Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 28 (1989), No. 1, 151--163

Persistent URL: `http://dml.cz/dmlcz/120228`

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1989 MATHEMATICA XXVIII VOL. 94

Katedra matematiky
pedagogické fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Doc.RNDr. František Zapletal

ON TWO-DIMENSIONAL LINEAR SPACES OF CONTINUOUS FUNCTIONS OF THE SAME CHARACTER

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(Received February 22, 1988)

This paper presents a necessary and sufficient condition for the existence of a global transformation of a strongly regular space of continuous functions onto a strongly regular space of continuous functions. The results obtained are then applied to spaces of solutions of second order linear differential equations of a general form

$$y'' + a(t)y' + b(t)y = 0 , \qquad \text{(ab)}$$

where $a, b \in C^{(0)}(j)$ and of the Sturm form

$$(p(t)y')' + q(t)y = 0 , \qquad \text{(pq)}$$

where $p, q \in C^{(0)}(j)$, $py' \in C^{(1)}(j)$, $p(t) \neq 0$ in j, whereby $C^{(0)}(j)$ and $(C^{(1)}(j))$ respectively denote a set of all continuous functions and a set of all functions with a continuous first derivate, on the interval j.

1. Global transformation and a two-dimensional canonical space of continuous functions

In this section we shall follow the discussion of [3] from which we recall some definitions:

Let R be a field of real numbers and j be an open interval in R. Further let $y_1, y_2 \in C^{(0)}(j)$. We say that the functions y_1, y_2 are dependent on the interval j if there exist such numbers $k_1, k_2 \in R$, $k_1^2 + k_2^2 > 0$ that the identity $k_1y_1(t) + k_2y_2(t) \equiv 0$ holds for every $t \in j$. If for any two numbers $k_1, k_2 \in R$, $k_1^2 + k_2^2 > 0$ and for any subinterval $j_1, j_1 \subset j$, the relation $k_1y_1(t) + k_2y_2(t) \not\equiv 0$ holds on j, we say that the functions y_1, y_2 are independent on j.

Suppose $y_1, y_2 \in C^{(0)}(j)$ are independent functions on j. A linear space of functions with a basis (y_1, y_2) over the field R will be called a two-dimensional space of continuous functions or also a two-dimensional space of continuous functions generated by the functions y_1, y_2 with the definition interval j.

Definition 1. Let j, J be open intervals in R. Let S_1 and S_2 be the spaces of continuous functions generated by the functions y_1, y_2 with the definition interval j, and by the functions Y_1, Y_2 with the definition interval J, respectively. We say, the space S_2 globally transforms itself onto the space S_1 if there exist

a) a bijection $h : j \longrightarrow J$, $h \in C^{(0)}(j)$,

b) a function $f \in C^{(0)}(j)$, $f(t) \neq 0$ for $t \in j$,

c) a matrix $A = \| a_{ik} \|$, $i,k = 1,2$, $a_{ik} \in R$, $\det A \neq 0$

to the vectors $y = (y_1, y_2)^T$, $Y = (Y_1, Y_2)^T$ such that the equality

$$y(t) = A\, f(f)\, Y[h(t)] \qquad (1)$$

holds for every $t \in j$, where $(.,.)^T$ denotes a transposed vector to the vector $(.,.)$.

The mapping of the column vector Y on the column vector y defined by (1) will be denoted by $\mathcal{T}$ and written as $\mathcal{T}Y = y$, $\mathcal{T} = \langle Af, h \rangle$. The mapping $\mathcal{T}$ will be called the global transformation of the space S_2 onto the space S_1.

In [2] the global transformation is shown as an equivalence relation on the set of the two-dimensional spaces of continuous functions.

Definition 2. The two-dimensional space of continuous functions S^* generated by the functions cos s, sin s, $s \in J$ will be called the two-dimensional canonical space of continuous functions with the definition interval J.

Theorem 1. Suppose S^* is a two-dimensional canonical space of continuous functions with a definition interval J and that $Y_1, Y_2 \in S^*$. If T_o, T_1 are two neighbouring zeros of the function Y_1, i.e. $Y_1(T_o) = Y_1(T_1) = 0$ and $Y_1(T) \neq 0$ for any $T \in (T_o, T_1)$, then the function Y_2 has exactly one zero in the interval (T_o, T_1) provided the functions Y_1, Y_2 are independent. If the functions Y_1, Y_2 are dependent, then both functions have all their zeros in common.

Proof. Since the functions cos s, sin $s \in S^*$ form a basis of the space S^*, there exist numbers $k_1, k_2 \in R$ such that

$$Y_1 = k_1 \cos s + k_2 \sin s \tag{2}$$

and numbers $\ell_1, \ell_2 \in R$ such that

$$Y_2 = \ell_1 \cos s + \ell_2 \sin s , \tag{3}$$

$s \in J$. The functions Y_1, Y_2 defined by equations (1) and (2) may be written as

$$Y_1 = K_1 \cos (s - K_2), \; s \in J, \tag{4}$$

$$Y_2 = L_1 \sin (s - L_2), \; s \in J, \tag{5}$$

where K_1, K_2, L_1, L_2 are constants given by the formulas

$$K_1 = \sqrt{k_1^2 + k_2^2} \text{ and for } K_2 \text{ we have } \cos K_2 = k_1/K_1, \quad \sin K_2 = k_2/K_1,$$

$$L_1 = \sqrt{\ell_1^2 + \ell_2^2} \text{ and for } L_2 \text{ we have } \cos L_2 = \ell_1/L_1, \quad \sin L_2 = -\ell_2/L_1.$$

It follows from (4) and (5) that the distance of any two neighbouring zeros of any function of the space S^* is equal to π. Thus, if T_0 and T_1 are two neighbouring zeros of the function Y_1, then their distance is equal to π. Since the distance of the neighbouring zeros of the function Y_2 is also equal to π, there may occur two cases:

1. If the function Y_2 has a zero with the function Y_1 in common in the interval J, then all their zeros are in common which occurs exactly when Y_1 and Y_2 are dependent functions.

2. If the functions Y_1, Y_2 are independent and T_0, T_1 are two neighbouring zeros of the function Y_1, then the function Y_2 has exactly one zero in the interval (T_0, T_1).

Indeed, if there were lying no zero of the function Y_2 in (T_0, T_1), then the neighbouring zeros of the function Y_2 would have a distance greater than π, which is impossible. There must therefore lie at least one zero of the function Y_2 in (T_0, T_1). However, two zeros of the function Y_2 cannot lie in (T_0, T_1) because the distance between the neighbouring zeros of the function Y_2 would be less than π. Hence, between any two neighbouring zeros of the function Y_1 there lies exactly one zero of the function Y_2. Similarly may be shown that there lies exactly one zero of the function Y_1 between any two neighbouring zeros of the function Y_2.

Definition 3. The fact observed above may be expressed by saying that the zeros of independent functions of a canonical space S^* of continuous functions separate.

2. Two-dimensional spaces of continuous functions of the same character

Definition 4. Suppose S^* is a two-dimensional canonical space of continuous functions with the definition interval J. We denote by M a set of all two-dimensional spaces of continuous functions such that the following expression holds: If $S \in M$, then there exists a global transformation of the canonical spaces S^* onto S.

Lemma 1. Suppose $j = (a,b)$, $J = (A,B)$ are open intervals, h is a bijection $h: j \longrightarrow J$, and that $h \in C^{(0)}(j)$. Then $h = h(t)$, $t \in j$ is a strictly monotonic function in j.

Proof. Let $t_1, t_2 \in (a,b)$ be arbitrary points. Then either $h(t_1) < h(t_2)$ or $h(t_2) < h(t_1)$, for h is a bijection.

Let $h(t_1) < h(t_2)$. If it were true that $h(t_0) > h(t_1)$ for any point $t_0 \in (a, t_1)$, then, with respect to Darboux's property of the continuous function, the intervals of functional values $\langle h(t_1), h(t_2) \rangle$ and $\langle h(t_1), h(t_0) \rangle$ would be incident and h would not be a bijection. From this it also follows that $\lim_{t \to a+} h(t) = A$. Similarly may be shown that $\lim_{t \to b-} h(t) = B$. The function h is thus increasing in the interval j.

In case of $h(t_2 < h(t_1)$ we then have, by similar reasoning to that above, $\lim_{t \to a+} h(t) = B$, $\lim_{t \to b-} h(t) = A$, and therefore the function h is decreasing in the interval j.

Theorem 2. Suppose S^* is a two-dimensional canonical space of continuous functions with the definition interval J generated by the functions $\cos s$, $\sin s$, and $S \in M$ is a two-dimensional space of continuous functions generated by the functions y_1, y_2 with the definition interval j. Moreover, let the space S^* be globally transformed onto the space S so that the basis $(\cos s, \sin s) \in S^*$ is transformed onto the basis $(y_1(t), y_2(t)) \in S$ by the formula

$$y(t) = A\, f(t)\, Y[h(t)] \,, \tag{6}$$

where $y = (y_1, y_2)^T$, $Y = (\cos s, \sin s)^T$, $h: j \to J$, $h \in C^{(0)}(j)$. Let $k_1, k_2 \in R$ be arbitrary numbers.

Then, by (6) the zeros of the function $k_1 y_1(t) + k_2 y_2(t) \in \in S$ and those of the function $k_1 (a_{11} \cos s + a_{12} \sin s) + + k_2(a_{21} \cos s + a_{22} \sin s) \in S^*$, where $s = h(t)$, $t \in j$, are schlicht mapped onto themselves by the function h.

Proof. Multiplying out (6) by the vector $k = (k_1, k_2)$, $k_1, k_2 \in R$ we obtain

$$k_1 y_1(t) + k_2 y_2(t) = f(t)\left[k_1(a_{11} \cos h(t) + a_{12} \sin h(t)) + \right.$$
$$\left. + k_2(a_{21} \cos h(t) + a_{22} \sin h(t))\right]$$

or with respect to (4)

$$k_1 y_1(t) + k_2 y_2(t) = K_1 f(t) \cos\left[h(t) - K_2\right], \tag{7}$$

where

$$K_1 = \sqrt{(k_1 a_{11} + k_2 a_{21})^2 + (k_1 a_{12} + k_2 a_{22})^2}$$

$$\cos K_2 = (k_1 a_{11} + k_2 a_{21})/K_1, \quad \sin K_2 = (k_1 a_{12} + k_2 a_{22})/K_1.$$

We see that at the points $t_i \in J$, at which the expression $h(t) - - K_2$ takes the values $-\frac{\pi}{2} + i\pi$, i being an integer, the right hand side of equation (7) equals to zero, and thus also the left hand side of equation (7) equals to zero. So the points t_i are just only common zeros of the function $k_1 y_1(t) + + k_2 y_2(t)$ and of the function $\cos\left[h(t) - K_2\right]$, because h is a strictly monotonic function and $K_1 \neq 0$, $f(t) \neq 0$ for $t \in j$.

Definition 5. Suppose $S \in M$ is a two-dimensional space of continuous functions generated by the functions y_1, y_2 with a definition interval j. We call the space S of finite (infinite) type on j, according as the functions of S have finite (infinite) many zeros in j.

The space S is called of finite type (m), m being positive integer, on j, according as at least one function of the space S has m zeros in j and no function of S has more than m zeros in j.

The space S of finite type (m) is called general if there exist functions in S having m zeros in j, and two independent functions having (m-1) zeros in j.

The space S of finite type (m) is called special if there exists a function in S having (m-1) zeros in j and any other function independent of this function has exactly m zeros in j.

The spaces S of infinite type are split up into onesided, resp. bothsided oscillatory on j according as at least one function of the space S has infinitely many zeros in j for which exactly one of the endpoints a, b of j, resp. exactly both endpoints a, b of j are the cluster points of zeros of that function.

We thus distinquish two-dimensional spaces of continuous functions from the set M according to the type: finite, infinite and according to the kind: general, special, onesided oscillatory, (bothsided) oscillatory.

Spaces of the same kind and type are called of the same character. (See [1], [4].)

We consider a two-dimensional canonical space S^* of continuous functions with the definition interval J, where J is an open interval (A,B).

To simplify the situation and considerations we study the following four basic cases of the definition interval (A,B). (Conf. [1], Canonical forms of the differential equation (q).)

I a) $A = 0,\ B = -\frac{1}{2}\pi + m\pi,\ m \geq 1$, integral,

b) $A = 0,\ B = m\pi,\ m \geq 1$, integral,

II a) $A = 0,\ B = +\infty$, resp. $A = -\infty,\ B = 0$,

b) $A = -\infty,\ B = +\infty$.

In the cases denoted by I the functions of the space S^* have a finite number of zeros as it follows from the separation theorem of zeros of the functions of the space S^*, and from the following consideration.

In the case a) there exist two independent functions $\sin s$, $\sin (s - \frac{\pi}{2})$ in S^* having exactly (m-1) zeros in J and besides the functions $\sin(s - c)$, where $0 < c < \frac{1}{2}\pi$, have then exactly m zeros.

In the case b) besides the function sin s and on it dependent functions, having exactly (m-1) zeros in J, all other functions have exactly m zeros in S^*.

Taking, for instance, the function $\sin s \in S^*$ into consideration for which $\sin A = \sin 0 = 0$ it can be easily seen from the Separation theorem of zeros that in the cases under II the functions of the space S^* have infinitely many of zeros.

In the case a) the zeros of each function of S^* have one and only one cluster point $+\infty$, resp. $-\infty$ in the interval considered.

In the case b) the zeros of each function of S^* have exactly two cluster points, namely, $-\infty$ and $+\infty$ in the interval considered.

So we are able, concurrently with Definition 5, to express the following

Theorem 3. The spaces $S \in M$ with the definition interval j which globally transform themselves onto a canonical space of continuous functions S^*, are in the cases

I.	of finite type (m),	ad a)	general
		ad b)	special
II.	of infinite type,	ad a)	onesided oscillatory (to the right or to the left)
		ad b)	bothsided oscillatory.

Main result:

Theorem 4. Suppose S_1 and S_2 are spaces of continuous functions belonging to the set M, with the definition intervals j_1 and j_2, respectively. A necessary and sufficient condition of the global transformation of the space S_2 onto the space S_1 is that the spaces be of the same character.

Proof. Let the space S_2 be globally transformed onto the space S_1. Then there exist to the bases $(y_1, y_2) \in S_1$, $(z_1, z_2) \in S_2$

a bijection $h: j_1 \to j_2$, $h \in C^{(0)}(j_1)$,

a function $f \in C^{(0)}(j_1)$, $f(t) \neq 0$ for $t \in j_1$,

a matrix $A = \| a_{ik} \|$, $i,k = 1,2$, $\det A \neq 0$

such that

$$y(t) = A\, f(t)\, z[h(t)]$$

for $t \in j_1$, where $y = (y_1, y_2)^T$, $z = (z_1, z_2)^T$.

Denoting $\tilde{z} = (a_{11}z_1 + a_{12}z_2, a_{21}z_1 + a_{22}z_2)^T$, then

$$y(t) = f(t)\, \tilde{z}[h(t)] \quad . \tag{8}$$

If we set $k = (k_1, k_2)$, $k_1, k_2 \in R$, we get from (8)

$$k\, y(t) = f(t)\, k\, \tilde{z}[h(t)]$$

or

$$k_1 y_1(t) + k_2 y_2(t) = f(t)[k_1(a_{11}z_1 + a_{12}z_2) + k_2(a_{21}z_1 + a_{22}z_2)].$$

It follows from this that the roots of the corresponding functions in the mapping $\mathcal{T} = \langle Af, h \rangle$ map schlicht on themselves, which gives the same character of the spaces S_1 and S_2.

Suppose conversely the spaces S_1, S_2 are of the same character, $(y_1, y_2) \in S_1$, $(z_1, z_2) \in S_2$ are bases, j_1 is a definition interval of S_1 and j_2 is a definition interval of S_2. It then follows from Definition 4 and from Theorem 3 that the spaces S_1 and S_2 globally transform themselves onto the canonical space of continuous functions S^* with the definition interval J with takes one of the four cases given in Ia - IIb.

So, there exist

a bijection $h: j_1 \longrightarrow J$, $h \in C^{(0)}(j_1)$,

a function $f \in C^{(0)}(j_1)$, $f(t) \neq 0$ for $t \in j_1$,

a matrix $A = \| a_{ik} \|$, $i,k = 1,2$, $\det A \neq 0$

such that

$$y(t) = A\, f(t)\, Y\left[h(t)\right] , \tag{9}$$

where $Y = (Y_1, Y_2)^T$, $Y_1 = \cos s$, $Y_2 = \sin s$, $s \in J$
and also

a bijection $H: j_2 \longrightarrow J$, $H \in C^{(0)}(j_2)$,

a function $F \in C^{(0)}(j_2)$, $F(T) \neq 0$ for $T \in j_2$,

a matrix $B = \| b_{ik} \|$, $i,k = 1,2$, $\det B \neq 0$

such that

$$z(T) = B\, F(T)\, Y\left[H(T)\right] , \tag{10}$$

where $Y = (Y_1, Y_2)^T$, $Y_1 = \cos s$, $Y_2 = \sin s$, $s \in J$.
From (9) and (10) we obtain

$$y(t) = A\, B^{-1} \frac{f(t)}{F\left[H^{-1}h(t)\right]} \cdot z\left[H^{-1}h(t)\right] ,$$

where H^{-1} is an inverse function to H, B^{-1} is an inverse matrix to the matrix B.

Since

$H^{-1}\left[h(t)\right]$ is a bijection : $j_1 \longrightarrow j_2$, $H^{-1}\left[h(t)\right] \in C^{(0)}(j_1)$,

$f(t)/F\left[H^{-1}h(t)\right] \in C^{(0)}(j_1)$, $f(t)/F\left[H^{-1}h(t)\right] \neq 0$ for $t \in j_1$,

$A\, B^{-1}$ is a second order regular matrix,

so the space S_2 globally transforms itself onto the space S_1.

Remark 1. In [3] the spaces belonging to the set M are named two-dimensional strongly regular spaces of continuous functions.

3. Spaces of solutions of differential equations (ab), (pq)

As special cases of strongly regular spaces of continuous functions we introduced in [3] the spaces of solutions of second order linear differential equations (ab) and (pq).

The spaces of solutions of the differential solutions (ab) and (pq) are denoted by S_{ab}, and by S_{pq}, respectively.

If we apply Theorem 3 and 4 to the cases of spaces S_{ab} and S_{pq}, we may express assertions analogous to those proved for the spaces of solutions of the Jacobian second order linear differential equation in [1].

Theorem 5. The spaces S_{ab} (S_{pq}) with the definition interval j which globally transform themselves onto the canonical space S^{*} with the definition interval $J = (A,B)$, are in the

case I
- a) $A = 0$, $B = -\frac{1}{2}\pi + m\pi$, $m \geq 1$ integral, of finite type (m) and this general,
- b) $A = 0$, $B = m\pi$, $m \geq 1$ integral, of finite type (m) and this special,

II
- a) $A = 0$, $B = +\infty$ of infinite type, and this onesided right oscillatory,
  resp. $A = -\infty$, $B = 0$ of infinite type, and this onesided left oscillatory,
- b) $A = -\infty$, $B = +\infty$ of infinite type, and this bothsided oscillatory.

Theorem 6. Suppose S_{ab}, S_{AB} (S_{pq}, S_{PQ}) are the spaces of solutions of differential equations (ab), (AB), ((pq),(PQ)) with the definition intervals j resp. J. A necessary and sufficient condition of the global transformation of the space S_{AB} onto the space S_{ab} (S_{PQ} onto S_{pq}) is that the spaces S_{ab}, S_{AB} (S_{pq},S_{PQ}) be of the same character.

Summary

In the present paper there is shown that a necessary and sufficient condition for the existence of a global transforma-

tion of two-dimensional strongly regular spaces of continuous functions onto themselves is that the spaces must be of the same character.

The result is applied to spaces of solutions of second order linear differential equations of general and Sturm forms.

Souhrn

LINEÁRNÍ PROSTORY SPOJITÝCH FUNKCÍ DIMENZE 2
TÉHOŽ CHARAKTERU

V práci je dokázáno, že nutná a postačující podmínka globální transformace dvou silně regulárních prostorů spojitých funkcí dimenze 2 na sebe je, aby prostory byly téhož charakteru. Výsledek je aplikován na prostory řešení lineárních diferenciálních rovnic 2.řádu obecného a Sturmova tvaru.

Р е з ю м е

ЛИНЕЙНЫЕ ДВУМЕРНЫЕ ПРОСТРАНСТВА НЕПРЕРЫВНЫХ ФУНКЦИЙ
ТОГО ЖЕ ХАРАКТЕРА

В настоящей работе доказано, что необходимое и достаточное условие глобальной трансформации двух сильно регулярных двумерных пространств непрерывных функций на себя есть, чтобы пространства были того же самого характера.

Результат применяется к пространствам решений линейных дифференциальных уравнений 2-ого порядка общей и Штурмовой формы.

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