x,x',x'') . \qquad (4.1)$$

Let there exist a constant $a \in R - (0)$ and a constant $H > 0$ such that the inequality

$$|h(x) - ax| \leqq H \qquad (A_0)$$

is satisfied for all $x \in (-\infty,+\infty)$.

Let there exist constants $H_1 \geqq 0$, $H_0 > 0$ such that the inequality

$$|h_1(y)| \leqq H_1|y| + H_0 \qquad (A_1)$$

holds for all $y \in (-\infty,+\infty)$. If

$$(E + Q_2)w_0 + (F + H_1 + Q_1)w_0^2 < 1 ,$$

then the equation (4.1) has a w-periodic solution.

Proving this theorem we proceed accordingly to the proof of Theorem 3. For an estimate of integral

$$\int_0^w x'''^2(t)dt$$

we use besides (2), (3) and (4_0) the inequality

$$\left|\int_0^w \{h[x(t)] - ax(t)\} x'''(t)dt\right| \leqq H\sqrt{w}\sqrt{\int_0^w x'''^2(t)dt}$$

holding with regard to (A_0) and

$$\left|\int_0^w h_1[x'(t)]x'''(t)dt\right| \leq H_1 w_0^2 \int_0^w x'''^2(t)dt + $$
$$+ H_0\sqrt{w}\sqrt{\int_0^w x'''^2(t)dt}$$

holding with regard to (A_1).

Some modifications of the foregoing theorem are

Theorem 4.1. Let (2), (3), (4_0) hold in the differential equation (4.1). Let there exist a constant $a \in R - (0)$ such that the inequality

$$|h(x) - a^3x| \leq H ,$$

where $H > 0$, holds for all $x \in (-\infty,+\infty)$ and the inequality

$$|h_1(x') - 3a^2x'| \leq \hat{H}_1|x'| + H_1 ,$$

where $\hat{H}_1 \geq 0$, $H_1 > 0$, holds for all $x' \in (-\infty,+\infty)$. If

$$(E + Q_2)w_0 + (F + \hat{H}_1 + Q_1 + 3a^2)w_0^2 < 1 ,$$

then the equation (4.1) has a w-periodic solution.

Process of the proof is quite analogical to that of the Theorem 2.2. The differential equation (4.1) belong now to the system

$$x''' + m\{e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_2(x'') -$$
$$- 3ax'' + h_1(x') - 3a^2x' + h(x) - a^3x -$$
$$- q(t,x,x',x'')\} + 3ax'' + 3a^2x' + a^3x = 0 , \qquad (S_4)$$

which, similarly as (S_3), contain for $m = 0$ the linear homogeneous differential equation [cf.(5)] with the triple root $-a$ of the corresponding characteristic equation.

When we start the proof with an estimate of integral

$$\int_0^w x'''^2(t)dt$$

(see the proving process of Theorem 1) then for an estimate of integral

$$\int_0^w x^2(t)dt$$

to bring to a close may be used both bounding constant

$$\overline{H}_2 = \max|h_2(x'')| \quad \text{for} \quad |x''| \leqq D'' > 0 \quad [\text{cf.}(12)]$$

and

$$\overline{H}_1 = \max|h_1(x')| \quad \text{for} \quad |x'| \leqq D' > 0 \quad [\text{cf.}(11)]$$

in the inequalities

$$\left|\int_0^w h_2[x''(t)]x(t)dt\right| \leqq \overline{H}_2\sqrt{w}\sqrt{\int_0^w x^2(t)dt}$$

and

$$\left|\int_0^w h_1[x'(t)]x(t)dt\right| \leqq \overline{H}_1\sqrt{w}\sqrt{\int_0^w x^2(t)dt}$$

[the last instead the inequality

$$\left|\int_0^w h_1[x'(t)]x(t)dt\right| \leqq (\hat{H}_1 D_1 + H_1\sqrt{w})\sqrt{\int_0^w x^2(t)dt} ,$$

where

$$\int_0^w x'^2(t)dt \leqq D_1^2 , \quad D_1 > 0 \quad [\text{cf.}(10)] \text{]}.$$

Theorem 4.2. Let (2), (3), (4_0) hold in the differential equation (4.1). Let there exist a constant $a \in R - (0)$ such that the inequality

$$|h(x) - ax| \leqq H ,$$

where $H > 0$, is satisfied for all $x \in (-\infty,+\infty)$.

Let $h_1(y) \in C^1(-\infty,+\infty)$ and let

$$h_1'(y) \leqq H^* := \begin{cases} 0 \text{ if } h_1'(y) \leqq 0 \\ H_1 > 0 \text{ if } 0 < h_1'(y) \leqq H_1 \end{cases}$$

hold for all $y \in (-\infty,+\infty)$. If

$$(E + Q_2)w_0 + (F + H^* + Q_1)w_0^2 < 1 ,$$

then the equation (4.1) has a w-periodic solution.

The proof is quite analogical to that of the Theorem 2.3.

Special cases of Theorem 4 are

Theorem 4.3.1. Let (2), (3), (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_2(x'') + h_1(x') + + ax = q(t,x,x',x'') , \qquad (4.2.1)$$

where $a \in R - (0)$ is an arbitrary given constant.

Let there hold one of the following two assumptions:

1) for all $y \in (-\infty,+\infty)$ is satisfied the inequality
$|h_1(y)| \leqq H_1|y| + H$, where $H_1 \geqq 0$, $H > 0$,
as well as
$(E + Q_2)w_0 + (F + H_1 + Q_1)w_0^2 < 1$

2) $h_1(y) \in C^1(-\infty,+\infty)$ is such that for all $y \in (-\infty,+\infty)$ holds

$$h_1'(y) \leq H^* := \begin{cases} 0 & \text{if } h_1'(y) \leq 0 \\ H_1' > 0 & \text{if } 0 < h_1'(y) \leq H_1' \end{cases}$$

as well as

$$(E + Q_2)w_0 + (F + H^* + Q_1)w_0^2 < 1 .$$

Then the equation (4.2.1) has a w-periodic solution.

<u>Theorem 4.3.2.</u> Let (2), (3), (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_2(x'') + $$
$$+ h_1(x') + h(x) + ax = q(t,x,x',x'') , \qquad (4.2.2)$$

where $a \in R - (0)$ is an arbitrary given constant.

Let there hold one of the following four assumptions:

1) for all $x \in (-\infty,+\infty)$ is satisfied the inequality

$|h(x)| \leq H$, where $H > 0$,

and for all $y \in (-\infty,+\infty)$ is satisfied the inequality

$|h_1(y)| \leq H_1|y| + H_0$, where $H_1 \geq 0$, $H_0 > 0$,

as well as

$$(E + Q_2)w_0 + (F + H_1 + Q_1)w_0^2 < 1$$

2) $h(x) \in C^1(-\infty,+\infty)$ is such that for all $x \in (-\infty,+\infty)$ holds

$$- h(x)x \leq H^* := \begin{cases} 0 & \text{if } 0 \leq h(x)x \\ H > 0 & \text{if } -H \leq h(x)x < 0 \end{cases}$$

and

$|h'(x)| \leq H'$, where $H' > 0$,

for all $y \in (-\infty,+\infty)$ holds

$|h_1(y)| \leq H_1|y| + H_0$, where $H_1 \geq 0$, $H_0 > 0$,

as well as

$$(E + Q_2)w_0 + (F + H_1 + Q_1)w_0^2 + H'w_0^3 < 1$$

3) for all $x \in (-\infty,+\infty)$ holds

$|h(x)| \leqq H$, where $H > 0$,

and $h_1(y) \in C^1(-\infty,+\infty)$ is such that for all $y \in (-\infty,+\infty)$ holds

$$h_1'(y) \leqq H_1^* := \begin{cases} 0 \text{ if } h_1'(y) \leqq 0 \\ H_1' > 0 \quad \text{if } 0 < h_1'(y) \leqq H_1' \end{cases}$$

as well as

$$(E + Q_2)w_0 + (F + H_1^* + Q_1)w_0^2 < 1$$

4) $h(x) \in C^1(-\infty,+\infty)$ is such that for all $x \in (-\infty,+\infty)$ holds

$$-h(x)x \leqq H^* := \begin{cases} 0 \text{ if } 0 \leqq h(x)x \\ H > 0 \text{ if } \quad -H \leqq h(x)x < 0 \end{cases}$$

and

$|h'(x)| \leqq H'$, where $H' > 0$,

$h_1(y) \in C^1$ $(-\infty,+\infty)$ is such that for all $y \in (-\infty,+\infty)$ holds

$$h_1'(y) \leqq H_1^* := \begin{cases} 0 \text{ if } h_1'(y) \leqq 0 \\ H_1' > 0 \text{ if } 0 < h_1'(y) \leqq H_1' \end{cases}$$

as well as

$$(E + Q_2)w_0 + (F + H_1^* + Q_1)w_0^2 + H'w_0^3 < 1 .$$

Then the equation (4.2.2) has a w-periodic solution.

To prove the both theorems we may use the elements of the proving procedure of the all foregoing theorems admissible with regard to the corresponding assumptions.

Closing the part IV. we present the theorem with a more generalized form of the function g in (1).

Theorem 4.4. Let (2), (3), (4_0) hold in the differential equation

$$x''' + e(t,x,x',x'')x'' + f(t,x,x',x'')x' + h_2(t,x'') + $$
$$+ h_1(t,x') + h(t,x) = q(t,x,x',x'') \ . \qquad (4.4)$$

Let there exist constants $a \in R - (0)$ and $H > 0$ such that for all $t \in (-\infty,+\infty)$ and for all $x \in (-\infty,+\infty)$ is satisfied the inequality

$$|h(t,x) - ax| \leqq H \ .$$

Let for all $t \in (-\infty,+\infty)$ and for all $y \in (-\infty,+\infty)$

$$|h_1(t,y)| \leqq \hat{H}_1|y| + H_1 \ , \text{ where } \hat{H}_1 \geqq 0, \ H_1 > 0 \ ,$$

and for all $t \in (-\infty,+\infty)$ and for all $z \in (-\infty,+\infty)$

$$|h_2(t,z)| \leqq \hat{H}_2|z| + H_2 \ , \text{ where } \hat{H}_2 \geqq 0, \ H_2 > 0 \ ,$$

hold. If

$$(E + \hat{H}_2 + Q_2)w_0 + (F + \hat{H}_1 + Q_1)w_0^2 < 1 \ ,$$

then the equation (4.4) has a w-periodic solution.

Let us note that in the case of

$$g = h_2(x'') + h_1(x') + h(t,x) - q$$
$$\text{or } g = h_2(x'') + h_1(t,x') + h(x) - q$$
$$\text{or } g = h_2(t,x'') + h_1(x') + h(x) - q$$

occuring in the differential equation (1) it is possible to modify the relevant theorem on existence of a periodic solution in view of the appropriate assumptions of $h_1(x')$ or $h(x)$ respectively.

On existence of a periodic solution to (1) with a general term g we may give the general

Theorem 4.5. Let (2) and (3) hold in the differential equation (1). Let there exist constants $a \in R - (0)$ and $G > 0$ such that for all $t,y,z \in (-\infty,+\infty)$ and for all $x \in (-\infty,+\infty)$ holds the inequality

$$|g(t,x,y,z) - ax| \leqq G \ .$$

If

$$(E + Fw_o)w_o < 1 ,$$

then the equation (1) has a w-periodic solution.

Proving this theorem with respect to their assumptions, we go under one´s way of accustomed procedure to

$$\int_0^w x'''^2(t)dt \leqq D_3^2 , \quad \text{where } D_3 := \frac{G\sqrt{w}}{K} > 0$$

with $K = 1 - (E + Fw_o)w_o > 0$, so that

$$\int_0^w x''^2(t)dt \leqq D_2^2 , \text{ where } D_2 := w_o D_3 > 0$$

$$\int_0^w x'^2(t)dt \leqq D_1^2 , \text{ where } D_1 := w_o D_2 > 0 ,$$

from whose [cf.(11) and (12)]

$$|x''(t)| \leqq D'' , \text{ where } D'' := \sqrt{w}\, D_2 > 0$$

$$|x'(t)| \leqq D' , \text{ where } D' := \sqrt{w}\, D_1 > 0$$

and further

$$\int_0^w x^2(t)dt \leqq D_o^2 , \text{ where } D_o := \frac{1}{|a|} (ED_2 + FD_1)\sqrt{w} > 0 ,$$

from whose [cf.(14)]

$$|x(t)| \leqq D , \text{ where } D := \left[\frac{D_o}{\sqrt{w}} + \sqrt{w}\, D_1 \right] > 0 .$$

So that, with regard to the inequality $|x^{(j)}(t)| \leqq \max (D,D',D'')$ holding for $j = 0,1,2$ [cf.(15)] and to $a \in R - (0)$, $a \neq 0$, the sufficient conditions on existence of a w-periodic solution to (1) are fulfilled.

SOUHRN

K EXISTENCI PERIODICKÉHO ŘEŠENÍ NELINEÁRNÍ DIFERENCIÁLNÍ ROVNICE TŘETÍHO ŘÁDU

VLADIMÍR VLČEK

Existence periodického řešení nelineární diferenciální rovnice 3.řádu (1) je postupně vyšetřována s ohledem na různé tvary jejího posledního členu, tj. funkce g. Ve větách jsou uvedeny podmínky k zajištění stejnoměrné ohraničenosti všech řešení (včetně jejich derivací) jistého jednoparametrického systému diferenciálních rovnic, což vzhledem k užité metodě důkazu stačí k existenci periodického řešení uvažované rovnice. Současně je ukázáno, nakolik a jakým způsobem podmínky kladené na jednotlivé členy v rovnici (1) ovlivňují příslušné ohraničující konstanty.

РЕЗЮМЕ

К СУЩЕСТВОВАНИЮ ПЕРИОДИЧЕСКОГО РЕШЕНИЯ НЕЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ 3-ОГО ПОРЯДКА

В. ВЛЧЕК

Существование периодического решения нелинейного дифференциального уравнения (1) изучается постепенно принимая во внимание разные формы его последнего члена, именно функции g . В теоремах приведены условия гарантирующие равномерную ограниченность всех решений /и их производных/ совершенной однопараметрической системы дифференциальных уравнений, что - имея в виду примененный метод доказательства - достаточно к существованию периодического решения (1). Одновременно показывается как требования к отдельным членам уравнения (1) влияют на соответственные ограничивающие постоянные.

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