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A NOTE ON DISCONJUGATE LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH PERIODIC COEFFICIENTS

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Dedicated to Academician O. Borůvka on his 80th birthday

Introduction

O. Borůvka in [3–6] and F. Neuman in [14, 15] brought into relation the Floquet theory for both-sided oscillatory equations (q): $y'' = q(t)y$, $q \in C^0(\mathbf{R})$, $q(t + \pi) = q(t)$ for $t \in \mathbf{R}(= (-\infty, \infty))$ with the phase and dispersion theory. The present paper brings into relation the Floquet theory for nonoscillatory equations (q) with the theory of hyperbolic and parabolic phases [7–11].

2. Fundamental concepts and lemmas

We shall investigate differential equations of type

$$y'' = q(t)y, \qquad q \in C^0(\mathbf{R}), \qquad q(t + \pi) = q(t) \qquad \text{for } t \in \mathbf{R}, \tag{q}$$

being nonoscillatory on $\mathbf{R}$. Then it follows from [12] that an equation (q) is disconjugate (on $\mathbf{R}$), which implies that every (nontrivial) solution of (q) has at most one zero on $\mathbf{R}$. The trivial solution will be excluded from our consideration.

Let $x \in \mathbf{R}$ and y_1, y_2 be solutions of (q), $y_1(x) = y_2'(x) = 0$, $y_1'(x) = y_2(x) = 1$. According to the Floquet theory every equation (q) may be associated with the algebraic quadratic equation

$$\lambda^2 - A\lambda + 1 = 0, \qquad (A := y_1'(x + \pi) + y_2(x + \pi)),$$

whose roots (generally complex) ϱ, $\frac{1}{\varrho}$ ($\varrho \neq 0$) are called the *characteristic multipliers* of (q). In the case of a disconjugate equation (q) we come to the following.

Lemma 1. *Let* (q) *be a disconjugate equation. Then the equation* (q) *has only positive characteristic multipliers* $\varrho, \frac{1}{\varrho}$ *and there exist independent solutions* u, v *of* (q) *satisfying either*

$$u(t + \pi) = \varrho \,.\, u(t), \qquad v(t + \pi) = \frac{1}{\varrho} \,.\, v(t), \qquad \varrho \neq 1, \tag{1}$$

or

$$u(t + \pi) = u(t) + v(t), \qquad v(t + \pi) = v(t), \qquad \varrho = 1. \tag{2}$$

In case of $\varrho = 1$ *the equation* (q) *does not possess all* π*-periodic solutions.*

Proof. If (q) has complex characteristic multipliers, then if tollows from [4, 12] that (q) is oscillatory. Consequently the disconjugate equation (q) may have only real characteristic multipliers $\varrho, \frac{1}{\varrho}$. If $\varrho < 0$, there exists according to the Floquet theory a solution u of (q) satisfying $u(t + \pi) = \varrho \,.\, u(t)$ for $t \in \mathbf{R}$. Then for (every) $t_1 \in \mathbf{R}$ we have $u(t_1 + \pi) \,.\, u(t_1) \leqq 0$. Thus the solution u has at least one zero on $\langle t_1, t_1 + \pi \rangle$. Naturally, $-\infty$ and ∞ represent the cluster points of the zeros of u and the equation (q) is oscillatory, which is a contradiction. Therefore $\varrho > 0$. Let $\varrho = 1$ and let all solutions of (q) be π-periodic. Let u be a solution of (q). Then $u(t) \neq 0$ for $t \in \mathbf{R}$. Let $t_0 \in \mathbf{R}$ and put $v(t) := u(t) \int_{t_0}^{t} \frac{ds}{u^2(s)}$ for $t \in \mathbf{R}$. Since v is a solution of (q), it represents a π-periodic function. Now from $v(t + \pi) = u(t + \pi) \int_{t_0}^{t+\pi} \frac{ds}{u^2(s)} = u(t) \left(\int_{t_0}^{t} \frac{ds}{u^2(s)} + \int_{t}^{t+\pi} \frac{ds}{u^2(s)} \right) = v(t) + u(t) \int_{t}^{t+\pi} \frac{ds}{u^2(s)}$ we have $u(t) \int_{t}^{t+\pi} \frac{ds}{u^2(s)} = 0$ for $t \in \mathbf{R}$, which is a contradiction. Consequently (q) does not possess all π-periodic solutions. It follows from the Floquet theory that there exist independent solutions u, v of (q) satisfying either (1) or (2).

Remark 1. If we have $\varrho \neq 1$ for the characteristic multipliers $\varrho, \frac{1}{\varrho}$ of a disconjugate equation (q), then obviously, there always exist independent solutions u, v of (q) satisfying (1), $u(t) > 0$, $v(t) > 0$ for $t \in \mathbf{R}$ and $|uv' - u'v| = 2$.

Let us say with [7] that a function $\alpha \in C^0(\mathbf{i})$, $\mathbf{i} \subset \mathbf{R}$ is a (first) *hyperbolic phase* of (q) on $\mathbf{i}$, if there exist independent solutions u, v of (q) satisfying $|u(t)| < |v(t)|$ on $\mathbf{i}$ and

$$\operatorname{tgh} \alpha(t) = \frac{u(t)}{v(t)} \qquad \text{for } t \in \mathbf{i}.$$

Then $\alpha \in C^3(\mathbf{i})$, $\alpha'(t) \neq 0$ and $q(t) = -\frac{\alpha'''(t)}{2\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 + \alpha'^2(t)$ for $t \in \mathbf{i}$. If for a hyperbolic phase α of (q) is $\mathbf{i} = \mathbf{R}$ and $\lim_{t \to -\infty} \alpha(t) = -\operatorname{sign} \alpha' . \infty$, $\lim_{t \to -\infty} \alpha(t) = \operatorname{sign} \alpha' . \infty$, then (q) is a *pure disconjugate equation* (on $\mathbf{R}$).

Lemma 2. *Let* (q) *be a disconjugate equation. Then there exist independent solutions* u, v *of* (q) *satisfying* $u(t) > 0$, $v(t) > 0$ *for* $t \in \mathbf{R}$ *and* $|uv' - u'v| = 2$ *iff there exists a hyperbolic phase* α *of* (q) *defined on* $\mathbf{R}$ *where*

$$u(t) = \frac{\exp \alpha(t)}{\sqrt{|\alpha'(t)|}}, \qquad v(t) = \frac{\exp(-\alpha(t))}{\sqrt{|\alpha'(t)|}}, \qquad t \in \mathbf{R}. \tag{3}$$

Proof. ($\Rightarrow$) Let u, v be independent solutions of a disconjugate equation (q) satisfying $u(t) > 0$, $v(t) > 0$ for $t \in \mathbf{R}$ and $|uv' - u'v| = 2$. We set $u_1(t) := u(t) - v(t)$, $v_1(t) := u(t) + v(t)$, $t \in \mathbf{R}$. Then u_1, v_1 are independent solutions of (q), $v_1(t) > 0$, $|u_1(t)| < |v_1(t)|$ for $t \in \mathbf{R}$ and $|u_1 v_1' - u_1' v_1| = 2|uv' - u'v| = 4$. We define on $\mathbf{R}$ a hyperbolic phase α of (q) by the relation $\operatorname{tgh} \alpha(t) = \frac{u_1(t)}{v_1(t)}$. Then $u_1(t) = \frac{2 \sinh \alpha(t)}{\sqrt{|\alpha'(t)|}}$, $v_1(t) = \frac{2 \cosh \alpha(t)}{\sqrt{|\alpha'(t)|}}$ as follows from [7]. Hence $u(t) = \frac{1}{2}(u_1(t) + v_1(t)) = \frac{\exp \alpha(t)}{\sqrt{|\alpha'(t)|}}$, $v(t) = \frac{1}{2}(v_1(t) - \mu_1(t)) = \frac{\exp(-\alpha(t))}{\sqrt{|\alpha'(t)|}}$.

($\Leftarrow$) Let α be a hyperbolic phase of a disconjugate equation (q) on $\mathbf{R}$ and let u, v be the function defined by (3). Following Theorem 5 [8] u, v are independent solutions of (q), $u(t) > 0$, $v(t) > 0$ for $t \in \mathbf{R}$. It is easy to verify that $|uv' - u'v| = 2$.

Say with [10, 11] that a function $\alpha \in C^0(\mathbf{i})$, $\mathbf{i} \subset \mathbf{R}$ represents a (first) *parabolic phase* of (q) on $\mathbf{i}$ if there exist independent solutions u, v of (q), $v(t) \neq 0$ for $t \in \mathbf{i}$ satisfying

$$\alpha(t) = \frac{u(t)}{v(t)} \qquad \text{for } t \in \mathbf{i}.$$

Then $\alpha \in C^3(\mathbf{i})$, $\alpha'(t) \neq 0$ and $q(t) = -\frac{\alpha'''(t)}{2\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2$ for $t \in \mathbf{i}$. If for a parabolic phase α of (q) is $\mathbf{i} = \mathbf{R}$ and $\lim_{t \to -\infty} \alpha(t) = -\operatorname{sign} \alpha' . \infty$, $\lim_{t \to \infty} \alpha(t) = \operatorname{sign} \alpha' . \infty$, then (q) is a *special disconjugate equation* (on $\mathbf{R}$).

Let $\mathfrak{H}$ be the set of all functions h, $h \in C^3(\mathbf{R})$, $h'(t) \neq 0$ and $h(t + \pi) = h(t) + \pi . \operatorname{sign} h'$ for $t \in \mathbf{R}$.

Definition 1. *Say, equations* (q_1) *and* (q_2) *have the same behaviour if they are either pure disconjugate with the same characteristic multipliers or are special disconjugate.*

3. Main results

Theorem 1. *An equation* (q) *is disconjugate and* $\varrho, \frac{1}{\varrho}$ *are its characteristic multipliers,* $\varrho > 1$, *iff* (q) *is pure disconjugate and there exists a hyperbolic phase* α *of* (q) *on* **R** *satisfying*

$$\alpha(t + \pi) = \alpha(t) + a, \qquad t \in \mathbf{R}, \tag{4}$$

where $a = \ln \varrho$ (> 0).

Proof. ($\Leftarrow$) Let $\varrho, \frac{1}{\varrho}$ be the characteristic multipliers of a disconjugate equation (q), $\varrho > 1$. Then, following Lemma 1 and Remark 1, there exist independent solutions u, v of (q), $u(t) > 0, v(t) > 0$ for $t \in \mathbf{R}$ and $|uv' - u'v| = 2$ satisfying (1). By Lemma 2 there exists a hyperbolic phase α of (q) on **R** for which (3) applies. Since

$$\frac{u(t + \pi)}{v(t + \pi)} = \varrho^2 \frac{u(t)}{v(t)},$$

we have

$$\exp 2\alpha(t + \pi) = \varrho^2 . \exp 2\alpha(t) = \exp 2(\alpha(t) + a),$$

where $a = \ln \varrho$ (> 0). Hence $\alpha(t + \pi) = \alpha(t) + a$ and (q) is pure disconjugate.

($\Leftarrow$) Let α be a hyperbolic phase of a pure disconjugate equation (q) on **R** satisfying (4), where $a = \ln \varrho$ (> 0). Then (q) is a disconjugate, the functions u, v defined by (3) are independent solutions of (q) and

$$u(t + \pi) = \frac{\exp \alpha(t + \pi)}{\sqrt{|\alpha'(t + \pi)|}} = \frac{\exp(\alpha(t) + a)}{\sqrt{|\alpha'(t)|}} = e^a u(t),$$

$$v(t + \pi) = \frac{\exp(-\alpha(t + \pi))}{\sqrt{|\alpha'(t + \pi)|}} = \frac{\exp(-\alpha(t) - a)}{\sqrt{|\alpha'(t)|}} = e^{-a} v(t).$$

From this it follows that $e^a = \varrho$, $e^{-a} = \frac{1}{\varrho}$ are the characteristic multipliers of (q) and $\varrho > 1$.

Theorem 2. *Let* (q_1) *be a pure disconjugate equation and let* $\varrho, \frac{1}{\varrho}$ *be its characteristic multipliers,* $\varrho > 1$. *Let* α *be a hyperbolic phase of* (q_1) *on* **R** *satisfying* (4). *Then equations* (q_1) *and* (q_2) *have the same behaviour iff there exists a hyperbolic phase* α_2 *of* (q_2) *satisfying*

$$\alpha_2(t) = \operatorname{sign} h' . \alpha(h(t)), \qquad t \in \mathbf{R}$$

for a $h \in \mathfrak{H}$.

Proof. ($\Rightarrow$) Let (q_1) be a pure disconjugate equation and let equations (q_1) and (q_2) possess the same behaviour. Following Theorem 1 there exist then a hyperbolic phase α_2 of (q_2) on $\mathbf{R}$ satisfying $\alpha_2(t+\pi) = \alpha_2(t) + a$, where $a = \ln \varrho$ (> 0). We put $h(t) := \alpha^{-1}(\alpha_2(t))$ for $t \in \mathbf{R}$. Then $\operatorname{sign} h' = 1$ and $h(t+\pi) = \alpha^{-1}(\alpha_2(t+\pi)) = \alpha^{-1}(\alpha_2(t) + a) = \alpha^{-1}(\alpha_2(t)) + \pi = h(t) + \pi$. We see that $h \in \mathfrak{H}$.

($\Leftarrow$) Let $h \in \mathfrak{H}$ and $\alpha_2(t) := \operatorname{sign} h' \,.\, \alpha(h(t))$ be a hyperbolic phase of (q_2) on $\mathbf{R}$. Then (q_2) is a pure disconjugate equation, $\alpha_2(t+\pi) = \operatorname{sign} h' \,.\, \alpha(h(t+\pi)) = \operatorname{sign} h' \,.\, \alpha(h(t) + \pi \,.\, \operatorname{sign} h') = \operatorname{sign} h' \,.\, (\alpha(h(t)) + a \,.\, \operatorname{sign} h') = \alpha_2(t) + a (a = \ln \varrho > 0)$ and the formula $q_2(t) = -\dfrac{\alpha_2'''(t)}{2\alpha_2'(t)} + \dfrac{3}{4}\left(\dfrac{\alpha_2''(t)}{\alpha_2'(t)}\right)^2 + \alpha'^2(t)$ yields $q_2(t+\pi) = q_2(t)$. Next we obtain from Theorem 1 that (q_2) possesses the characteristic multipliers $\varrho, \dfrac{1}{\varrho}$. Hence the equations (q_1) and (q_2) have the same behaviour.

Theorem 3. *An equation* (q) *is disconjugate and has the characteristic multipliers* $\varrho, \dfrac{1}{\varrho}$, $\varrho = 1$, *iff* (q) *is special disconjugate and there exists a parabolic phase* α *of* (q) *on* $\mathbf{R}$ *satisfying*

$$\alpha(t+\pi) = \alpha(t) + \operatorname{sign} \alpha', \qquad t \in \mathbf{R}. \tag{5}$$

Proof. ($\Rightarrow$) Let (q) be a disconjugate equation and let $\varrho, \dfrac{1}{\varrho}$ be its characteristic multipliers, $\varrho = 1$. Following Lemma 1 there exists a π-periodic solution v of (q), $v(t) \neq 0$ for $t \in \mathbf{R}$. Let $t_0 \in \mathbf{R}$ and put $u(t) := v(t) \displaystyle\int_{t_0}^{t} \frac{ds}{v^2(s)}$, $t \in \mathbf{R}$. Then u is a solution of (q). It follows from $\left(\displaystyle\int_t^{t+\pi} \frac{ds}{v^2(s)}\right)' = \dfrac{1}{v^2(t+\pi)} - \dfrac{1}{v^2(t)} = 0$ that $\displaystyle\int_t^{t+\pi} \frac{ds}{v^2(s)} = a$ for $t \in \mathbf{R}$, where a is a constant, $a > 0$. Put $\alpha(t) := \dfrac{u(t)}{a \,.\, v(t)}$, $t \in \mathbf{R}$. Then α is a parabolic phase of (q) on $\mathbf{R}$, $\operatorname{sign} \alpha' = 1$,

$$\alpha(t+\pi) = \frac{u(t+\pi)}{a \,.\, v(t+\pi)} = \frac{1}{a}\int_{t_0}^{t+\pi} \frac{ds}{v^2(s)} = \alpha(t) + \frac{1}{a}\int_t^{t+\pi} \frac{ds}{v^2(s)} =$$
$$= \alpha(t) + \operatorname{sign} \alpha'$$

and (q) is special disconjugate.

($\Leftarrow$) Let α be a parabolic phase of a special disconjugate equation (q) on $\mathbf{R}$ satisfying (5). Then (q) is a disconjugate equation. Putting $u(t) := \dfrac{\alpha(t)}{\sqrt{|\alpha'(t)|}} \,.\, \operatorname{sign} \alpha'$, $v(t) := \dfrac{1}{\sqrt{|\alpha'(t)|}}$, $t \in \mathbf{R}$, then u, v are independent solutions of (q) and

$$u(t+\pi) = \frac{\alpha(t+\pi)}{\sqrt{|\alpha'(t+\pi)|}} \,.\, \operatorname{sign}\alpha' = \frac{\alpha(t) + \operatorname{sign}\alpha'}{\sqrt{|\alpha'(t)|}} \,.\, \operatorname{sign}\alpha' =$$

$$= \frac{\alpha(t)}{\sqrt{|\alpha'(t)|}} \,.\, \operatorname{sign}\alpha' + \frac{1}{\sqrt{|\alpha'(t)|}} = u(t) + v(t),$$

$$v(t+\pi) = \frac{1}{\sqrt{|\alpha'(t+\pi)|}} = \frac{1}{\sqrt{|\alpha'(t)|}} = v(t).$$

Hence $\varrho, \frac{1}{\varrho}$ with $\varrho = 1$ are the characteristic multipliers of (q).

Theorem 4. *Let* (q_1) *be a special disconjugate equation and let* α *be its parabolic phase satisfying* (5). *Then equations* (q_1) *and* (q_2) *have the same behaviour iff there exists a parabolic phase* α_2 *of* (q_2) *on* $\mathbf{R}$ *satisfying*

$$\alpha_2(t) = \alpha(h(t)), \qquad t \in \mathbf{R},$$

for a $h \in \mathfrak{H}$.

Proof. ($\Rightarrow$) Let (q_1) be a special disconjugate equation and let (q_1) and (q_2) possess the same behaviour. Let α be a parabolic phase of (q_1) satisfying (5). Then $\alpha^{-1}(t + \nu) = \alpha^{-1}(t) + \nu\pi \,.\, \operatorname{sign}\alpha'$ with $\nu = \pm 1$. Following Theorem 3 there exist a parabolic phase α_2 of (q_2) on $\mathbf{R}$ satisfying $\alpha_2(t + \pi) = \alpha_2(t) + \operatorname{sign}\alpha_2$. We set $h(t) := \alpha^{-1}(\alpha_2(t))$ for $t \in \mathbf{R}$. Then $\operatorname{sign} h' = \operatorname{sign}\alpha' \,.\, \operatorname{sign}\alpha_2'$ and $h(t + \pi) = \alpha^{-1}(\alpha_2(t + \pi)) = \alpha^{-1}(\alpha_2(t) + \operatorname{sign}\alpha_2') = \alpha^{-1}(\alpha_2(t)) + \pi \,.\, \operatorname{sign}\alpha' \,.\, \operatorname{sign}\alpha_2' = h(t) + \pi \,.\, \operatorname{sign} h'$. Hence $\alpha_2 = \alpha h$, where $h \in \mathfrak{H}$.

($\Leftarrow$) Let $h \in \mathfrak{H}$ and $\alpha_2(t) := \alpha(h(t))$ for $t \in \mathbf{R}$ be a parabolic phase of (q_2) on $\mathbf{R}$. Since $\operatorname{sign}\alpha_2' = \operatorname{sign}\alpha' \,.\, \operatorname{sign} h'$ we have $\alpha_2(t + \pi) = \alpha(h(t + \pi)) = \alpha(h(t) + \pi \times \operatorname{sign} h') = \alpha(h(t)) + \operatorname{sign} h' \,.\, \operatorname{sign}\alpha' = \alpha_2(t) + \operatorname{sign}\alpha_2'$ and the formula $q_2(t) = -\frac{\alpha_2'''(t)}{2\alpha_2'(t)} + \frac{3}{4}\left(\frac{\alpha_2''(t)}{\alpha_2'(t)}\right)^2$ yields $q_2(t + \pi) = q_2(t)$. Consequently the equations (q_1) and (q_2) are special disconjugate and have the same behaviour.

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Souhrn

POZNÁMKA K DISKONJUGOVANÝM DIFERENCIÁLNÍM ROVNICÍM 2. ŘÁDU S PERIODICKÝMI KOEFICIENTY

SVATOSLAV STANĚK

V práci jsou vyšetřovány diferenciální rovnice typu

$$y'' = q(t)\,y, \qquad q \in C^0(\mathbf{R}), \qquad q(t + \pi) = q(t) \qquad \text{pro } t \in \mathbf{R}, \tag{q}$$

které jsou diskonjugované na $\mathbf{R}$. Funkce $\alpha \in C^0(\mathbf{R})$ se nazývá hyperbolická fáze rovnice (q) na $\mathbf{i}$ ($\subset \mathbf{R}$) jestliže existují její nezávislá řešení u, v: $|u(t)| < |v(t)|$ a $\operatorname{tgh} \alpha(t) = \dfrac{u(t)}{v(t)}$ pro $t \in \mathbf{i}$. Jestliže rovnice (q) má hyperbolickou fázi α na $\mathbf{R}$ pro niž platí $\lim_{t \to -\infty} \alpha(t) = -\operatorname{sign} \alpha' . \infty$, $\lim_{t \to \infty} \alpha(t) = \operatorname{sign} \alpha' . \infty$, pak (q) je ryze diskonjugovaná rovnice $\mathbf{R}$. Funkce $\beta \in C^0(\mathbf{R})$ se nazývá parabolická fáze rovnice (q) na $\mathbf{i}$ jestliže existují její nezávislá řešení $u_1, v_1 : v_1(t) \neq 0$ a $\beta(t) = \dfrac{u_1(t)}{v_1(t)}$ pro $t \in \mathbf{i}$. Jestliže rovnice (q) má parabolickou fázi β na $\mathbf{R}$ pro niž platí $\lim_{t \to -\infty} \beta(t) = -\operatorname{sign} \beta' . \infty$, $\lim_{t \to \infty} \beta(t) = \operatorname{sign} \beta' . \infty$, pak (q) je speciálně diskonjugovaná rovnice na $\mathbf{R}$.

Podle Floquetovy teorie lze ke každé rovnici (q) přiřadit jistou kvadratickou rovnici, jejíž kořeny $\varrho, \frac{1}{\varrho}$ ($\varrho \neq 0$) se nazývají charakteristické kořeny rovnice (q). V práci je dokázáno:

Rovnice (q) je diskonjugovaná a $\varrho, \frac{1}{\varrho}$ ($\varrho > 1$) jsou její charakteristické kořeny právě když (q) je ryze diskonjugovaná rovnice a existuje hyperbolická fáze α rovnice (q) na **R** pro niž

$$\alpha(t + \pi) = \alpha(t) + a, \qquad t \in \mathbf{R},$$

kde $a = \ln \varrho$ (> 0).

Rovnice (q) je diskonjugovaná a $\varrho, \frac{1}{\varrho}$ ($\varrho = 1$) jsou její charakteristické kořeny právě když (q) je speciálně diskonjugovaná rovnice a existuje parabolická fáze β rovnice (q) na **R** pro niž

$$\beta(t + \pi) = \beta(t) + \operatorname{sign} \beta', \qquad t \in \mathbf{R}.$$

Současně jsou také nalezeny všechny diskonjugované rovnice (q), které mají stejné charakteristické kořeny.

Резюме

ЗАМЕЧАНИЕ К ЛИНЕЙНЫМ ДИФФЕРЕНЦИАЛЬНЫМ УРАВНЕНИЯМ ВТОРОГО ПОРЯДКА БЕЗ СОПРЯЖЕННЫХ ТОЧЕК С ПЕРИОДИЧЕСКИМИ КОЭФФИЦИЕНТАМИ

СВАТОСЛАВ СТАНЕК

В работе исследуются дифференциальные уравнения без сопряженных точек на **R** типа

$$y'' = q(t)\, y,\ q \in C^{\circ}(\mathbf{R}),\ q(t + \pi) = q(t) \text{ для } t \in \mathbf{R}. \qquad \text{(q)}$$

Функция $\alpha \in C^{\circ}(\mathbf{R})$ называется гиперболической фазой уравнения (q) на i($\subset \mathbf{R}$) если существуют его независимые решения u, v: $|u(t)| < |v(t)|$ и $\operatorname{tgh} \alpha(t) = \frac{u(t)}{v(t)}$ для $t \in$ i. Если уравнение (q) имеет гиперболическую фазу α на **R** для которой имеет место $\lim_{t \to -\infty} \alpha(t) = -\operatorname{sign} \alpha' . \infty$, $\lim_{t \to \infty} \alpha(t) = \operatorname{sign} \alpha' . \infty$, то (q) является уравнением совершенно без сопряженных точек на **R**. Функция $\beta \in C^{\circ}(\mathbf{R})$ называется параболической фазой уравнения (q) на i если существуют его независимые решения u_1, v_1 : $v_1(t) \neq 0$ и $\beta(t) = \dfrac{u_1(t)}{v_1(t)}$ для $t \in \mathrm{i}$. Если уравнение (q) имеет параболическую фазу β на $\mathbf{R}$ для которой имеет место $\lim\limits_{t \to -\infty} \beta(t) = -\operatorname{sign} \beta' . \infty$, $\lim\limits_{t \to \infty} \beta(t) = \operatorname{sign} \beta' . \infty$, то (q) является уравнением специально без сопряженных точек на $\mathbf{R}$.

В теории Флоке к каждому уравнению (q) присоединяется квадратическое уравнение, корни $\varrho, \dfrac{1}{\varrho}$ $(\varrho \neq 0)$ которого называются характеристические корни уравнения (q). В работе показано:

(q) есть уравнение без сопряженных точек и его характеристические корни равны $\varrho, \dfrac{1}{\varrho}$ (> 1) только в случае, когда (q) есть уравнение совершенно без сопряженных точек и существует гиперболическая фаза α уравнения (q) на $\mathbf{R}$ так что

$$\alpha(t + \pi) = \alpha(t) + a, \quad t \in \mathbf{R}$$

где $a = \ln \varrho$ (> 0).

(q) есть уравнение без сопряженных точек и его характеристические корни равны $\varrho, \dfrac{1}{\varrho}$ $(\varrho = 1)$ только в том случае, когда (q) есть уравнение специально без сопряженных точек и существует параболическая фаза β уравнения (q) на $\mathbf{R}$ так что

$$\beta(t + \pi) = \beta(t) + \operatorname{sign} \beta', \quad t \in \mathbf{R}.$$

Одновременно показаны все неколеблющиеся уравнения (q) которые имеют одинаковые характеристические корни.