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TO THE THEORY OF LINEAR DIFFERENCE EQUATIONS

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This paper is devoted to the theory of linear difference equations with constant coefficients from the viewpoint of Borůvka's theory of the 1st kind central dispersions of the 2nd order linear differential equations of Jacobi's type. There is shown a natural generalization of a linear difference equation to a certain functional equation considered on an infinite or finite cyclic group of functions, which are the 1st kind central dispersions of the above mentioned differential equation either bothside oscillatory or special of a finite type on the interval $(-\infty, \infty)$.

1. Consider a bothside oscillatory 2nd order linear differential equation of Jacobi's type

$$y'' = q(t)\, y, \tag{q}$$

where $q \in C^0(-\infty, \infty)$.

The symbol $C^0(-\infty, \infty)$ denotes a set of real continuous functions on the interval $(-\infty, \infty)$.

Let $\varphi = \varphi(t)$ be the basic 1st kind central dispersion of (q). Let $n = 1, 2, \ldots$. Let $\varphi_n = \varphi_n(t)$ denote the n-th 1st kind central dispersion, which is the n-times composite basic central dispersion of the 1st kind, whereby $\varphi_1(t) = \varphi(t)$. We set $\varphi_0(t) = t$. With this notation $\varphi_{-n} = \varphi_{-n}(t)$ is an inverse function to the function φ_n.

Under the given assumption, the 1st kind central dispersions are known to form an infinite cyclic group $\{\varphi_n\}_{n=-\infty}^{\infty}$ of increasing functions of class C^3 mapping the interval $(-\infty, \infty)$ onto the interval $(-\infty, \infty)$; thereby the function φ_1 is a generating function, the group operation is the composition of functions.

2. Let the differential equations

$$y'' = q(t)\, y, \tag{q}$$

$$Y'' = Q(t)\, Y \tag{Q}$$

be bothside oscillatory on the interval $(-\infty, \infty)$ and the coefficients q, Q be from class $C^\infty(-\infty, \infty)$. For $j = 0, \pm 1, \pm 2, \ldots$ let $\varphi_j = \varphi_j(t)$ and $\Phi_j = \Phi_j(t)$

be the j-th central dispersions of the 1st kind of the differential equation (q) and (Q), respectively.

The 1st kind central dispersions of (q) and (Q) are known to satisfy Kummer's nonlinear differential equation of the 3rd order

$$-\{\varphi, t\} + q(\varphi)\,\varphi'^2(t) = q(t) \tag{q, q}$$

and

$$-\{\Phi, t\} + Q(\Phi)\,\Phi'^2(t) = Q(t), \tag{Q, Q}$$

respectively. Whereby

$$\{\varphi, t\} = \frac{1}{2}\frac{\varphi'''(t)}{\varphi'(t)} - \frac{3}{4}\frac{\varphi''^2(t)}{\varphi'^2(t)},$$

$$\{\Phi, t\} = \frac{1}{2}\frac{\Phi'''(t)}{\Phi'(t)} - \frac{3}{4}\frac{\Phi''^2(t)}{\Phi'^2(t)}.$$

E. Barvínek in [2] proved a **Theorem** which we will use in our considerations below. The Theorem reads: *Let X, Z, ζ be solutions of Kummer's differential equations*

$$-\{X, t\} + q(X)\,X'^2(t) = Q(t) \tag{q, Q}$$

and

$$-\{Z, t\} + Q(Z)\,Z'^2(t) = Q(t) \tag{Q, Q}$$

and

$$-\{\zeta, t\} + q(\zeta)\,\zeta'^2(t) = q(t), \tag{q, q}$$

respectively.

Then the equality

$$X[Z(t)] = \zeta[X(t)] \tag{1}$$

holds: for all $X \in$ (q, Q) exactly if $Z(t) = \zeta(t) = t$,

for all increasing solutions $X \in$ (q, Q) exactly if $Z = \Phi_n$, $\zeta = \varphi_n$,

for all decreasing solutions $X \in$ (q, Q) exactly if $Z = \Phi_n$, $\zeta = \varphi_{-n}$.

An immediate consequence of this Theorem is:

If we set $q(t) = -\pi^2$, we may use the foregoing Theorem to the basic 1st kind central dispersion φ of the differential equation $(-\pi^2)$, which is the function $\varphi = t + 1$, and to the basic 1st kind central dispersion Φ of the differential equation (Q). Then for any increasing solution $X \in (-\pi^2, \text{Q})$ we have the equality

$$X[\Phi(t)] = X(t) + 1,$$

and for any decreasing solution we have the equality

$$X[\Phi(t)] = X(t) - 1,$$

because $\Phi_{-1} = t - 1$.

If φ_n denotes the n-th central dispersion of the 1st kind of $(-\pi^2)$, which, as we know, is the function $\varphi_n = t + n$ and Φ_n denotes the n-th central dispersion of the 1st kind of (Q) for $n = 1, 2, 3, \ldots$, then for any increasing solution $X \in (-\pi^2, \text{Q})$ we have the equality

$$X[\Phi_n(t)] = X(t) + n \tag{2}$$

and for any decreasing solution $X \in (-\pi^2, \mathrm{Q})$ we have the equality

$$X[\Phi_n(t)] = X(t) - n,$$

because $\Phi_{-n}(t) = t - n$.

3. Consider a functional equation

$$a_0 f[\Phi_n(t)] + a_1 f[\Phi_{n-1}(t)] + \dots + a_n f[\Phi_0(t)] = 0, \tag{3}$$

where a_j denotes real constants for $j = 0, 1, \dots, n$ in assuming $a_0 \neq 0$, $a_n \neq 0$, and Φ_j denotes the j-th central dispersion of the 1st kind of (Q).

Definition. *Equation* (3) *will be called the homogeneous linear difference equation of the n-th order with constant coefficients on the cyclic group* $\{\Phi_n\}_{n=-\infty}^{\infty}$ *of the 1st kind central dispersions of the differential equation* (Q).

Theorem. *Let* $\Phi = \Phi_1$ *be the basic 1st kind central dispersion of the differential equation* (Q). *Let X be an increasing solution of the functional equation*

$$X[\Phi(t)] = X(t) + 1. \tag{A}$$

Let λ_0 *be the root of the so-called characterist equation*

$$a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_n = 0 \tag{4}$$

for the functional equation (3), *where* a_j, $j = 0, 1, \dots, n$, *are the coefficients of equation* (3). *Then the function*

$$f = \lambda_0^{X(t)} \tag{5}$$

is a solution of the functional equation (3).

Proof. Let us now try to find a solution of equation (3) in the form $f = \lambda^{X(t)}$. Inserting this into (3), we get

$$a_0\lambda^{X[\Phi_n(t)]} + a_1\lambda^{X[\Phi_{n-1}(t)]} + \dots + a_n\lambda^{X[\Phi_0(t)]} = 0.$$

Since by (2) $X[\Phi_j(t)] = X(t) + j$ for $j = 0, 1, \dots, n$, we get after substituting

$$\lambda^{X(t)}[a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_n] = 0.$$

From this it follows that λ is the root of the characteristic equation (4).

Analogous we may prove the following Theorem.

Theorem. *Let* $\Phi = \Phi_1$ *be the basic 1st kind central dispersion of* (Q). *Let X be the decreasing solution of the functional equation*

$$X[\Phi(t)] = X(t) - 1. \tag{A*}$$

Let μ_0 *be the root of the reciprocal characteristic equation*

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0 \tag{4*}$$

for the functional equation (3), *where* a_j, $j = 0, 1, \ldots, n$, *are the coefficients of equation* (3). *Then the function*

$$f = \mu_0^{X(t)} \tag{5*}$$

is a solution of the functional equation (3).

Example. Consider instead of the differential equation (Q) the differential equation $(-\pi^2)$, i.e. the differential equation

$$Y'' = (-\pi^2)\, Y. \tag{$-\pi^2$}$$

As we know, the basic 1st kind central dispersion of equation $(-\pi^2)$ is the function $\Phi(t) = t + 1$. The j-th central dispersion of the 1st kind is the function $\Phi_j = t + j$ for $j = 0, 1, \ldots, n$. The increasing solution of the functional equation

$$X(t + 1) = X(t) + 1$$

is the function $X(t) = t$, which can be verified by substitution. It follows that the homogeneous linear difference equation of the n-th order with constant coefficients

$$a_0 f(t + n) + a_1 f(t + n - 1) + \ldots + a_n f(t) = 0 \tag{6}$$

is a special case of the functional equation (3) and its solution with respect to formula (5) has the form

$$f(t) = \lambda_0^t,$$

where λ_0 is the root of the characteristic equation (4).

If μ_0 is the root of the reciprocal characteristic equation (4*), then the solution of the difference equation (6) may be expressed with respect to formula (5*) in the form

$$f(t) = \mu_0^{-t},$$

because the function $X(t) = -t$ is a decreasing solution of the functional equation

$$X(t + 1) = X(t) - 1.$$

It can be easily seen that the following **Theorems** are true.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be real different* (*simple*) *roots of the characteristic equation* (4). *Let* X *be an increasing solution of the functional equation* (A). *Then the function*

$$f(t) = c_1 \lambda_1^{X(t)} + c_2 \lambda_2^{X(t)} + \ldots + c_n \lambda_n^{X(t)}$$

is a solution of the functional equation (3), *where* c_j, $j = 1, 2, \ldots, n$, *are real constants.*

If the characteristic equation (4) *has a simple complex root* λ_0, *then the two linearly independent solutions of the functional equation* (3) *have the form*

$$f_1(t) = |\lambda_0|^{X(t)} \sin\left[\arg \lambda_0\, X(t)\right],$$
$$f_2(t) = |\lambda_0|^{X(t)} \cos\left[\arg \lambda_0\, X(t)\right].$$

Let the characteristic equation (4) *have an s-fold root* λ_0, *where* $1 \leqq s \leqq n$. *Then the functional equation* (3) *has s linearly independent solutions in the form*

$$\lambda_0^{X(t)},\ X(t)\,\lambda_0^{X(t)},\ \ldots,\ X^{s-1}(t)\,\lambda_0^{X(t)}.$$

5. Let the differential equations

$$y'' = q(t)\,y, \tag{q}$$

$$Y'' = Q(t)\,Y \tag{Q}$$

be bothside oscillatory on the interval $(-\infty, \infty)$. Let the coefficients q, Q be from class $C^0(-\infty, \infty)$.

Consider two linear difference equations of the n-th order with the same coefficients $a_0, a_1, \ldots, a_n$, the former on the group $\{\varphi_n\}_{n=-\infty}^{\infty}$ of the 1st kind central dispersions of (q), the latter on the group $\{\Phi_n\}_{n=-\infty}^{\infty}$ of the 1st kind central dispersions of (Q):

$$a_0 g[\varphi_n(t)] + a_1 g[\varphi_{n-1}(t)] + \ldots + a_n g[\varphi_0(t)] = 0, \tag{7}$$

$$a_0 f[\Phi_n(t)] + a_1 f[\Phi_{n-1}(t)] + \ldots + a_n f[\Phi_0(t)] = 0, \tag{8}$$

where $a_0 \neq 0$, $a_n \neq 0$ and for $j = 0, 1, \ldots, n$, $\varphi_j \in (\mathrm{q}, \mathrm{q})$, $\Phi_j \in (\mathrm{Q}, \mathrm{Q})$ are the j-th central dispersions of the 1st kind of (q) or (Q).

Theorem. *Let* $X = X(t)$ *be an increasing solution of the differential equation* (q, Q). *Let* $g = g(t)$ *be a solution of the linear difference equation* (7). *Then the function* $f = g[X(t)]$ *is a solution of the linear difference equation* (8).

Proof. Equation (7) may be written in the form

$$a_0 g[\varphi_n X(t)] + a_1 g[\varphi_{n-1} X(t)] + \ldots + a_n g[\varphi_0 X(t)] = 0. \tag{7*}$$

Since the functions φ_j, Φ_j, X satisfy the assumptions of Barvínek's theorem, the equalities

$$X[\Phi_j(t)] = \varphi_j[X(t)]$$

hold for $j = 0, 1, \ldots, n$ on the interval $(-\infty, \infty)$. Thus, equation (7*) may be written in the form

$$a_0 g[X\Phi_n(t)] + a_1 g[X\Phi_{n-1}(t)] + \ldots + a_n g[X\Phi_0(t)] = 0. \tag{8*}$$

Equations (8) and (8*) are identical exactly if $f = gX$. Therefore, if g is a solution of (7), then the function $f = gX$ is a solution of equation (8).

Theorem. *Let* $X = X(t)$ *be a decreasing solution of the differential equation* (q, Q). *Let* $g = g(t)$ *be a solution of the linear difference equation*

$$a_n g[\varphi_n(t)] + a_{n-1} g[\varphi_{n-1}(t)] + \ldots + a_0 g[\varphi_0(t)] = 0. \tag{7_1}$$

Then the function $f = g[X(t)]$ *is a solution of the linear difference equation* (8).

Proof. Equation (7_1) may be written in the form

$$a_0 g[\varphi_{-n}(t)] + a_1 g[\varphi_{-n+1}(t)] + \dots + a_n g[\varphi_0(t)] = 0$$

and also

$$a_0 g[\varphi_{-n} X(t)] + a_1 g[\varphi_{-n+1} X(t)] + \dots + a_n g[\varphi_0 X(t)] = 0. \qquad (7_1^*)$$

Since the functions φ_j, Φ_j, X satisfy the assumptions of Barvínek's theorem, the equalities

$$X[\Phi_j(t)] = \varphi_{-j}[X(t)]$$

hold for $j = 0, 1, \dots, n$ on the interval $(-\infty, \infty)$. Thus, equation (7_1^*) may be written in the form

$$a_0 g[X\Phi_n(t)] + a_1 g[X\Phi_{n-1}(t)] + \dots + a_n g[X\Phi_0(t)] = 0.$$

The last equation and equation (8) are identical exactly if $f = gX$. Thus, if g is a solution of (7_1), then the function $f = gX$ is a solution of equation (8).

6. We now show a certain generalization of the classical difference equation with constant coefficients. To do this requires the following consideration:

Consider a linear differential equation of the 2nd order of Jacobi's type

$$Y'' = \frac{1 - m^2}{(1 + t^2)^2} Y \qquad (9)$$

for $t \in (-\infty, \infty)$, where $m \geqq 2$ is a natural number.

Setting

$$Y_1 = (t^2 + 1)^{1/2} \sin (m \operatorname{arctg} t), \qquad Y_2 = (t^2 + 1)^{1/2} \cos (m \operatorname{arctg} t), \qquad (10)$$

$t \in (-\infty, \infty)$, then Y_1 and Y_2 are two independent solutions of (9).

Theorem. *Let c_1, c_2 be nonzero real numbers. Let*

$$Y = c_1 Y_1 + c_2 Y_2 \qquad (11)$$

be a particular solution of (9), *where Y_1 and Y_2 are given by formulas of* (10). *Let t_0 be the zero of* (11), *i.e.* $Y(t_0) = 0$. *Then exactly the points*

$$\Phi_j(t_0) = \operatorname{tg}\left(\operatorname{arctg} t_0 + \frac{j\pi}{m}\right), \qquad j = 0, 1, \dots, m - 1 \qquad (12)$$

are all zeros of (11) *on the interval* $(-\infty, \infty)$.

Proof. Since the function $m \operatorname{arctg} t$ is strictly monotone on the interval $(-\infty, \infty)$ and $m \operatorname{arctg} t \in \left(-\frac{\pi}{2} m, \frac{\pi}{2} m\right)$ for $t \in (-\infty, \infty)$, we see that every solution of (11) of the differential equation (9) has exactly m zeros on $(-\infty, \infty)$ if $c_1 \neq 0$, $c_2 \neq 0$.

Remark 1. Solutions Y_1 and Y_2 in case of m being even and odd, respectively, and the solutions linearly dependent on them, have only $(m - 1)$ zeros on $(-\infty, \infty)$. In such cases we complete the set of $(m - 1)$ zeros in the mentioned solutions by one improper point. Thus in (12) we define the value $\operatorname{tg}\left(\frac{\pi}{2} + \varkappa\pi\right)$, $\varkappa$ an integer, being equal to the point at infinity.

It can be easily seen that the points $\Phi_j(t_0)$, $j = 0, 1, \ldots, m - 1$, are mutually different.

Theorem. *Let $m \geqq 2$ be a natural number. Let ν be an integer. Let*

$$\Phi_\nu(t) = \operatorname{tg}\left(\operatorname{arctg} t + \frac{\nu\pi}{m}\right), \tag{13}$$

$t \in (-\infty, \infty)$. Then

$$\Phi_\nu(t) = \Phi_j(t) \qquad \textit{holds, where} \qquad j = \nu (\operatorname{mod} m), 0 \leqq j \leqq m - 1$$

and the functions $\Phi_0, \Phi_1, \ldots, \Phi_{m-1}$ form a cyclic group of the order m. The group operation is the composition of functions; Φ_1 is a generating element of the group, $\Phi_m(t) = \Phi_0(t) = t$ is the neutral element of the group.

Proof. Let μ, ν be integers. It holds $\Phi_\mu[\Phi_\nu(t)] = \Phi_{\mu+\nu}(t)$, because $\Phi_\mu[\Phi_\nu(t)] = \operatorname{tg}\left[\operatorname{arctg} \Phi_\nu(t) + \frac{\mu\pi}{m}\right] = \operatorname{tg}\left[\operatorname{arctg}\left[\operatorname{tg}\left(\operatorname{arctg} t + \frac{\nu\pi}{m}\right)\right] + \frac{\mu\pi}{m}\right] = \operatorname{tg}\left[\left(\operatorname{arctg} t + \frac{\nu\pi}{m}\right) + \frac{\mu\pi}{m}\right] = \operatorname{tg}\left(\operatorname{arctg} t + \frac{(\mu + \nu)}{m}\right) = \Phi_{\mu+\nu}(t)$.

Thereby $\Phi_0(t) = t$, the inverse element to the function $\Phi_\nu(t)$ is the function $\Phi_\nu(t)$, because $\Phi_\nu[\Phi_{-\nu}(t)] = \Phi_0(t) = t$. Next $\Phi_m(t) = \operatorname{tg}\left(\operatorname{arctg} t + \frac{m\pi}{m}\right) = \operatorname{tg}(\operatorname{arctg} t + \pi) = \operatorname{tg}(\operatorname{arctg} t) = t$.

Thus the functions $\Phi_0, \Phi_1, \ldots, \Phi_{m-1}$ form a cyclic group of the order m, because there exists an integer j for every integer ν such that $j = \nu(\operatorname{mod} m)$ and $0 \leqq j \leqq m - 1$.

Definition. *The function $\Phi_j = \Phi_j(t)$ for $j = 0, 1, \ldots, m - 1$ will be called the j-th central dispersion of the* 1*st kind of the differential equation* (9).

Remark 2. Let us note that the functions $\Phi_\nu = \Phi_\nu(t)$ may be written in the form of linear broken functions, namely

$$\Phi_\nu(t) = \frac{t + \operatorname{tg}\dfrac{\nu\pi}{m}}{\left(-\operatorname{tg}\dfrac{\nu\pi}{m}\right)t + 1}.$$

In case of $\frac{\nu\pi}{m} = \frac{\pi}{2} + \varkappa\pi$, $\varkappa$ an integer, we set $\Phi_\nu = -\frac{1}{t}$.

Remark 3. Let us denote by $J_1, J_2, \ldots, J_m$ the following m intervals:

$$J_1 \quad = \left(-\infty, -\operatorname{cotg}\frac{\pi}{m}\right),$$

$$J_2 = \left(-\text{cotg}\frac{\pi}{m}, -\text{cotg}\frac{2\pi}{m}\right),$$
$$\vdots$$
$$J_j = \left(-\text{cotg}\frac{(j-1)\pi}{m}, -\text{cotg}\frac{j\pi}{m}\right),$$
$$\vdots$$
$$J_{m-1} = \left(-\text{cotg}\frac{(m-2)\pi}{m}, -\text{cotg}\frac{(m-1)}{m}\right) = \left(\text{cotg}\frac{2\pi}{m}, \text{cotg}\frac{\pi}{m}\right),$$
$$J_m = \left(\text{cotg}\frac{\pi}{m}, \infty\right).$$

Let now

$$\Phi_1 = \frac{t + \text{tg}\frac{\pi}{m}}{\left(-\text{tg}\frac{\pi}{m}\right)t + 1}$$

be a generating element of a cyclic group. Let us recall that

$$\lim_{|t|\to\infty} \Phi_1 = \text{cotg}\frac{\pi}{m}$$

$$\Phi_1' = \frac{1}{t^2}, \qquad \text{or} \quad \Phi_1' = \frac{1 + \text{tg}^2\frac{\pi}{m}}{\left[\left(-\text{tg}\frac{\pi}{m}\right)t + 1\right]^2} \qquad \text{for } m = 2, \text{ or } m > 2.$$

It is easily seen that the image of the j-th interval J_j, $j = 1, 2, \dots, m-1$ in the mapping Φ_1 is the interval J_{j+1} and the image of the m-th interval J_m is the interval J_1, which can be indicated by writing: For $j = 1, 2, \dots, m-1$ we have

$$\Phi_1(J_j) = \Phi_1\left(-\text{cotg}\frac{(j-1)\pi}{m}, -\text{cotg}\frac{j\pi}{m}\right) =$$
$$= \left(-\text{cotg}\frac{j\pi}{m}, -\text{cotg}\frac{(j+1)\pi}{m}\right) = J_{j+1};$$

For $j = m$ we have

$$\Phi_1(J_m) = \Phi_1\left(\text{cotg}\frac{\pi}{m}, \infty\right) = \left(-\infty, -\text{cotg}\frac{\pi}{m}\right) = J_1.$$

The 1st kind central dispersions Φ_j, $j = 0, 1, \dots, m-1$ of (9) have the following properties:

$$\Phi_j(t) - \Phi_{j-1}(t) > 0 \text{ in } (J_1 \cup \dots \cup J_{m-j}) \cup (J_{m-j+2} \cup \dots \cup J_m) =$$
$$= \left(-\infty, \text{cotg}\frac{j\pi}{m}\right) \cup \left(\text{cotg}\frac{(j-1)\pi}{m}, \infty\right),$$

$$\Phi_j(t) - \Phi_{j-1}(t) < 0 \quad \text{in} \quad J_{m-j+1} = \left(\operatorname{cotg}\frac{j\pi}{m}, \operatorname{cotg}\frac{(j-1)\pi}{m}\right)$$

for $j = 1, 2, \ldots, m - 1$;

$$\Phi_m(t) - \Phi_{m-1}(t) > 0 \quad \text{in} \quad (J_2 \cup \ldots \cup J_m) = \left(-\operatorname{cotg}\frac{\pi}{m}, \infty\right),$$

$$\Phi_m(t) - \Phi_{m-1}(t) < 0 \quad \text{in} \quad J_1 = \left(-\infty, -\operatorname{cotg}\frac{\pi}{m}\right),$$

where $\Phi_m(t) = \Phi_0(t)$, for $j = m$.

Let $m \geqq 2$ be a natural number. Let k be a natural number, for which $1 \leqq k \leqq \leqq m - 1$. Let us consider the functional equation

$$a_0 f[\Phi_k(t)] + a_1 f[\Phi_{k-1}(t)] + \ldots + a_k f[\Phi_0(t)] = 0, \tag{14}$$

where a_j are real constants, $j = 0, 1, \ldots, k$, $a_0 \neq 0$, $a_k \neq 0$, and $\Phi_j = \operatorname{tg}\left(\operatorname{arctg} t + \frac{j\pi}{m}\right)$ denotes the 1st kind central dispersion of (9).

Definition. *The functional equation* (14) *will be called a homogeneous linear difference equation of the k-th order with constant coefficients on the finite cyclic group* $\{\Phi_j\}_{j=0}^{m-1}$ *of the 1st kind central dispersions of* (9).

Theorem. *Let* $\Phi = \operatorname{tg}\left(\operatorname{arctg} t + \frac{\pi}{m}\right)$ *be the 1st kind central dispersion of* (9). *Let* λ_0 *be a root of the characteristic equation*

$$a_0 \lambda^k + a_1 \lambda^{k-1} + \ldots + a_k = 0, \tag{15}$$

where $a_j, j = 0, 1, \ldots, k$ *are the coefficients of* (14), $a_0 \neq 0$, $a_k \neq 0$. *Then the function*

$$X = \frac{m}{\pi} \operatorname{arctg} t \tag{16}$$

is an increasing solution of the functional equation

$$X\left[\operatorname{tg}\left(\operatorname{arctg} t + \frac{\pi}{m}\right)\right] = X(t) + 1 \tag{17}$$

and the function

$$f = \lambda_0^{\frac{m}{\pi} \operatorname{arctg} t} \tag{18}$$

is a solution of the functional equation (14).

Proof. The fact that function (16) is the solution of equation (17) may be verified by direct substitution.

To prove the second assertion we first mention that $X[\Phi_j(t)] = X(t) + j$, where $X = \frac{m}{\pi} \operatorname{arctg} t$, $\Phi_j = \operatorname{tg}\left(\operatorname{arctg} t + \frac{j\pi}{m}\right)$ holds for $j = 0, 1, \ldots, k$, because

$X[\Phi_j(t)] = \frac{m}{\pi} \operatorname{arctg}\left[\operatorname{tg}\left(\operatorname{arctg} t + \frac{j\pi}{m}\right)\right] = \frac{m}{\pi} \operatorname{arctg} t + j = X(t) + j$. Searching for a solution of the functional equation (14) in the form $f = \lambda^{\frac{m}{\pi} \operatorname{arctg} t}$, we get after substitution

$$\lambda^{\frac{m}{\pi} \operatorname{arctg} t}(a_0 \lambda^k + a_1 \lambda^{k-1} + \dots + a_k) = 0.$$

It then follows that λ is a root of the characteristic equation (15).

It is fairly easy to show that the following **Theorems** hold:

Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be different real (simple) roots of the characteristic equation (15). *Let $X = \frac{n}{\pi} \operatorname{arctg} t$ be an increasing solution of the functional equation* (17). *Then the function*

$$f(t) = c_1 \lambda_1^{\frac{m}{\pi} \operatorname{arctg} t} + \dots + c_k \lambda_k^{\frac{m}{\pi} \operatorname{arctg} t},$$

where c_j, $j = 1, 2, \dots, k$, are real constants, is a solution of the functional equation (14).

If the characteristic equation (15) *has a simple complex root λ_0, then*

$$f_1(t) = |\lambda_0|^{\frac{m}{\pi} \operatorname{arctg} t} \sin\left(\arg \lambda_0 \frac{m}{\pi} \operatorname{arctg} t\right),$$

$$f_2(t) = |\lambda_0|^{\frac{m}{\pi} \operatorname{arctg} t} \cos\left(\arg \lambda_0 \frac{m}{\pi} \operatorname{arctg} t\right)$$

are two linearly independent solutions of (14).

Let the characteristic equation (15) *have an s-fold root λ_0, $1 \leqq s \leqq k$. Then*

$$\lambda_0^{\frac{m}{\pi} \operatorname{arctg} t}, \frac{m}{\pi} \operatorname{arctg} t \lambda_0^{\frac{m}{\pi} \operatorname{arctg} t}, \dots, \left(\frac{m}{\pi} \operatorname{arctg} t\right)^{s-1} \lambda_0^{\frac{m}{\pi} \operatorname{arctg} t}$$

are s linearly independent solutions of the functional equation (14).

Remark 4. The differential equation (9) is a special equation of the finite type m [1], p. 63.

7. Consider a 2nd order linear differential equation of Jacobi's type

$$Y'' = Q(t) \,.\, Y, \tag{Q}$$

where $Q \in C^0(-\infty, \infty)$, being a special equation of the finite type m.

Let $r_1(s_1)$ be the left (right) basic number of the 1st kind, of the differential equation (Q).

Recall that $r_1 = \inf R_1$ and $s_1 = \sup S_1$, where $R_1(S_1)$ denote the set of numbers in $(-\infty, \infty)$ admitting conjugate numbers of the 1st kind from the left (right). with respect to (Q).

Definition. *The function* $\Phi = \Phi(t)$ *will be called the 1st kind basic central dispersion of the special differential equation* (Q) *of the finite type m, if it immediately at the right associates to every point* $t_0 \in (-\infty, s_1)$ *the following zero of a solution* $Y \in$ (Q), *for which* $Y(t_0) = 0$.

Let us remark that the linearly dependent solutions have the same zeros and vice versa, solutions having the same zeros are linearly dependent, so that the function Φ is uniquely defined.

As can be easily seen, the function Φ increases in the interval $(-\infty, s_1)$ from the value r_1 to ∞ and in the interval (s_1, ∞) from the value $-\infty$ to r_1. It holds

$$\lim \Phi(t) = \infty \quad \text{for } t \to s_{1-}, \qquad \lim \Phi(t) = -\infty \quad \text{for } t \to s_{1+}$$
$$\lim \Phi(t) = r_1 \quad \text{foı } t \to \infty, \qquad \lim \Phi(t) = r_1 \quad \text{for } t \to -\infty.$$

The function Φ is of class C^3 on the set $(-\infty, s_1) \cup (s_1, \infty)$.

Remark 1. If $Y \in$ (Q) has only $(m - 1)$ zeros in the interval $(-\infty, \infty)$ (i.e. in case of $Y(r_1) = Y(s_1) = 0$), then the set of $(m - 1)$ zeros of the solution Y will be completed by one improper point at infinity.

Remark 2. It is easily seen that the function Φ cyclically orders the m-tuple of zeros of any solution $Y \in$ (Q) or the $(m - 1)$-tuple of zeros and the improper point at infinity.

Let j be a natural number. We denote by $\Phi_j = \Phi_j(t)$ the j-times composite function $\underbrace{\Phi \ldots \Phi(t)}_{j\text{-times}}$. The function Φ_j then associates to every point from the cyclic ordered m-tuples considered the following j-th element of the m-tuple in that cyclic ordering.

Convention. We have defined the functions Φ_j for $j = 1, 2, \ldots$. Thereby $\Phi_1(t) = \Phi(t)$ denotes the basic central dispersion of the 1st kind. We set $\Phi_0(t) = t$. The symbol Φ_{-j} stands for the inverse function to Φ_j. Thus the function $\Phi_\nu = \Phi_\nu(t)$ is defined for any integer ν.

Theorem. *Let* ν *be an integer. Let the function* $\Phi_\nu = \Phi_\nu(t)$ *have the stated meaning. Then* $\Phi_\nu(t) = \Phi_j(t)$, *where* $j = \nu (\text{mod } m)$, $0 \leqq j \leqq m - 1$ *and the functions* $\Phi_0, \Phi_1, \ldots, \Phi_{m-1}$ *form a cyclic group of the order m.*

The group operation is the composition of functions, the function Φ_1 *is a generating element of the group,* $\Phi_n(t) = \Phi_0(t) = t$ *is the neutral element of the group.*

For $j = 1, 2, \ldots, m - 1$ *the function* $\Phi_j = \Phi_j(t)$ *increases in the interval* $(-\infty, \Phi_{-j}(s_1))$ *from the value* $\Phi_j(r_1)$ *to* ∞ *and in the interval* $(\Phi_{-j}(s_1), \infty)$ *from* $-\infty$ *to the value* $\Phi_j(r_1)$. *Here* $\Phi_j \in C^{(\infty)}$.

The proof of this Theorem is analogous to that of the Theorem considered in the foregoing paragraph.

Let $m \geqq 2$ be a natural number. Be k a natural number for which $1 \leqq k \leqq m - 1$. Consider the functional equation

$$a_0 f[\Phi_k(t)] + a_1 f[\Phi_{k-1}(t)] + \dots + a_k f[\Phi_0(t)] = 0, \tag{19}$$

where a_j denotes real constants, $a_0 \neq 0$, $a_k \neq 0$, and Φ_j stands for the j-th central dispersion of the 1st kind of the differential equation (Q), for $j = 0, 1, \dots, k$.

Definition. *The functional equation* (19) *will be called a homogeneous linear difference equation of the k-th order with constant coefficients on the finite cyclic group* $\{\Phi_j\}_{j=0}^{m-1}$ *of the 1st kind central dispersions of the special differential equation* (Q) *being of the finite type m.*

Theorem. *Let* $\Phi = \Phi(t)$ *be the basic 1st kind central dispersion of the special differential equation* (Q) *of the finite type. Let* $X = X(t)$ *be an increasing solution of the functional equation*

$$X[\Phi(t)] = X(t) + 1 \tag{20}$$

Be λ_0 *a root of the characteristic equation*

$$a_0\lambda^k + a_1\lambda^{k-1} + \dots + a_k = 0 \tag{21}$$

relative to the functional equation (19). *Then the function*

$$f = \lambda_0^{X(t)} \tag{22}$$

is a solution of the functional equation (19).

The proof is analogous to that of the Theorem in paragraph 3.

It is easy to show that the following **Theorems** hold:

Let $\lambda_1, \lambda_2, \dots, \lambda_k$ *be the different real (simple) roots of the characteristic equation* (21). *Let* $X = X(t)$ *be an increasing solution of the functional equation* (20). *Then the function*

$$f(t) = c_1\lambda_1^{X(t)} + c_2\lambda_2^{X(t)} + \dots + c_k\lambda_k^{X(t)},$$

where $c_j, j = 1, 2, \dots, k$ *are real constants, is a solution of the functional equation* (19).

If the characteristic equation (21) *has a simple complex root* λ_0, *then two linearly independent solutions of* (19) *are of the form*

$$f_1(t) = |\lambda_0|^{X(t)} \sin[\arg \lambda_0 X(t)],$$
$$f_2(t) = |\lambda_0|^{X(t)} \cos[\arg \lambda_0 X(t)].$$

Let the characteristic equation (21) *have an s-fold root* λ_0, *where* $1 \leqq s \leqq k$. *Then the functional equation* (19) *has s linearly independent solutions of the form*

$$\lambda_0^{X(t)},\ X(t)\,\lambda_0^{X(t)}, \dots, X^{s-1}(t)\,\lambda_0^{X(t)}.$$

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Souhrn

PŘÍSPĚVEK K TEORII LINEÁRNÍCH DIFERENČNÍCH ROVNIC

MIROSLAV LAITOCH

Uvažuje se funkční rovnice tvaru

$$a_0 f[\varphi_n(t)] + a_1 f[\varphi_{n-1}(t)] + \dots + a_n f[\varphi_0(t)] = 0, \tag{3}$$

kde $a_0 \neq 0$, $a_n \neq 0$. Přitom $\varphi_0(t) = t$ a funkce $\varphi_j(t)$, $j = 1, 2, \dots, n$ značí j-tou centrální dispersi 1. druhu příslušnou k lineární diferenciální rovnici 2. řádu Jacobiho typu

$$y'' = q(t) \,.\, y,$$

kde $q \in C^0$ $(-\infty, \infty)$, a diferenciální rovnice je buď oboustranně oscilatorická nebo speciální typu m.

Ukazuje se zvláště, že řešení $f(t)$ rovnice (3) lze vyjádřit ve tvaru

$$f(t) = \lambda_0^{X(t)},$$

kde λ_0 je kořen charakteristické rovnice

$$a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_n = 0$$

a $X(t)$ je rostoucí řešení Abelovy funkční rovnice

$$X[\varphi(t)] - X(t) = 1,$$

kde $\varphi = \varphi(t)$ je základní centrální disperse 1. druhu příslušná k diferenciální rovnici (q), tj. $\varphi(t) = \varphi_1(t)$.

V článku se definuje centrální disperse 1. druhu příslušná k diferenciální rovnici (q), která je speciální konečného typu m a odvozují se explicitní vyjádření řešení funkční rovnice (3) v závislosti na kvalitě kořenů charakteristické rovnice.

Резюме

ЗАМЕТКА К ТЕОРИИ ЛИНЕЙНЫХ КОНЕЧНО-РАЗНОСТНЫХ УРАВНЕНИЙ

МИРОСЛАВ ЛАЙТОХ

Рассматривается функциональное уравнение

(3) $$a_0 f[\varphi_n(t)] + a_1 f[\varphi_{n-1}(t)] + \dots + a_n f[\varphi_0(t)] = 0,$$

где $a_0 \neq 0$, $a_n \neq 0$. Далее $\varphi_0(t) = t$ и функции $\varphi_j(t)$, $j = 1, 2, \ldots, n$ обозначают j-тую центральную дисперсию 1-го рода, соответствующую линейному дифференциальному уравнению 2-го рода типа Якоби

$$\text{(q)} \qquad y'' = q(t) \, . \, y,$$

где $q \in C^{(0)}(-\infty, \infty)$, которое или осцилирует в обе стороны или специальное конечного типа m.

Показывается особенно, что решение $f(t)$ уравнения (3) можно выразить в виде

$$f(t) = \lambda_0^{X(t)},$$

где λ_0 корень характеристического уравнения

$$a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0$$

и $X(t)$ возрастающее решение функционального уравнения Абеля

$$X[\varphi(t)] - X(t) = 1,$$

где $\varphi = \varphi(t)$ основная центральная дисперсия 1-го рода, соответствующая дифференциальному уравнению (q), т, е. $\varphi(t) = \varphi_1(t)$.

В работе определяется центральная дисперсия 1-го рода, соответствующая дифференциальному уравнению (q), которое является специальным конечного типа m и выводятся явные виды решений функционального уравнения (3) в зависимости от качества корней характеристического уравнения.

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