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ON A COINCIDENCE OF THE DIFFERENTIAL EQUATION $y''-q(t)y=r(t)$ WITH ITS ASSOCIATED EQUATION

MILOSLAV FIALKA

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1. Introduction

Academician O.Borůvka introduced the concept of the n-th (n=1,2,...) associated equation to a given differential equation and was the first who studied it especially for n=1 (see f.i. [1] p.8). Its significance has been largely proved in the theory of global properties of solutions of ordinary linear second-order differential equations built up by O.Borůvka and his research school. M.Laitoch [2] introduced the concept of the first associated equation

$$y'' = Q_1(t)y \qquad (Q_1)$$

to a homogeneous linear second-order differential equation of the Jacobian form

$$y'' = q(t)y, \quad q \in C^2(j), \quad q(t) < 0 \quad \text{for} \quad t \in j, \qquad (q)$$

of the basis (α,β), where $j = (a,b)(-\infty \leqq a < b \leqq \infty)$, $\alpha,\beta \in R(=(-\infty, \infty))$, $\alpha^2 + \beta^2 > 0$.

Let the symbol (q) resp. (Q_1) resp. $[R_1]$ refer conveniently either to the given differential equation or to the set of solutions of that equation. Trivial solutions of homogeneous equations (q) resp. (Q_1) will be left out of our consideration.

We know from [2] that the carrier $Q_1(t)$ of the associated equation (Q_1) is of the form

$$Q_1 = q + \frac{\alpha\beta q'}{\alpha^2 - \beta^2 q} + \sqrt{\alpha^2 - \beta^2 q}\,\left(\frac{1}{\sqrt{\alpha^2 - \beta^2 q}}\right)'' \qquad (1)$$

and if $u \in (q)$, then

$$U = (\alpha u + \beta u')(\alpha^2 - \beta^2 q)^{-1/2} \in (Q_1) \qquad (2)$$

2. Basic concepts and lemmas

Definition 1. By an associate equation to the nonhomogeneous linear differential equation

$$y'' - q(t)y = r(t),\ q \in C^2(j),\ q(t) < 0 \text{ for } t \in j,\ r \in C^1(j), \qquad [r]$$

of the basis (α,β), where $j=(a,b)(-\infty \leqq a < b \leqq \infty)$, $\alpha,\beta \in R$, $\alpha^2 + \beta^2 > 0$, we mean the equation

$$y'' - Q_1(t)y = R_1(t), \qquad [R_1]$$

where Q_1 is defined by (1) and

$$R_1(t) = \frac{\alpha r + \beta r'}{\sqrt{\alpha^2 - \beta^2 q}} + 2\beta\left(\frac{1}{\sqrt{\alpha^2 - \beta^2 q}}\right)' r,\quad t \in j. \qquad (3)$$

Lemma 1. Suppose $y \in [r]$. Then the function

$$Y(t) = (\alpha y + \beta y')(\alpha^2 - \beta^2 q)^{-1/2} \in [R_1]. \qquad (4)$$

Proof. This will be carried out by a direct calculation.

Lemma 2. Suppose $Y \in [R_1]$. Then there exists exactly one solution $y \in [r]$ such that

$$\frac{\alpha y + \beta y'}{\sqrt{\alpha^2 - \beta^2 y}} = Y . \tag{5}$$

Proof. Let $t_0 \in j$, $Y(t_0) = Y_0$, $Y_0'(t_0) = Y_0'$. To prove the above Lemma it will suffice to show the existence of exactly one solution y_0, y_0' of the system of equations

$$\begin{aligned} &\frac{\alpha y_0 + \beta y_0'}{\sqrt{\alpha^2 - \beta^2 q(t_0)}} = Y_0 \\ &\frac{\beta[q(t_0)y_0 + r(t_0)] + \alpha y_0'}{\sqrt{\alpha^2 - \beta^2 q(t_0)}} + (\alpha y_0 + \beta y_0')\left(\frac{1}{\sqrt{\alpha^2 - \beta^2 q(t)}}\right)'_{t=t_0} = Y_0' . \end{aligned} \tag{6}$$

If y_0, y_0' is the solution of system (6), then for the solution $y \in [r]$ satisfying the initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

there holds (5) and from the uniqueness of the solution of system (6) being equivalent to the system of nonhomogeneous linear equations

$$\begin{aligned} &\alpha y_0 + \beta y_0' = Y_0 \sqrt{\alpha^2 - \beta^2 q(t_0)} \\ &\beta q(t_0) y_0 + \alpha y_0' = \Big\{ -\beta r(t_0) + \sqrt{\alpha^2 - \beta^2 q(t_0)} \Big[Y_0' - \\ &\qquad - Y_0 \sqrt{\alpha^2 - \beta^2 q(t_0)} \left(\frac{1}{\sqrt{\alpha^2 - \beta^2 q(t)}}\right)'_{t=t_0} \Big] \Big\} \end{aligned} \tag{7}$$

we obtain the uniqueness of the solution $y \in [r]$ satisfying (5). System (7) has exactly one solution y_0, y_0' for the determinant of this system is $\alpha^2 - \beta^2 q(t_0) > 0$.

3. Main results

Theorem 1. There exists a one-to-one mapping between the set of all solutions of the equation $[r]$ and the set of all solutions of the equation $[R_1]$ given by relation (5).

Proof. This immediately follows from the foregoing two Lemmas.

Remark 1. If we specially consider the basis (0,1) but with $q(t) \neq 0$ in the interval j, $q \in C^2(j)$, we then obtain the differential equation

$$y'' - \left[q(t) + \sqrt{|q(t)|}\left(\frac{1}{\sqrt{|q(t)|}}\right)''\right] y = \frac{r'(t)}{\sqrt{|q(t)|}} +$$
$$+ 2r(t)\left(\frac{1}{\sqrt{|q(t)|}}\right)' ,$$

which from another aspect has been considered by S.Staněk in [4].

In what follows we will investigate a problem in which the differential equation $[r]$ is coinciding with its associated equation.

If we consider a special basis $(\alpha, 0)$, where $\alpha > 0$, then the equation $[r]$ is assigned to itself by this basis, because as we know from (1), $Q_1 = q$ and by (3) $R_1 = r$. Therefore this case is of no interest for us.

M.Laitoch has shown in [3] that the differential equation (q), $\alpha^2 - \beta^2 q(t) \neq 0$, in the interval j is assigned to itself by the basis (α, β) exactly if the carrier q assumes the following expressions:

1) $q = \mu^2 - k \qquad k \neq 0, \quad \mu \neq 0$

2) $q = \mu^2 - 16\mu^2\left[\exp(-2\mu t + A) - 2\mu B\right]^{-2} \qquad \mu \neq 0$

3) $q = \mu^2 + 16\mu^2\left[\exp(-2\mu t + A) - 2\mu B\right]^{-2} \qquad \mu \neq 0$

4) $q = -(At + B)^{-2} \qquad A \neq 0$

5) $q = (At + B)^{-2} \qquad A \neq 0$,

where $A,B \in R$, $\beta \neq 0$, $\mu := \frac{\alpha}{\beta}$. If $\mu B > 0$, then in the above expressions 2 and 3 there is naturally assumed that $\frac{A-\ln(2\mu B)}{2\mu} \notin j$ and in expressions 4 and 5 then $-B/A \notin j$. It next follows from [3] that expressions 1, 2 and 3 occur by $\alpha \neq 0$ and expressions 4 and 5 by $\alpha = 0$.

Bacause of our assumption $q(t) < 0$, we can consider only the first (with $k > \mu^2$), second and fourth case.

Theorem 2. Suppose $A,B,C,\alpha,\beta \in R$, $C \neq 0$, $\alpha^2+\beta^2 > 0$, $\beta \neq 0$, $\mu := \frac{\alpha}{\beta}$. We set for $\mu \neq 0$, $\mu B \in (-2,\infty)(\mu B \in (0,\infty);\ B \in (2,\infty))$ that $t^* := \frac{A-\ln(2\mu B+4)}{2\mu}$ $(\bar{t} := \frac{A-\ln(2\mu B)}{2\mu}$; $t^{**} := \frac{A-\ln(2\mu B-4)}{2\mu})$.
The differential equation [r] coincides with its associated equation $[R_1]$ by the basis (α,β) exactly if the functions q,r assume in the open interval j the following expressions: There is either

(I) $\quad q = \mu^2-k, \quad k \in (\mu^2, \infty), \quad \mu \neq 0,$

$$r = C \exp\left[(\sqrt{k}\ \operatorname{sgn}\beta - \mu)t\right] \tag{8}$$

or

(II) $\quad q = -(At+B)^{-2} \quad A \neq 0, \ -\frac{B}{A} \notin j,$

and then for the function r exactly one of the following possibilities exists:

either

$$r = C\left|At+B\right|^{-\left(\frac{\operatorname{sgn}\beta}{A}+2\right)}, \quad \text{if } j \subset \left(-\infty, -\frac{B}{A}\right) \tag{9}$$

or

$$r = C\left|At+B\right|^{\left(\frac{\operatorname{sgn}\beta}{|A|}-2\right)}, \quad \text{if } j \subset \left(-\frac{B}{A}, \infty\right) \tag{10}$$

or

(III) $\quad q = \mu^2-16\mu^2\left[\exp(-2\mu t+A) - 2\mu B\right]^{-2}$ for $\mu B \in (-2,\infty)$, $\mu \neq 0, t^* \notin j$,

and then:

(1) if $B = 0$, we get

$$r = C \exp\left[3\mu t + 2\operatorname{sgn}\alpha\ \exp(2\mu t - A)\right]; \tag{11}$$

(2) if $\mu B \in (2, \infty)$, we get
either

$$r = C\left[\exp(-2\mu t+A) - 2\mu B\right]^{D-2} \exp\left[(2D-1)\mu t\right], \tag{12}$$

if j is a part of the interval with the endpoints t^{*}, $\bar{t}$

or

$$r = C\left|\exp(-2\mu t+A)-2\mu B\right|^{-(D+2)}\exp\left[-(2D+1)\mu t\right] , \tag{13}$$

if j is a part of the interval with the endpoints $\bar{t}$, t^{**},

where

$$D := -\frac{\operatorname{sgn}\alpha}{\mu B} ; \tag{14}$$

(3) if $\mu B \in (0,2\rangle$, then the function r is given either by (12) or by (13) according as j is or is not a part of the interval with the endpoints t^{*}, $\bar{t}$.

(4) if $\mu B \in (-2,0)$, then the function r is given by (12).

Proof. The identity $R_1(t) \equiv r(t)$ in (3) is fulfilled exactly if the function satisfies the differential equation

$$r = \operatorname{sgn}\beta\left[\frac{\mu r + r'}{(\mu^2-q)^{1/2}} + \frac{q'r}{(\mu^2-q)^{3/2}}\right] ,$$

which is a first order differential equation with separable variables. Here the function q assumes one of the three admissible forms given prior the theorem. This equation may be modified to the form

$$r' - h(t)r = 0 , \tag{15}$$

where we set

$$h(t) = -(\mu^2-q)^{-1}q' + \operatorname{sgn}\beta(\mu^2-q)^{1/2} -\mu . \tag{16}$$

The general solution of equation (15) may then be written in the form

$$r = C\exp\left[\int h(t)dt\right] , \qquad C \in R, \quad C \neq 0, \tag{17}$$

where in proving this theorem we use the symbol $\int h(t)dt$ to denote an arbitrary but a fixed of the primitive functions to the function h(t).

By the admissible forms of the function q we separate the remaining part of the proof in three parts:

(I) If $q = \mu^2 - k$, where $k \in (\mu^2, \infty)$, then $h(t) = = \sqrt{k}\,\mathrm{sgn}\,\beta - \mu$, whence from (17) we obtain (8).

(II) If $q = -(At+B)^{-2}$, $A \neq 0$, $-\frac{B}{A} \notin j$, i.e. the case in which by [3] $\alpha = 0$, then

$$h(t) = \frac{q'}{q} + \mathrm{sgn}\beta\sqrt{-q} = -\frac{2A}{At+B} + \frac{\mathrm{sgn}\,\beta}{|At+B|}\ .$$

Analyzing the values sgn(A) and sgn(At+B) yields

$$h(t) = -\mathrm{sgn}\ A(\mathrm{sgn}\beta + 2A\ \mathrm{sgn}\ A)\ \frac{1}{At+B}$$

for $t \in j \subset (-\infty, -\frac{B}{A})$ and

$$h(t) = \mathrm{sgn}\ A\ (\mathrm{sgn}\beta - 2A\ \mathrm{sgn}\ A)\ \frac{1}{At+B}$$

for $t \in j \subset (-\frac{B}{A}, \infty)$,

which yields upon integration (9) and (10).

(III) Let now

$$q = \mu^2 - 16\mu^2\left[\exp(-2\mu t+A) - 2\mu B\right]^{-2},\ \bar{t} \notin j\ . \qquad (18)$$

Denoting

$$E(t) := \exp(-2\mu t+A),\ \varepsilon(t) := \mathrm{sgn}(E(t)-2\mu B),\ t \in R, \qquad (19)$$

gives

$$q' = -64\mu^3\ \frac{E(t)}{(E(t)-2\mu B)^3}\ ,$$

$$h(t) = \frac{4\mu E(t)}{E(t)-2\mu B} + 4\varepsilon(t)\mathrm{sgn}\alpha\ \frac{\mu}{E(t)-2\mu B} - \mu\ . \qquad (20)$$

It can be deduced very simply from (18) that the assumption $q < 0$ on j is satisfied exactly if the condition

$$2\mu B - 4 < E(t) < 2\mu B + 4 , \qquad t \in j \tag{21}$$

is fulfilled.

As can be seen from (20) it is necessary to distinguish two cases at the function h(t) according as $\varepsilon = 1$ or $\varepsilon = -1$. The case $\varepsilon = 0$ does not occur. ($\bar{t} \notin j$). Evidently

$$\varepsilon(t) = 1, \quad t \in R, \quad \mu B < 0 , \tag{22}$$

$$\varepsilon(t) = \begin{cases} 1 & \text{for } \mu > 0 \wedge t < \bar{t} \vee \mu < 0 \wedge t > \bar{t}, \quad \mu B > 0, \\ -1 & \text{for } \mu > 0 \wedge t > \bar{t} \vee \mu < 0 \wedge t < \bar{t}, \quad \mu B > 0. \end{cases} \tag{23}$$

Condition (21) is fulfilled if either

$$2\mu B - 4 \leqq 0 \wedge 0 < E(t) < 2\mu B + 4, \qquad t \in j \tag{P}$$

or

$$0 < 2\mu B - 4 < E(t) < 2\mu B + 4, \qquad t \in j . \tag{Q}$$

It then follows from (P) that $\mu B \in (-2, 2\rangle$.

Let $\mu B \in (-2, 0\rangle$. (Pi)

If $\mu > 0$, then $t > t^{*}$, i.e. $j \subset (t^{*}, \infty)$,
if $\mu < 0$, then $t < t^{*}$, i.e. $j \subset (-\infty, t^{*})$;

Let $\mu B \in (0, 2\rangle$. (Pii)

Now besides $t^{*} \notin j$, also $\bar{t} \notin j$.

If $\mu > 0$, then $t^{*} < \bar{t}$, $t^{*} < t$, i.e. either $j \subset (t^{*}, \bar{t})$ or $j \subset (\bar{t}, \infty)$,

if $\mu < 0$, then $\bar{t} < t^{*}$, $t < t^{*}$, i.e. either $j \subset (-\infty, \bar{t})$ or $j \subset (\bar{t}, t^{*})$;

Let $\mu B \in (2, \infty)$. (Q)

If $\mu > 0$ ($\mu < 0$), then

$t^{*} < t < t^{**}$, $t^{*} < \bar{t}$, $\bar{t} < t^{**}$

($t^{**} < t < t^{*}$, $\bar{t} < t^{*}$, $t^{**} < \bar{t}$).

From this we get:

If $\mu > 0$, then either $j \subset (t^{*}, \bar{t})$ or $j \subset (\bar{t}, t^{**})$;

if $\mu < 0$, then either $j \subset (t^{**}, \bar{t})$ or $j \subset (\bar{t}, t^{*})$.

Taking into consideration the values ε given by (22) and (23), then integrating the function h(t) and on using (17) we obtain the respective forms of the function r in the assertion (III) of our theorem.

For $\mu B = 0 (\mu \neq 0)$ we have namely

$$\int h(t)dt = 3\mu t + 2\operatorname{sgn}\alpha \, . \, \exp(2\mu t - A)$$

and for $\mu B \in (-2,0) \cup (0,\infty)$ we obtain successively

$$\begin{aligned}\int h(t)dt &= 4\mu\left(-\frac{1}{2\mu}\right)\ell n\left|E(t)-2\mu B\right| + 4\varepsilon\operatorname{sgn}\alpha\left(-\frac{1}{4\mu B}\right)\ell n\left|1-2\mu B E^{-1}(t)\right| - \mu t = \\ &= -2\ell n\left|E(t)-2\mu B\right| - \varepsilon\frac{\operatorname{sgn}\alpha}{\mu B}\,\ell n\,\frac{|E(t)-2\mu B|}{E(t)} - \mu t = \\ &= (\varepsilon D-2)\ell n\left|E(t) - 2\mu B\right| - \varepsilon D(-2\mu t+A) - \mu t = \\ &= (\varepsilon D-2)\ell n\left|E(t) - 2\mu B\right| + (2\varepsilon D-1)\mu t - \varepsilon AD ,\end{aligned}$$

where the constant D is defined in (14).

Conversely, the proof of the fact that the functions q,r are assuming the forms of (I), or (II) or (III), and then $R(t) \equiv r(t)$ in (3), is quite evident and is thus omitted. This completes the proof.

Remark 2. In using the same notation as in the foregoing theorem enables us to express formulas (9) and (10) for the corresponding μB and j by one formula, i.e.

$$r = C\left|At + B\right|^{\frac{\operatorname{sgn}\beta}{A}\operatorname{sgn}(At+B)-2} ,$$

and analogous, formulas (12) and (13) may be expressed for the corresponding μB and j by the formula

$$r = C\left|\exp(-2\mu t+A) - 2\mu B\right|^{F-2} \exp\left[(2F-1)\mu t\right] ,$$

where

$$F = -\frac{\operatorname{sgn}\alpha}{\mu B}\operatorname{sgn}\left[\exp(-2\mu t+A) - 2\mu B\right] .$$

Example. Choosing a special basis (0,1), i.e. if $\alpha = 0$, then there arises the case (II) of the foregoing theorem. Choosing moreover A = 1, B = 0, then we obtain the Euler differential equation

$$y'' + \frac{1}{t^2}y = 0 ,$$

whose fundamental system of solutions in the interval $(0,\infty)$ is formed by the functions $u = \sqrt{t}\ \sin(\sqrt{3/4}\ \ln t)$, $v = \sqrt{t}\cos(\sqrt{3/4}\ \ln t)$ (see [5], p.279). Then, according to (10), exactly all differential equations

$$y'' + \frac{1}{t^2}y = \frac{C}{t}, \qquad C\in R, \qquad C \neq 0 ,$$

will be in the interval $(0,\infty)$ associated to themselves, by the basis (0,1).

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O SPLYNUTÍ DIFERENCIÁLNÍ ROVNICE $y'' - q(t)y = r(t)$ SE SVOU PRŮVODNÍ ROVNICÍ

Souhrn

Průvodní rovnicí k nehomogenní diferenciální rovnici

$$y'' - q(t)y = r(t), \quad q \in C^2(j), \quad q < 0 \text{ na } j, \; r \in C^1(j) \qquad [r]$$

při bázi (α, β), kde $j = (a,b)(-\infty \leqq a < b \leqq \infty)$, $\alpha, \beta \in (-\infty, \infty)$, $\alpha^2 + \beta^2 > 0$, nazveme rovnici

$$y'' - Q_1(t)y = R_1(t) ,$$

kde

$$Q_1 = q + \frac{\alpha\beta q'}{\alpha^2 - \beta^2 q} + \sqrt{\alpha^2 - \beta^2 q}\left(\frac{1}{\sqrt{\alpha^2 - \beta^2 q}}\right)''$$

a

$$R_1 = \frac{\alpha r + \beta r'}{\sqrt{\alpha^2 - \beta^2 q}} + 2\beta\left(\frac{1}{\sqrt{\alpha^2 - \beta^2 q}}\right)' r .$$

Je ukázána existence vzájemně jednoznačného zobrazení mezi řešeními diferenciálních rovnic [r] a $[R_1]$ a dokázáno, ve kterých případech splyne rovnice [r] se svou průvodní rovnicí.

О СОВПАДЕНИИ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $y'' - q(t)y = r(t)$ СО СВОИМ СОПРОВОДИТЕЛЬНЫМ УРАВНЕНИЕМ

Резюме

Сопроводительным уравнением при базисе (α, β) где $\alpha, \beta \in R(=(-\infty, \infty))$, $\alpha^2 + \beta^2 > 0$, к дифференциальному уравнению для

$y'' - q(t)y = r(t)$, $q \in C^2(j)$, $q(t) < 0$ для $t \in j$, $r \in C^1(j)$, [r]
$j = (a,b)(-\infty \leqq a < b \leqq \infty)$, мы называем дифференциальное уравнение

$$y'' - Q_1(t)y = R_1(t) \qquad [R_1]$$

где $Q_1 = q + \dfrac{\alpha\beta q'}{\alpha^2 - \beta^2 q} + \sqrt{\alpha^2 - \beta^2 q}\left(\dfrac{1}{\sqrt{\alpha^2 - \beta^2 q}}\right)''$

и $R_1 = \dfrac{\alpha r + \beta r'}{\sqrt{\alpha^2 - \beta^2 q}} + 2\beta\left(\dfrac{1}{\sqrt{\alpha^2 - \beta^2 q}}\right)' r.$

Показано существование взаимнооднозначного отображения между решениями дифференциальных уравнений [r] и $[R_1]$ и доказано, в каких случаях совпадут уравнение [r] и его сопроводительное уравнение $[R_1]$.

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