Sborník prací Přírodovědecké fakulty University Palackého v Olomouci. Matematika

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Sborník prací Přírodovědecké fakulty University Palackého v Olomouci. Matematika, Vol. 21 (1982), No. 1, 63--67

Persistent URL: `http://dml.cz/dmlcz/120120`

1982 — ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM — TOM 73

Laboratoř výpočetní techniky Univerzity Palackého v Olomouci
Ředitel: RNDr. Milan Král, CSc.

TO CONJUGACY FUNCTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

JITKA LAITOCHOVÁ

(*Received March 30, 1981*)

Dedicated to my father on his 60th birthday

The present paper investigates sufficient conditions under which the conjugacy functions Δ_{φ_n}, Δ_{ψ_n}, $\Delta_{\varphi_{-n}}$, $\Delta_{\psi_{-n}}$ are concave or convex. It generalizes the results given in [4] where the conjugacy functions Δ_{φ_1} and Δ_{ψ_1} were studied.

W. Leighton defined the conjugacy function Δ_{φ_1} in [2] and treated the problem of Δ_{φ_1} increasing and decreasing being concave and convex, respectively. W. Leighton's results may be generalized in applying methods and results from the dispersion theory established by O. Borůvka [1].

Let us consider a second order linear differential equation in the Jacobian form

$$y'' = q(t)\, y, \qquad q \in C^{\circ}(\mathbf{R}) \tag{q}$$

oscillatory on $\mathbf{R}$, i.e. $-\infty$ and $+\infty$ are the limit points of the zeros of any solution of (q). Trivial solutions not being considered.

Following [1] let us denote by φ_n (φ_{-n}) the n-th, $(-n$-th) central dispersion of the 1st kind relative to (q) and by ψ_n, (ψ_{-n}) the n-th $(-n$-th) central dispersion of the 2nd kind relative to (q), where $q(t) < 0$ for $t \in \mathbf{R}$; n be a natural number.

Following [2] and [4] let us define the functions $\Delta_{\varphi_n}(t) = \varphi_n(t) - t$, $\Delta_{\psi_n}(t) = \psi_n(t) - t$, $\Delta_{\varphi_{-n}}(t) = \varphi_{-n}(t) - t$, $\Delta_{\psi_{-n}}(t) = \psi_{-n}(t) - t$, $t \in \mathbf{R}$.

Definition. *The function Δ_{φ_n} $(\Delta_{\varphi_{-n}})$ will be called the n-th $(-n$-th) conjugacy function of the 1st kind relative to the differential equation (q). The function Δ_{ψ_n} $(\Delta_{\psi_{-n}})$ will be called the n-th $(-n$-th) conjugacy function of the 2nd kind relative to the differential equation (q).*

The monotonicity of the differences $\varphi_n(t) - t$, $\psi_n(t) - t$, $\varphi_{-n}(t) - t$, $\psi_{-n}(t) - t$ was treated in [1].

The comparison theorem ([3], p. 277) yields the following

Lemma 1. *Let for*

$$y'' = Q(t)\,y \tag{Q}$$

$$y'' = \bar{Q}(t)\,y \tag{$\bar{Q}$}$$

$Q(t) \geqq \bar{Q}(t)$ for $t \in \mathbf{R}$, whereby the sign of equality does not identically hold in any interval of the type $(t, \varphi_1(t))$ and let φ_n (φ_{-n}) be the n-th $(-n$-th$)$ central dispersion of the 1st kind relative to the differential equation (Q) and $\bar{\varphi}_n$ $(\bar{\varphi}_{-n})$ be the n-th $(-n$-th$)$ central dispersion of the 1st kind relative to the differential equation $(\bar{Q})$. Then

$$\varphi_n(t) > \bar{\varphi}_n(t), \qquad \varphi_{-n}(t) < \bar{\varphi}_{-n}(t) \qquad \textit{for } t \in \mathbf{R}. \tag{1}$$

Proof follows by complete induction. The first statement in (1) for $n = 1$ is the consequence of the comparison theorem. If the first relation in (1) is true for $n - 1$, we will show that this is true for n too:

$$\varphi_1[\varphi_{n-1}(t)] > \bar{\varphi}_1[\varphi_{n-1}(t)] > \bar{\varphi}_1[\bar{\varphi}_{n-1}(t)]$$

that is

$$\varphi_n(t) > \bar{\varphi}_n(t).$$

The comparison theorem similarly yields the second statement in (1) for $n = 1$. If the second statement in (1) is true for $n - 1$, we will show that this is true for n too:

$$\varphi_{-1}[\varphi_{-(n-1)}(t)] < \bar{\varphi}_{-1}[\varphi_{-(n-1)}(t)] < \bar{\varphi}_{-1}[\bar{\varphi}_{-(n-1)}(t)]$$

that is

$$\varphi_{-n}(t) < \bar{\varphi}_{-n}(t).$$

Let $q \in C^2(\mathbf{R})$, $q(t) < 0$ for $t \in \mathbf{R}$. In accordance with O. Borůvka ([1], p. 8 and onwards) we introduce the differential equation (q_1) associated to (q) as the equation

$$y'' = q_1(t)\,y \tag{q_1}$$

where $q_1(t) = q(t) + \sqrt{|q(t)|}\,(1/\sqrt{|q(t)|})''$, $t \in \mathbf{R}$.

For each solution y_1 of the differential equation (q_1) the function $y_1\sqrt{|q(t)|}$ represents the derivative y' of precisely one solution y of (q).

Lemma 2 ([1]). *Let $q \in C^2(\mathbf{R})$, $q(t) < 0$ for $t \in \mathbf{R}$. The central dispersion of the second kind related to a differential equation (q) is the central dispersion of the first kind relative to the differential equation (q_1) associated to (q).*

In conformity with [2] we put

$$h(t) := \sqrt{-q^{-1}(t)}, \qquad t \in \mathbf{R}.$$

Then $q_1(t) = q(t) + \sqrt{|q(t)|}\,(1/\sqrt{|q(t)|})'' = q(t) + \dfrac{h''(t)}{h(t)}$.

Lemma 3. *Let $q \in C^2(\mathbf{R})$, $q(t) < 0$ for $t \in \mathbf{R}$, $t_0 \in \mathbf{R}$ be an arbitrary number. Let us put $h(t) := \sqrt{-q^{-1}(t)}$, $t \in \mathbf{R}$. Then the following implications hold on $\mathbf{R}$:*

$$h''(t) < 0 \Rightarrow \psi_n(t_0) < \varphi_n(t_0),\ \psi_{-n}(t_0) > \varphi_{-n}(t_0),$$
$$h''(t) > 0 \Rightarrow \psi_n(t_0) > \varphi_n(t_0),\ \psi_{-n}(t_0) < \varphi_{-n}(t_0),$$
$$h''(t) \equiv 0 \Rightarrow \psi_n(t_0) = \varphi_n(t_0),\ \psi_{-n}(t_0) = \varphi_{-n}(t_0).$$

Proof. If $h'' < 0$, then $q_1 - q = h''/h < 0$ and thus $q_1 < q$. In applying Lemmas 1 and 2 we get $\psi_n(t_0) < \varphi_n(t_0)$ and $\psi_{-n}(t_0) > \varphi_{-n}(t_0)$. If $h'' > 0$, then $q_1 - q = h''/h > 0$ and therefore $q_1 > q$. Applying Lemmas 1 and 2 we get $\psi_n(t_0) > \varphi_n(t_0)$ and $\psi_{-n}(t_0) < \varphi_{-n}(t_0)$. If $h'' \equiv 0$, then $q_1 - q = h''/h \equiv 0$ and thus $q_1 \equiv q$, so that $\psi_n(t_0) = \varphi_n(t_0)$ and $\psi_{-n}(t_0) = \varphi_{-n}(t_0)$.

Theorem 1. *Let $q \in C^2(\mathbf{R})$, $q(t) < 0$ for $t \in \mathbf{R}$. Let us put $q_1(t) = q(t) + \sqrt{|q(t)|} \left(\frac{1}{\sqrt{|q(t)|}}\right)''$, $t \in \mathbf{R}$ and let $q_1(t) < 0$ on $\mathbf{R}$. Let us put further $h(t) := \sqrt{-q^{-1}(t)}$, $h_1(t) := \sqrt{-q_1^{-1}(t)}$, $t \in \mathbf{R}$. Then there hold the following implications on $\mathbf{R}$:*

(2) $h''(t) < 0 \Rightarrow \Delta_{\varphi_n}$ *is concave,* $\Delta_{\varphi_{-n}}$ *is concave*
(3) $h''(t) > 0 \Rightarrow \Delta_{\varphi_n}$ *is convex,* $\Delta_{\varphi_{-n}}$ *is convex*
(4) $h_1''(t) < 0 \Rightarrow \Delta_{\psi_n}$ *is concave,* $\Delta_{\psi_{-n}}$ *is concave*
(5) $h_1''(t) > 0 \Rightarrow \Delta_{\psi_n}$ *is convex,* $\Delta_{\psi_{-n}}$ *is convex*

Proof for (2) and (3): Let $\nu = \pm 1, \pm 2, \ldots$. Since $\Delta_{\varphi_\nu}(t) = \varphi_\nu(t) - t$ we have $\Delta'_{\varphi_\nu}(t) = \varphi'_\nu(t) - 1 = [u^2[\varphi_\nu(t)]/u^2(t)] - 1$ providing that $u(t_0) \neq 0$. Further

$$\Delta''_{\varphi_\nu}(t) = \varphi''_\nu(t) =$$
$$= \{2u[\varphi_\nu(t)]\, u'[\varphi_\nu(t)]\, \varphi'_\nu(t)\, u^2(t) - u^2[\varphi_\nu(t)]\, 2u(t)\, u'(t)\}/u^4(t) =$$
$$= \{2u^2[\varphi_\nu(t)]/u^2(t)\}\, u[\varphi_\nu(t)]\, u'[\varphi_\nu(t)] - u(t)\, u'(t)/u^2(t)\}.$$

If $t_0 \in j$ is an arbitrary number and u is such a solution that $u'(t_0) = 0$, then $u(t_0) \neq 0$. Then we have

$$\Delta''_{\varphi_\nu}(t_0) = \varphi''_\nu(t_0) = \{2u^2[\varphi_\nu(t_0)]/u^2(t_0)\}\, \{u[\varphi_\nu(t_0)] \cdot u'[\varphi_\nu(t_0)]/u^2(t_0)\}.$$

Let $n = 1, 2, 3, \ldots$

For $u(t_0) \gtrless 0$ holds $u[\varphi_n(t_0)] \gtrless 0$ for n being even and $u[\varphi_n(t_0)] \lessgtr 0$ for n being odd. In case of $\psi_n(t_0) < \varphi_n(t_0)$, then $u'[\varphi_n(t_0)] \gtrless 0$ for n being odd and $u'[\varphi_n(t_0)] \lessgtr 0$ for n being even; in case of $\psi_n(t_0) > \varphi_n(t_0)$, then $u'[\varphi_n(t_0)] \lessgtr 0$ for n being odd and $u'[\varphi_n(t_0)] \gtrless 0$ for n being even.

If $h'' < 0$, then $\psi_n(t_0) < \varphi_n(t_0)$. From this follows that $u[\varphi_n(t_0)]\, u'[\varphi_n(t_0)] < 0$ and therefore $\Delta''_{\varphi_n}(t_0) = \varphi''_n(t_0) < 0$. Thus the function $\Delta_{\varphi_n}(t)$ is concave.

If $h'' > 0$, then $\psi_n(t_0) > \varphi_n(t_0)$. From this follows that $u[\varphi_n(t_0)]\, u'[\varphi_n(t_0)] > 0$ and therefore $\Delta''_{\varphi_n}(t_0) = \varphi''_n(t_0) > 0$. Thus the function $\Delta_{\varphi_n}(t)$ is convex.

For $u(t_0) \gtrless 0$ is $u[\varphi_{-n}(t_0)] \gtrless 0$ for n even and $u[\varphi_{-n}(t_0)] \lessgtr 0$ for n odd. If $\psi_{-n}(t_0) < \varphi_{-n}(t_0)$, then $u'[\varphi_{-n}(t_0)] \gtrless 0$ for n even and $u'[\varphi_n(t_0)] \lessgtr 0$ for n odd. If $\psi_{-n}(t_0) > \varphi_{-n}(t_0)$, then $u'[\varphi_{-n}(t_0)] \lessgtr 0$ for n even and $u'(\varphi_{-n}(t_0)) \gtrless 0$ for n odd.

If $h'' < 0$ then $\psi_{-n}(t_0) > \varphi_{-n}(t_0)$. From this follows that $u[\varphi_{-n}(t_0)]\, u'[\varphi_{-n}(t_0)] < < 0$ and therefore $\Delta''_{\varphi_{-n}}(t_0) = \varphi''_{-n}(t_0) < 0$. Thus the function $\Delta_{\varphi_{-n}}(t)$ is concave.

If $h'' > 0$, then $\psi_{-n}(t_0) < \varphi_{-n}(t_0)$. From this follows that $u[\varphi_{-n}(t_0)]\, u'[\varphi_{-n}(t_0)] > > 0$ and therefore $\Delta''_{\varphi_{-n}}(t_0) = \varphi''_{-n}(t_0) > 0$. Thus the function $\Delta_{\varphi_{-n}}(t)$ is convex.

Proof for (4) and (5) proceeds analogous for the differential equation (q_1).

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Souhrn

PŘÍSPĚVEK KE KONJUGOVANÝM FUNKCÍM LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC 2. ŘÁDU

JITKA LAITOCHOVÁ

V článku jsou definovány n-té ($-n$-té) konjugované funkce 1. druhu příslušné k diferenciální rovnici $(q)\ y'' = q(t)\, y$ vztahy $\Delta_{\varphi_n}(t) = \varphi_n(t) - t$, $\Delta_{\varphi_{-n}}(t) = \varphi_{-n}(t) - - t$ a n-té ($-n$-té) konjugované funkce 2. druhu příslušné k diferenciální rovnici (q) vztahy $\Delta_{\psi_n}(t) = \psi_n(t) - t$ a $\Delta_{\psi_{-n}}(t) = \psi_{-n}(t) - t$, kde φ_n (φ_{-n}) je n-tá ($-n$-tá) centrální disperze 1. druhu a ψ_n (ψ_{-n}) je n-tá ($-n$-tá) centrální disperze 2. druhu [1].

Jsou nalezeny postačující podmínky konkávnosti a konvexnosti těchto funkcí.

Резюме

ЗАМЕТКА О СОПРЯЖЕННЫХ ФУНКЦИЯХ ЛИНЕЙНЫХ ДИФФЕРЕНЦИЯЛЬНЫХ УРАВНЕНИЙ 2-ГО РОДА

ЙИТКА ЛАЙТОХОВА

В работе определены н-тые (-н-тые) сопряженные функции 1-го рода соответствующие дифференциальному уравнению $(q) y'' = q(t) y$ соотношениями

$$\Delta_{\varphi_n}(t) = \varphi_n(t) - t, \qquad \Delta_{\varphi_{-n}}(t) = \varphi_{-n}(t) - t$$

и н-тые (−н-тые) сопряженные функции 2-го рода соответствующие дифференциальному уравнению (q) соотношениями

$$\Delta_{\psi_n}(t) = \psi_n(t) - t, \qquad \Delta_{\psi_{-n}}(t) = \psi_{-n}(t) - t,$$

где φ_n (φ_{-n}) является н-той (−*н*-той) центральной дисперсией 1-го рода и ψ_n (ψ_{-n}) является *н*-той (−*н*-той) центральной дисперсией 2-го рода [1].

Найдены достаточные условия выпуклости и вогнутости этих функций.