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Jaroslav Hančl

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THE FOURIER METHOD FOR THE LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE n-TH ORDER

JAROSLAV HANČL

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The Fourier method for linear partial differential equations with constant coefficients is well-known in mathematics. Combinating this method with Kummer transformation it is possible to solve a greater class of linear partial differential equations. This paper is a continuation of paper 3 .

1. Introduction

We will solve the following equation

$$\sum_{k=o}^{n} a_k(t) \frac{\partial^k u}{\partial t^k} = \sum_{k=o}^{n} b_k(y) \frac{\partial^k u}{\partial y^k} \qquad (1)$$

in intervals $t \in J_1$, $y \in J_2$ where $J_1 = \langle t_o, t_1 \rangle$ or $J_1 = \langle t_o, \infty)$, t_o, t_1 are real numbers and $J_2 = \langle y_o, y_1 \rangle$ or $J_2 = \langle y_o, \infty)$, y_o, y_1 are also real numbers.

2. Solutions and assumptions

Let B and C are real numbers such that for every $t \in J_1$ and $y \in J_2$ hold $Ba_n(t)$, $Cb_n(y) > 0$ and $a_n(t) \in C^{(n)}(J_1)$, $b_n(y) \in C^{(n)}(J_2)$, $a_{n-1}(t) \in C^{(n-1)}(J_1)$, $b_{n-1}(y) \in C^{(n-1)}(J_2)$.

Definition: $u(t,y)$ is a solution of equation (1) on $(t,y) \in J_1 \times J_2$ if and only if $u(t,y)$ satisfies equation (1) everywhere and $u(t,y) \in C^{(n)}(J_1,J_2)$.

3. Main result

Theorem 1: Let (1) be an equation and let us suppose above assumptions. Denote

$$x(t) = \int_{t_o}^{t} \left(\frac{1}{Ba_n(s)} \right)^{\frac{1}{n}} ds + K_1 \tag{2}$$

$$Y(y) = \int_{y_o}^{t} \left(\frac{1}{Cb_n(s)} \right)^{\frac{1}{n}} ds + K_2 \tag{3}$$

where K_1 and K_2 are positive real numbers and $t \in J_1$, $y \in J_2$. Let B_i, C_i $(i = 0,1,\dots,n-1)$ are real numbers such that $Q_1(z)$ is a solution of the equation

$$Z^n Q_1^{(n)}(Z) + \sum_{i=o}^{n-1} B_i Z^i Q_1^{(i)}(Z) = Bz^n Q_1(Z) \tag{4}$$

in the interval $z \in \langle k_1, \infty)$ and $Q_2(z)$ is a solution of the equation

$$Z^n Q_2^{(n)}(z) + \sum_{i=0}^{n-1} C_i Z^i Q^{(i)}(Z) = Cz^n Q_2(z) \tag{5}$$

in the interval $z \in \langle K_2, \infty)$.

Denote

$$F(t) = (x(t))^{\frac{B_{n-1}}{n}} (x'(t))^{-\frac{n-1}{2}} e^{-\int_{t_o}^{t} \frac{a_{n-1}(t)dt}{na_n(t)}} \tag{6}$$

$$G(y) = (Y(y))^{\frac{C_{n-1}}{n}} (Y'(y))^{-\frac{n-1}{2}} e^{-\int_{y_o}^{y} \frac{b_{n-1}(y)dy}{nb_n(y)}} \tag{7}$$

and let us suppose that for $a_k(t)$ and $b_k(y)$ $(k = 0,1,\dots,n)$ hold the following identities

$$a_k(t) = \begin{vmatrix} 1, & \dots & \dots & a_n(t) \\ \vdots & & \vdots & \\ \dfrac{\frac{\partial^n}{\partial\varrho^n}\left[(x(t+\varrho) - x(t))^k F(t+\varrho)\right]_{\varrho=0}}{k!\,(x'(t))^k F(t)}, & \dots & \dots & B_k a_n(t) \left(\frac{x'(t)}{x(t)}\right)^{n-k} \end{vmatrix} \tag{8}$$

thus $a_k(t) = |D|$ where $D = (d_{ij})_{i,j=1}^{n-k+1}$

$$d_{ij} = \begin{cases} \dfrac{\dfrac{\partial^{n+1}}{\partial \varrho^{n+1-j}} \left[(x(t+\varrho)-x(t))^{n-i+1} F(t+\varrho)\right]_{\varrho=0}}{(n+1-i)!\ (x'(t))^{n+1-i}\ F(t)} & \text{for } i \geq j,\ j \neq n-k+1 \\ B_{n+1-i}\ a_n(t) \left(\dfrac{x'(t)}{x(t)}\right)^{i-1} & \text{for } i \neq j,\ j = n-k+1 \\ a_n(t) & \text{for } i = 1,\ j = n-k+1 \\ 0 & \text{for } i < j,\ j \neq n-k+1 \end{cases}$$

and

$$b_k(y) = \begin{vmatrix} 1, & \dots & \dots & b_n(y) \\ \vdots & & \vdots & \\ \dfrac{\dfrac{\partial^n}{\partial \varrho^n} \left[(Y(y+\varrho) - Y(y))^k\ G(y+\varrho)\right]_{\varrho=0}}{k!\ (Y'(y))^k\ G(y)}, & \dots & \dots & C_k\ b_n(y) \left(\dfrac{Y'(y)}{Y(y)}\right)^{n-k} \end{vmatrix} \qquad (9)$$

thus $b_k(y) = |P|$ where $P = (p_{ij})_{i,j=1}^{n-k+1}$

$$p_{ij} \begin{cases} = \dfrac{\dfrac{\partial^{n+1-j}}{\partial \varrho^{n+1-j}} \left[(Y(y+\varrho)-Y(y))^{n+1-i} G(y+\varrho) \right]_{\varrho=0}}{(n+1-i)!\ (Y'(y))^{n+1-i}\ G(y)} & \text{for } i \geq j,\ j \neq n-k+1 \\ = C_{n+1-i}\ b_n(y) \left(\dfrac{Y'(y)}{Y(y)}\right)^{i-1} & \text{for } i \neq 1,\ j=n-k+1 \\ = b_n(y) & \text{for } i=1,\ j=n-k+1 \\ = 0 & \text{for } i < j,\ j \neq n-k+1 \end{cases}$$

We can make sure by the calculation that these identities agree with (4) and (5).

At the end let us suppose that the series

$$u(t,y) = \sum_{\lambda=1}^{\infty} A_\lambda Q_1(\lambda X(t)) F(t) Q_2(\lambda Y(y)) G(y) \in C^{(n)}(J_1, J_2) \quad (10)$$

(Where A_λ are real numbers for every natural number λ), converges and the k-th partial derivatives ($k = 0,1,\dots,n$) with respect to t and y term by term of the series (10) are convergent to the k-th partial derivatives with respect to t and y of the function $u(t,y)$. Then $u(t,y)$ is a solution of equation (1).

Proof : It is sufficient to prove that the single term of identity (10) is a solution of equation (1). It implies that it is sufficient to prove that equations

$$\sum_{k=0}^{n} a_k(t) \frac{\partial^k}{\partial t^k} (F(t) Q_1(\lambda X(t))) = \lambda^n F(t) Q_1(\lambda X(t)) \quad (11)$$

$$\sum_{k=0}^{n} b_k(y) \frac{\partial^k}{\partial y^k} (G(y) Q_2(\lambda Y(y))) = \lambda^n G(y) Q_2(\lambda Y(y)) \quad (12)$$

hold for every natural number λ . We will only prove that equation (11) holds. The proof that equation (12) holds is similar. Equation (11) is equivalent with the equation

$$\sum_{k=0}^{n} a_k(t) \sum_{j=0}^{k} \binom{k}{j} \frac{\partial^j}{\partial t^j} Q_1(\lambda x(t)) \, F^{(k-j)}(t) = \lambda^n F(t) Q_1(\lambda x(t)) \qquad (13)$$

Let us denote

$$\frac{\partial^j}{\partial t^j} Q_1(\lambda x(t)) = \sum_{i=0}^{j} S_i^j \, Q_1^{(i)}(\lambda x(t)) \, \lambda^i \,, \qquad (14)$$

where S_i^j is the function of derivatives of the function x(t). Substituting (14) in (13) we have

$$\sum_{k=0}^{n} a_k(t) \sum_{j=0}^{k} \binom{k}{j} F^{(k-j)}(t) \sum_{i=0}^{j} \lambda^i S_i^j \, Q_1^{(i)}(\lambda x(t)) = \lambda^n F(t) Q_1(\lambda x(t))$$

Changing indexes we obtain the form

$$\sum_{i=0}^{n} \lambda^i Q_1^{(i)}(\lambda x(t)) \sum_{k=i}^{n} a_k(t) \sum_{j=i}^{n} \binom{k}{j} S_i^j F^{(k-j)}(t) = \lambda^n F(t) Q_1(\lambda x(t)) \qquad (15)$$

Further substituting (4) to (15) we get

$$\sum_{i=0}^{n-1} \lambda^i Q_1^{(i)}(\lambda x(t)) \Big(\sum_{k=i}^{n} a_k(t) \sum_{j=i}^{k} \binom{k}{j} S_i^j F^{(k-j)}(t) -$$

$$- B_i x(t)^{i-n} a_n(t) S_n^n F(t) \Big) + B \lambda^n a_n(t) Q_1(\lambda x(t)) S_n^n F(t) =$$

$$= \lambda^n F(t) Q_1(\lambda x(t)) \qquad (16)$$

Equation (16) implies that it is sufficient to prove that equations

$$B\, a_n(t)\, S_n^n\, F(t) = F(t) \tag{17}$$

$$\sum_{k=i}^{n} a_k(t) \sum_{j=i}^{k} \binom{k}{j} S_i^j F^{(k-j)}(t) = B_i x(t)^{i-n} a_n(t) S_n^n F(t) \tag{18}$$

hold for every i = 0,1,...,n-1. Equation (17) is a consequence of equation (2). In 1866 Schlömilch 8 proved the identity

$$S_i^j = \frac{1}{i!} \frac{\partial^j}{\partial q^j} \left[(x(t+q) - x(t))^i \right] q = 0 \tag{19}$$

Substituting (19) in (18) we obtain for every i = 0,1,...,n-1

$$B_i\, x(t)^{i-n} a_n(t) (x'(t))^n F(t) =$$

$$= \sum_{k=i}^{n} a_k(t) \sum_{j=i}^{k} \binom{k}{j} \frac{\partial^j}{\partial q^j} \left[(x(t+q) - x(t))^i \right]_{q=o} F^{(k-j)}(t) =$$

$$= \sum_{k=i}^{n} \frac{1}{i!} a_k(t) \frac{\partial^k}{\partial q^k} \left[(x(t+q) - x(t))^i\, F(t+q) \right]_{q=o} \tag{20}$$

This is the system of n+1 linear equations with n+1 unknowns $a_o(t)$, $a_1(t)$,...,$a_n(t)$. If we solve this system with the help of determinants we get for k = 0,1,...,n

$$a_k(t) = \frac{|H_k|}{|H|}$$

where $H = (h_{ij})_{i,j=1}^{n+1}$, $H_k = (h_{ij}^k)_{i,j=1}^{n+1}$

$$h_{ij} = \begin{cases} \dfrac{\dfrac{\partial^{n+1-j}}{\partial \varrho^{n+1-j}} \left[(x(t+\varrho) - x(t))^{n-i+1} F(t+\varrho) \right]_{\varrho=0}}{(n-i+1)!} & \text{for } i \geq j \\ 0 & \text{for } i < j \end{cases}$$

and

$$
h_{ij}^{k}
\begin{cases}
= \dfrac{\dfrac{\partial^{n+1-j}}{\partial\rho^{n+1-j}} \left[(x(t+\rho)-x(t))^{n-i+1}F(t+\rho)\right]_{\rho=0}}{(n-i+1)!} & \text{for } i\geq j, j\neq n-k+1 \\
= 0 & \text{for } i<j, j\neq n-k+1 \\
= B_{n+1-j}\,(x(t))^{1-i}a_n(t)F(t)(x'(t))^n & \text{for } i\neq 1, j=n-k+1 \\
= a_n(t)(x'(t))^n F(t) & \text{for } i=1, j=n-k+1
\end{cases}
$$

Matrix H is a down triangle, thus

$$
|H| = \prod_{i=0}^{n} \frac{1}{i!}\frac{\partial^i}{\partial\rho^i}\left[(x(t+\rho)-x(t))^i F(t+\rho)\right]_{\rho=0} = \prod_{i=0}^{n} F(t)(x'(t))^i
$$

Similarly, because of $h_{ij}^{k} = 0$ for $i < j$, $j > n+1-k$, we can write the determinant $|H_k|$ as a product

$$
|H_k| = \begin{vmatrix}
\frac{1}{n!}\frac{\partial^n}{\partial\rho^n}\left[(x(t+\rho) - x(t)^n\, F(t+\rho)\right]_{\rho=0}, & \cdots & a_n(t)(x'(t))^n F(t) \\
\vdots & & \vdots \\
\frac{1}{k!}\frac{\partial^n}{\partial\rho^n}\left[(x(t+\rho) - x(t))^k\, F(t+\rho)\right]_{\rho=0}, & \cdots & B_k a_n(t)F(t)\,\frac{(x'(t))^n}{(x(t))^{n-k}}
\end{vmatrix}
$$

$$\cdot \prod_{i=0}^{k-1} F(t)(x'(t))^i$$

Thus $a_k(t) = \frac{|H_k|}{|H|} =$

$$= \begin{vmatrix} 1, & \cdots & \cdots & a_n(t) \\ \vdots & & \vdots & \\ \frac{\frac{\partial^n}{\partial \varrho^n}\left[(x(t+\varrho) - x(t))^k F(t+\varrho)\right]_{\varrho=0}}{k!\ (x'(t))^k F(t)}, & \cdots & \cdots & B_k\ a_n(t)\ \left(\frac{x'(t)}{x(t)}\right)^{n-k} \end{vmatrix}$$

and it holds because (8). It implies that identities (20), (18), (16) and (11) hold and thus $u(t, y)$ defined in (10) satisfied equation (1).

Theorem 2: Let (1) be an equation and let us suppose above assumptions. Denote

$$x(t) = \int_{t_0}^{t} \left(\frac{1}{B\ a_n(s)} \right)^{\frac{1}{n}} ds \qquad (21)$$

$$Y(y) = \int_{y_o}^{y} \left(\frac{1}{C\, b_n(s)} \right)^{\frac{1}{n}} ds \qquad (22)$$

for $t \in J_1$, $y \in J_2$ and let $Q_1(z)$ be a solution of the equation

$$Q_1^{(n)}(z) = B\, Q_1(z) \qquad (23)$$

in interval $z \in \langle 0, \infty)$ and let $Q_2(z)$ be a solution of the equation

$$Q_2^{(n)}(z) = C\, Q_2(z) \qquad (24)$$

in interval $z \in \langle 0, \infty)$. Further denote

$$F(t) = (B\, a_n(t))^{\frac{n-1}{2n}}\, e^{-\int_{t_o}^{t} \frac{a_{n-1}(s)}{n\, a_n(s)} ds} \qquad (25)$$

$$G(y) = (C\, b_n(y))^{\frac{n-1}{2n}}\, e^{-\int_{y_o}^{y} \frac{b_{n-1}(s)}{n\, b_n(s)} ds} \qquad (26)$$

Let us suppose, that for $a_k(t)$, $b_k(y)$ $(k = 0,1,\dots,n)$ (8) and (9) hold, where $x(t)$, $Y(y)$, $F(t)$ and $G(y)$ are defined in (21), (22), (25) and (26). Let the function $u(t,y)$ is defined as in (10) where $x(t)$, $Y(y)$, $F(t)$ and $G(y)$ are defined in (21), (22), (25) and (26). Let for $Q_1(z)$ (23) holds and for $Q_2(z)$ (24) holds. Let us assume that the function $u(t,y)$ has same convergent properties as in Theorem 1. Then $u(t,y)$ is a solution of partial differential equation (1).

The proof of Theorem 2 is similar to the proof of Theorem 1. We leave it out.

4. Some remarks

Remark 1: In sum (10) it is possible to take not only $\lambda \in \{1,2,\dots\}$ but $\lambda \in M$ where M is a countable subset of real positive numbers. The set M can be also finite. The condition

$u(t,y) \in C^{(n)}(J_1, J_2)$ is possible weaken. Then it is necessary to change same assumptions in Theorem 1 and 2.

Remark 2: We can also use the property of additivity of solutions of equation (1) i.e. if $u_1(t,y)$ and $u_2(t,y)$ are solutions of equation (1), then $u(t,y) = u_1(t,y) + u_2(t,y)$ is also solution of equation (1).

Remark 3: For $Q_1(z)$ and $Q_2(z)$ in (23) and (24) it is advantageous to take e.g. functions sin z, cos z, $e^z - e^{-z}$, because they have a lot of suitable zero points.

Souhrn

FOURIEROVA METODA PRO LINEÁRNÍ PARCIÁLNÍ DIFERENCIÁLNÍ ROVNICE n-TÉHO ŘÁDU

Článek popisuje možnost řešení lineární parciální diferenciální rovnice n-tého řádu

$$\sum_{k=0}^{n} a_k(t) \frac{\partial^k u}{\partial t^k} = \sum_{k=0}^{n} b_k(y) \frac{\partial^k u}{\partial y^k}$$

použitím kombinace Fourierovy metody a Kummerovy transformace. Na koeficienty $a_k(t)$ a $b_k(y)$ $(k = 0,1,\dots,n)$ jsou kladeny dodatečné podmínky.

Ukazuje se, že touto kombinací můžeme řešit širší okruh diferenciálních rovnic, než když používáme Fourierovu metodu a Kummerovu transformaci samostatně.

РЕЗЮМЕ

МЕТОД ФУРЬЕ ДЛЯ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ n-ТОГО ПОРЯДКА

В статье показывается возможность решить линейные дифференциальные уравнения в частных производных n-того порядка

$$\sum_{k=0}^{n} a_k(t) \frac{\partial^k u}{\partial t^k} = \sum_{k=0}^{n} b_k(y) \frac{\partial^k u}{\partial y^k}$$

при помощи комбинации метода Фурье и преобразования Куммера. Коэффициенты $a_k(t)$ и $b_k(y)$ $(k = 0,1,\ldots,n)$ должны выполнять данные условия.

Показывается, что при помощи этой комбинации можно решить более широкий класс дифференциальных уравнений чем пользованием метода Фурье и преобразования Куммера самостоятельно.

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RNDr. Jaroslav Hančl, CSc.
ÚTHP ČSAV
Místecká 17
703 00 Ostrava 3
Czechoslovakia

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