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A NOTE TO A CERTAIN PAIR OF PARAMETRIC INTEGRALS

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The mapping $I(z)$ defined by the parametric integral

$$\int_0^{+\infty} \exp(-\bar{z}t)\,\mathrm{d}t, \qquad \text{where } \operatorname{Re} z > 0 \tag{1}$$

stands for the inversion with respect to the unit circle with its centre at the origin 0. This mapping biuniquely carries over the exterior of the circle $|z| > 1$ onto its interior $0 < |z| < 1$ and reversely in the complex halfplane $\operatorname{Re} z > 0$, whereby the only self-adjoint points of this mapping are exactly all the points of the open half-circle having the equation $z(t) = \exp(it)$, where $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The convergence of the integral (1) is guaranted by the condition $\operatorname{Re} z > 0$; whereby the parametric integral (1) uniformly converges for all z which additionally satisfy the inequality

$$0 < \delta \leqq \operatorname{Re} z,$$

where δ is an arbitrary positive number, i.e. (1) converges uniformly in every half-plane imbedded in the right halfplane $\operatorname{Re} z > 0$.

Both parts $\operatorname{Re} I(z)$, $\operatorname{Im} I(z)$ of the mapping $I(z)$ are the parametric integrals of the form

$$\int_0^{+\infty} \exp(-xt)\cos(yt)\,\mathrm{d}t \qquad \text{and} \qquad \int_0^{+\infty} \exp(-xt)\sin(yt)\,\mathrm{d}t,$$

respectively, both converging [or uniformly converging] in the halfplane $\{x > 0\}$ [or $\{x \geqq \delta > 0\}$, where $\delta > 0$ is an arbitrary number]. These parametric integrals simultaneously represent the Laplace integral transformation with two independent real parameters $x \in (0, +\infty)$ and $y \in (-\infty, +\infty)$, the kernel of which constitutes the function $K(x, t) = \exp(-xt)$; the subjects of this transformation are the functions $f(y, t) = \cos(yt)$ and $g(y, t) = \sin(yt)$, $t \in \langle 0, +\infty)$, respectively.

To persue the properties of both images Re $I(z)$, Im $I(z)$ of this transformation clearly, let us denote both these real functions of both real variables x, y by $C(x, y)$, $S(x, y)$, respectively, i.e. for $\forall\ [x \in (0, +\infty), y \in (-\infty, +\infty)]$ let us write

$$C(x, y) = \int_0^{+\infty} \exp(-xt) \cos(yt)\, \mathrm{d}t,$$

$$S(x, y) \quad \int_0^{+\infty} \exp(-xt) \sin(yt)\, \mathrm{d}t$$

and moreover

$$E(x, y) = \int_0^{+\infty} \exp[-(x^2 + y^2)\, t]\, \mathrm{d}t = \int_0^{+\infty} \exp[(-|z|^2)\, t]\, \mathrm{d}t$$

for the two-parametric integral which, evidently, converges for $\forall\ [(x, y) \neq (0, 0)]$.

Theorem 1.

I. 1. For $\forall\ [x > 0, y \in (-\infty, +\infty)]$:

$$C(x, y) = C(x, -y) \tag{1.1}$$

$$S(x, y) = -S(x, -y) \tag{1.2}$$

$$yC(x, y) = xS(x, y) \tag{1.3}$$

2. For $\forall\ [x > 0, y > 0]$:

$$C(x, y) = S(y, x) \tag{2.1}$$

$$C(x, -y) = S(y, x) \tag{2.2}$$

$$C(x, y) = -S(y, -x) \tag{2.3}$$

3. For $\forall\ [x > 0]$:

$$C(x\ 1) = C\left(\frac{1}{x}, 1\right) \tag{3.1}$$

For $\forall\ [y \neq 0]$:

$$S(1, y) = S\left(1, \frac{1}{y}\right) \tag{3.2}$$

For $\forall\ [x > 0, \lambda > 0]$:

$$C(x, \lambda) = \frac{1}{\lambda} C\left(\frac{x}{\lambda}, 1\right) \tag{3.3}$$

For $\forall\ [\lambda > 0, y \in (-\infty, +\infty)]$:

$$S(\lambda, y) = \frac{1}{\lambda} S\left(1, \frac{y}{\lambda}\right) \tag{3.4}$$

4. For $\forall\ [(x, y) \neq (0, 0)]$:

$$E(x, y) = E(y, x) \tag{4.1}$$

For $\forall\,[x \neq 0]$:

$$E(x, x) = \frac{1}{2} E(x, 0) \tag{4.2}$$

For $\forall\,[x > 0,\, y \in (-\infty, +\infty)]$:

$$C(x, y) = xE(x, y) \tag{4.3}$$

$$S(x, y) = yE(x, y) \tag{4.4}$$

For $\forall\,[x > 0]$:

$$C(x, x) = S(x, x) = \sqrt{\frac{1}{2} E(x, x)}. \tag{4.5}$$

For $\forall\,[|x| \geqq 1]$:

$$E(x, 0) = C(1, \sqrt{x^2 - 1}) \tag{4.6}$$

For $\forall\,[|x| > 1]$:

$$E(x, 0) = S(\sqrt{x^2 + 1}, 1) \tag{4.7}$$

II. For $\forall\,[x > 0,\, y > 0]$:

1. $C(x, y) - C(y, x) = S(\sqrt{2xy}, x - y)\,[= -S(\sqrt{2xy}, y - x)]$,

whereby

$$C(x, y) - C(y, x) = \begin{cases} C(x - y, \sqrt{2xy}) & \text{if } x > y > 0 \\ 0 & \text{if } x = y > 0 \\ -C(y - x, \sqrt{2xy}) & \text{if } y > x > 0 \end{cases}$$

2. $S(x, y) - S(y, x) = S(\sqrt{2xy}, y - x)\,[= -S(\sqrt{2xy}, x - y)]$,

whereby

$$S(x, y) - S(y, x) = \begin{cases} -C(x - y, \sqrt{2xy}) & \text{if } x > y > 0 \\ 0 & \text{if } x = y > 0 \\ C(y - x, \sqrt{2xy}) & \text{if } y > x > 0 \end{cases}$$

3. $C(x, y) - S(x, y) = S(\sqrt{2xy}, x - y)\,[= -S(\sqrt{2xy}, y - x)]$,

whereby

$$C(x, y) - S(x, y) = \begin{cases} C(x - y, \sqrt{2xy}) & \text{if } x > y > 0 \\ 0 & \text{if } x = y > 0 \\ -C(y - x, \sqrt{2xy}) & \text{if } y > x > 0 \end{cases}$$

III. For $\forall\,[x > 0,\, y \in (-\infty, +\infty)]$:

1. $C^2(x, y) + S^2(x, y) = E(x, y)$

For $\forall\,[x > 0,\, y > 0]$:

2. $C^2(x, y) - S^2(x, y) = S(2xy, x^2 - y^2)\,[= -S(2xy, y^2 - x^2)]$,

whereby

$$C^2(x, y) - S^2(x, y) = \begin{cases} C(x^2 - y^2, 2xy) & \text{if } x > y > 0 \\ 0 & \text{if } x = y > 0 \\ -C(y^2 - x^2, 2xy) & \text{if } y > x > 0 \end{cases}$$

3. $2S(x, y)\, C(x, y) = C(2xy, x^2 - y^2)\, [= C(2xy, y^2 - x^2)]$,

whereby

$$2S(x, y)\, C(x, y) = \begin{cases} S(x^2 - y^2, 2xy) & \text{if } x > y > 0 \\ E(x, x) & \text{if } x = y > 0 \\ S(y^2 - x^2, 2xy) & \text{if } y > x > 0 \end{cases}$$

4. $C^2(x, y) = \frac{1}{2}[E(x, y) + S(2xy, x^2 - y^2)]$

$$\left\{= \frac{1}{2}[E(x, y) - S(2xy, y^2 - x^2)]\right\},$$

whereby

$$C^2(x, y) = \begin{cases} \frac{1}{2}[E(x, y) + C(x^2 - y^2, 2xy)] & \text{if } x > y > 0 \\ \frac{1}{2}E(x, x) & \text{if } x = y > 0 \\ \frac{1}{2}[E(x, y) - C(y^2 - x^2, 2xy)] & \text{if } y > x > 0 \end{cases}$$

5. $S^2(x, y) = \frac{1}{2}[E(x, y) - S(2xy, x^2 - y^2)]$

$$\left\{= \frac{1}{2}[E(x, y) + S(2xy, y^2 - x^2)]\right\},$$

whereby

$$S^2(x, y) = \begin{cases} \frac{1}{2}[E(x, y) - C(x^2 - y^2, 2xy)] & \text{if } x > y > 0 \\ \frac{1}{2}E(x, x) & \text{if } x = y > 0 \\ \frac{1}{2}[E(x, y) + C(y^2 - x^2, 2xy)] & \text{if } y > x > 0 \end{cases}$$

IV. For $\forall\, [x > 0, y > 0]\ \forall\, [n \in \boldsymbol{N}]$:

1. $E(x^n, y^n) = E(x^n - y^n, \sqrt{2x^n y^n})\, [= E(y^n - x^n, \sqrt{2x^n y^n})]$

2. $C(x^n, y^n) = \sqrt{\frac{x^n}{2y^n}}\, C(\sqrt{2x^n y^n}, x^n - y^n) \left[= \sqrt{\frac{x^n}{2y^n}}\, C(\sqrt{2x^n y^n}, y^n - x^n)\right],$

whereby

$$C(x^n, y^n) = \begin{cases} \sqrt{\dfrac{x^n}{2y^n}}\, S(x^n - y^n, \sqrt{2x^n y^n}) & \text{if } x > y > 0 \\ E(x^n, x^n) & \text{if } x = y > 0 \\ \sqrt{\dfrac{x^n}{2y^n}}\, S(y^n - x^n, \sqrt{2x^n y^n}) & \text{if } y > x > 0 \end{cases}$$

3. $S(x^n, y^n) = \sqrt{\dfrac{y^n}{2x^n}}\, C(\sqrt{2x^n y^n}, x^n - y^n) \left[= \sqrt{\dfrac{y^n}{2x^n}}\, C(\sqrt{2x^n y^n}, y^n - x^n) \right],$

whereby

$$S(x^n, y^n) = \begin{cases} \sqrt{\dfrac{y^n}{2x^n}}\, S(x^n - y^n, \sqrt{2x^n y^n}) & \text{if } x > y > 0 \\ E(x^n, x^n) & \text{if } x = y > 0 \\ \sqrt{\dfrac{y^n}{2x^n}}\, S(y^n - x^n, \sqrt{2x^n y^n}) & \text{if } y > x > 0 \end{cases}$$

Proof: It is fairly easy to verify the correctness of all formulas in the parts I. – IV. stated in the above Theorem by the following: both improper two-parametric integrals $C(x, y)$, $S(x, y)$ converge in the halfplane $\{x > 0,\ y \in (-\infty, +\infty)\}$, the integral $E(x, y)$ converge in the region $\boldsymbol{R}^2 - \{0\}$ and we get $C(x, y) = \dfrac{x}{x^2 + y^2}$, $S(x, y) = \dfrac{y}{x^2 + y^2}$ and $E(x, y) = \dfrac{1}{x^2 + y^2}$.

Theorem 2.

For $\forall\, [x \geqq \delta > 0,\ y \in (-\infty, +\infty)]\ \forall\, [n \in \boldsymbol{N}]$:

$$C_x^{(4n)}(x, y) - C_y^{(4n)}(x, y) = 0 \tag{1.a}$$

$$S_x^{(4n)}(x, y) - S_y^{(4n)}(x, y) = 0 \tag{1.b}$$

$$C_x^{(4n-1)}(x, y) - S_y^{(4n-1)}(x, y) = 0 \tag{1.c}$$

$$C_y^{(4n-1)}(x, y) + S_x^{(4n-1)}(x, y) = 0 \tag{1.d}$$

$$C_x^{(4n-2)}(x, y) + C_y^{(4n-2)}(x, y) = 0 \tag{1.e}$$

$$S_x^{(4n-2)}(x, y) + S_y^{(4n-2)}(x, y) = 0 \tag{1.f}$$

$$C_x^{(4n-3)}(x, y) + S_y^{(4n-3)}(x, y) = 0 \tag{1.g}$$

$$C_y^{(4n-3)}(x, y) - S_x^{(4n-3)}(x, y) = 0 \tag{1.h}$$

Remark:

The relation (1.e) or (1.f) denotes $\Delta^{2n-1}C(x, y) = 0$ [or $\Delta^{2n-1}S(x, y) = 0$], i.e. both functions $C(x, y)$, $S(x, y)$ are in every halfplane $\boldsymbol{D}\{x \geqq \delta > 0,\ y \in (-\infty, +\infty)\}$ $(2n - 1)$-harmonic (both being the solution of the *2nd* order Laplace homogeneous partial differential equation); the symbol Δ^{2n-1} denotes the $(2n - 1)$-times iteratep Laplace skalar differential operator of the $2nd$ order Δ [div grad].

Proof: Each of the continuous functions $C_x^{(n)}(x, y)$, $C_y^{(n)}(x, y)$, $S_x^{(n)}(x, y)$, $S_y^{(n)}(x, y)$, $n \in \boldsymbol{N}$, is defined by the respective improper two-parametric integral converging uniformly (and absolutely) in every halfplane of the form $\boldsymbol{D}\{x \in \langle\delta, +\infty), y \in \in (-\infty, +\infty)$, where $\delta > 0\}$. So, for $\forall [n \in \boldsymbol{N}] \forall [y \in \boldsymbol{R}] \forall [x \in \langle\delta, +\infty), \delta > 0]$:

$$| C_x^{(n)}(x, y) | = | (-1)^n \int_0^{+\infty} t^n \exp(-xt) \cos(yt) \, dt | \leqq$$

$$\leqq \int_0^{+\infty} t^n \exp(-xt) \, dt = \frac{n!}{x^{n+1}} \leqq \frac{n!}{\delta^{n+1}}.$$

Analogous it can be proved that by the same function $\varphi(n, \delta) = \dfrac{n!}{\delta^{n+1}}$ each of the remaining n^{th} derivatives

$$C_y^{(n)}(x, y) = \int_0^{+\infty} t^n \exp(-xt) \cos\left(yt + n\frac{\pi}{2}\right) dt,$$

$$S_x^{(n)}(x, y) = (-1)^n \int_0^{+\infty} t^n \exp(-xt) \sin(yt) \, dt,$$

$$S_y^{(n)}(x, y) = \int_0^{+\infty} t^n \exp(-xt) \sin\left(yt + n\frac{\pi}{2}\right) dt,$$

can be bounded for $\forall [n \in N] \forall [y \in (-\infty, +\infty)] \forall [x \in \langle\delta, +\infty), \delta > 0]$. The validity of all formulas (1.a)–(1.h) follows from the expressions of the functions $C_x^{(k)}(x, y)$, $C_y^{(k)}(x, y)$, $S_x^{(k)}(x, y)$, $S_y^{(k)}(x, y)$, $k \in \boldsymbol{N}$, in the form of the improper two-parametric integrals given above, by putting respectively $4n - i$, $n \in \boldsymbol{N}$, $i = 0,1,2,3$, in place of $k \in \boldsymbol{N}$.

Theorem 3.

For $\forall [x \geqq \delta > 0, y \in (-\infty, +\infty)] \forall [m \in \boldsymbol{N}] \wedge$

1. $[p = 0, 1, 2, \ldots, 2m]$:

$$C^{(4m)}_{x^{2(2m-p)}y^{2p}}(x, y) - C^{(4m)}_{x^{2p}y^{2(2m-p)}}(x, y) = 0, \tag{2.a}$$

$$S^{(4m)}_{x^{2(2m-p)}y^{2p}}(x, y) - S^{(4m)}_{x^{2p}y^{2(2m-p)}}(x, y) = 0. \tag{2.b}$$

2. $[p = 0, 1, 2, \ldots, 2m - 1]$:

$$C^{(4m-1)}_{x^{2(2m-p)-1}y^{2p}}(x, y) - S^{(4m-1)}_{x^{2p}y^{2(2m-p)-1}}(x, y) = 0, \tag{2.c}$$

$$C^{(4m-1)}_{x^{2p}y^{2(2m-p)-1}}(x, y) + S^{(4m-1)}_{x^{2(2m-p)-1}y^{2p}}(x, y) = 0. \tag{2.d}$$

3. $[p = 0, 1, 2, \ldots, 2m - 1]$:

$$C^{(4m-2)}_{x^{2(2m-p)-2}y^{2p}}(x, y) + C^{(4m-2)}_{x^{2p}y^{2(2m-p)-2}}(x, y) = 0, \tag{2.e}$$

$$S^{(4m-2)}_{x^{2(2m-p)-2}y^{2p}}(x, y) + S^{(4m-2)}_{x^{2p}y^{2(2m-p)-2}}(x, y) = 0. \tag{2.f}$$

4. $[p = 0, 1, 2, \ldots, 2(m-1)]$:

$$C^{(4m-3)}_{x^{2(2m-p)-3}y^{2p}}(x, y) + S^{(4m-3)}_{x^{2p}y^{2(2m-p)-3}}(x, y) = 0, \tag{2.g}$$

$$C^{(4m-3)}_{x^{2p}y^{2(2m-p)-3}}(x, y) - S^{(4m-3)}_{x^{2(2m-p)-3}y^{2p}}(x, y) = 0. \tag{2.h}$$

Proof: For $\forall\,[n \in N]\,\forall\,[0 \leqq k \leqq n]\,\forall\,[x > 0, y \in (-\infty, +\infty)]$:

$$C^{(n)}_{x^k y^{n-k}}(x, y) = (-1)^k \int_0^{+\infty} t^n \exp(-xt) \cos\left[yt + (n-k)\frac{\pi}{2}\right] \mathrm{d}t = C^{(n)}_{y^{n-k}x^k}(x, y),$$

$$S^{(n)}_{x^k y^{n-k}}(x, y) = (-1)^k \int_0^{+\infty} t^n \exp(-xt) \sin\left[yt + (n-k)\frac{\pi}{2}\right] \mathrm{d}t = S^{(n)}_{y^{n-k}x^k}(x, y).$$

(because of their continuity), both improper two-parametric integrals converging uniformly (and absolutely) in every halfplane $\boldsymbol{D}\{x \geqq \delta > 0, y \in (-\infty, +\infty\}$, where they both are bounded by the same function $\varphi(n, \delta) = \dfrac{n!}{\delta^{n+1}}$. The validity of all formulas (2.a) – (2.h) follows from the expressions of the functions $C^{(n)}_{x^k y^{n-k}}(x, y)$, $S^{(n)}_{x^k y^{n-k}}(x, y)$ by the given improper two-parametric integrals if we put respectively $4m - 1$, $m \in N$, $i = 0, 1, 2, 3$, in place of $n \in N$.

Remark:

In the formulas (2.a) – (2.h) we get for $p = 0$ the formulas (1.a) – (1.h) of the foregoing Theorem.

Theorem 4.

For $\forall\,[x \geqq \delta > 0, y \in (-\infty, +\infty)]\,\forall\,[m \in N] \wedge$

1. $[p = 1, 2, \ldots, 2m]$:

$$C^{(4m)}_{x^{2(2m-p)-1}y^{2p-1}}(x, y) + C^{(4m)}_{x^{2p-1}y^{2(2m-p)+1}}(x, y) = 0, \tag{3.a}$$

$$S^{(4m)}_{x^{2(2m-p)-1}y^{2p-1}}(x, y) + S^{(4m)}_{x^{2p-1}y^{2(2m-p)+1}}(x, y) = 0. \tag{3.b}$$

2. $[p = 1, 2, \ldots, 2m]$:

$$C^{(4m-1)}_{x^{2(2m-p)}y^{2p-1}}(x, y) + S^{(4m-1)}_{x^{2p-1}y^{2(2m-p)}}(x, y) = 0, \tag{3.c}$$

$$C^{(4m-1)}_{x^{2p-1}y^{2(2m-p)}}(x, y) - S^{(4m-1)}_{x^{2(2m-p)}y^{2p-1}}(x, y) = 0. \tag{3.d}$$

3. $[p = 1, 2, \ldots, 2m-1]$:

$$C^{(4m-2)}_{x^{2(2m-p)-1}y^{2p-1}}(x, y) - C^{(4m-2)}_{x^{2p-1}y^{2(2m-p)-1}}(x, y) = 0, \tag{3.e}$$

$$S^{(4m-2)}_{x^{2(2m-p)-1}y^{2p-1}}(x, y) - S^{(4m-2)}_{x^{2p-1}y^{2(2m-p)-1}}(x, y) = 0. \tag{3.f}$$

4. $[p = 1, 2, \ldots, 2m-1]$:

$$C^{(4m-3)}_{x^{2(2m-p)-2}y^{2p-1}}(x, y) - S^{(4m-3)}_{x^{2p-1}y^{2(2m-p)-2}}(x, y) = 0, \tag{3.g}$$

$$C^{4(m-3)}_{x^{2p-1}y^{2(2m-p)-2}}(x, y) + S^{(4m-3)}_{x^{2(2m-p)-2}y^{2p-1}}(x, y) = 0. \tag{3.h}$$

The proof is exactly the same as that of the foregoing Theorem.

Theorem 5.

For $\forall\,[x \in (0, +\infty), y \in (-\infty, +\infty)]\;\forall\,[n \in N]$:

$$C_x^{(2n)}(x, y) = (2n)! \sum_{i=0}^{n} (-1)^{n-i} \binom{2n+1}{2i+1} C^{2i+1}(x, y)\, S^{2(n-i)}(x, y), \tag{1.a}$$

$$C_x^{(2n-1)}(x, y) = (2n-1)! \sum_{i=0}^{n} (-1)^{n-i+1} \binom{2n}{2i} C^{2i}(x, y)\, S^{2(n-i)}(x, y), \tag{1.b}$$

$$S_x^{(2n)}(x, y) = (2n)! \sum_{i=0}^{n} (-1)^{n-i} \binom{2n+1}{2i} S^{2(n-i)+1}(x, y)\, C^{2i}(x, y), \tag{1.c}$$

$$S_x^{(2n-1)}(x, y) = (2n-1)! \sum_{i=0}^{n-1} (-1)^{i+1} \binom{2n}{2i+1} S^{2i+1}(x, y)\, C^{2(n-i)-1}(x, y), \tag{1.d}$$

$$C_y^{(2n)}(x, y) = (2n)! \sum_{i=0}^{n} (-1)^{n-i} \binom{2n+1}{2i} C^{2(n-i)+1}(x, y)\, S^{2i}(x, y), \tag{2.a}$$

$$C_y^{(2n-1)}(x, y) = (2n-1)! \sum_{i=0}^{n-1} (-1)^{i+1} \binom{2n}{2i+1} C^{2i+1}(x, y)\, S^{2(n-i)-1}(x, y), \tag{2.b}$$

$$S_y^{(2n)}(x, y) = (2n)! \sum_{i=0}^{n} (-1)^{n-i} \binom{2n+1}{2i+1} S^{2i+1}(x, y)\, C^{2(n-i)}(x, y), \tag{2.c}$$

$$S_y^{(2n-1)}(x, y) = (2n-1)! \sum_{i=0}^{n} (-1)^{n-i+1} \binom{2n}{2i} S^{2i}(x, y)\, C^{2(n-i)}(x, y). \tag{2.d}$$

Proof: The existence (and continuity) of all partial derivatives $C_x^{(n)}(x, y)$, $C_y^{(n)}(x, y)$, $S_x^{(n)}(x, y)$ and $S_y^{(n)}(x, y)$ for $\forall\,[n \in N]$ is ensured by the analyticity of the complex function $f(z) = z^{-1}$ in the whole halfplane $\mathrm{Re}\, z > 0$, where $\mathrm{Re}\, f(z) = C(x, y)$, $\mathrm{Im}\, f(z) = -S(x, y)$ [for both functions $C(x, y)$, $-S(x, y)$ are harmonic conjugate in the halfplane $\{x \in (0, +\infty), y \in (-\infty, +\infty)\}$]. We prove next the formula (1.a) only (the others may be proved in analogic form).

Since

$$C_x'(x, y) = S^2(x, y) - C^2(x, y)$$

and

$$S_x'(x, y) = -2C(x, y)\, S(x, y),$$

we get successively

$$\begin{aligned} C_x^{(2)}(x, y) &= 2[C^3(x, y) - 3C(x, y)\, S^2(x, y)], \\ C_x^{(4)}(x, y) &= 24[C^5(x, y) - 10C^3(x, y)\, S^2(x, y) + \\ &\quad + 5C(x, y)\, S^4(x, y)], \\ C_y^{(6)}(x, y) &= 720[C^7(x, y) - 21C^5(x, y)\, S^2(x, y) + \\ &\quad + 35C^3(x, y)\, S^4(x, y) - 7C(x, y)\, S^6(x, y)], \end{aligned}$$

etc., i.e.

$$C_x^{(2)}(x, y) = 2!\left[\binom{3}{3} C^3(x, y) - \binom{3}{1} C(x, y)\, S^2(x, y)\right],$$

$$C_x^{(4)}(x, y) = 4!\left[\binom{5}{5} C^5(x, y) - \binom{5}{3} C^3(x, y)\, S^2(x, y) + \binom{5}{1} C(x, y)\, S^4(x, y)\right],$$

$$C_x^{(6)}(x, y) = 6!\left[\binom{7}{7} C^7(x, y) - \binom{7}{5} C^5(x, y)\, S^2(x, y) + \right.$$
$$\left. + \binom{7}{3} C^3(x, y)\, S^4(x, y) - \binom{7}{1} C(x, y)\, S^6(x, y)\right]$$

etc.

We prove the validity of formula (1.a) by complete induction. Formula (1.a) obviously holds for $n = 1$; it can be seen from the induction assumption in the second step of the proof that (1.a) holds for any $n \in N$, and at the conclusion, that the formula (1.a) holds even for $n + 1$ (and in this way for $\forall\ [n \in N]$) we come by using the recurrence formula

$$C_x^{(2n+2)}(x, y) = C_x''[C_x^{(2n)}(x, y)],$$

valid for $\forall\ [n \in N]$.

Theorem 6.

For $\forall\ [x \in (0, +\infty),\ y \in (-\infty, +\infty)]\ \forall\ [n \in N]$:

$$E_x^{(2n)}(x, y) = (2n)!\, E(x, y) \sum_{i=0}^{n} (-1)^{n-i} \binom{2n+1}{2i} C^{2i}(x, y)\, S^{2(n-i)}(x, y), \tag{3.a}$$

$$E_x^{(2n-1)}(x, y) = (2n-1)!\, E(x, y) \sum_{i=0}^{n-1} (-1)^{n-i} \binom{2n}{2i+1} C^{2i+1}(x, y)\, S^{2(n-i-1)}(x, y), \tag{3.b}$$

$$E_y^{(2n)}(x, y) = (2n)!\, E(x, y) \sum_{i=0}^{n} (-1)^{n-i} \binom{2n+1}{2i} S^{2i}(x, y)\, C^{2(n-i)}(x, y), \tag{3.c}$$

$$E_y^{(2n-1)}(x, y) = (2n-1)!\, E(x, y) \sum_{i=0}^{n-1} (-1)^{n-i} \binom{2n}{2i+1} S^{2i+1}(x, y)\, C^{2(n-i-1)}(x, y). \tag{3.d}$$

Proof: In the halfplane $\{x \in (0, +\infty),\ y \in (-\infty, +\infty)\}$ the functions $E(x, y)$, $C(x, y)$ and $S(x, y)$ have (continuous) partial derivatives with respect to both variable x and y of an arbitrary order $n \in N$.

Since

$$E_x'(x, y) = -2E(x, y)\, C(x, y),$$
$$E_y'(x, y) = -2E(x, y)\, S(x, y),$$
$$C_x'(x, y) = S^2(x, y) - C^2(x, y) = -S_y'(x, y),$$
$$S_x'(x, y) = -2C(x, y)\, S(x, y) = C_y'(x, y),$$

it holds (for instance for the formula (3.b) and for the remaining formulas in an analogic form):

$$E_x^{(3)}(x, y) = -6E(x, y)\,[4C^3(x, y) - 4C(x, y)\, S^2(x, y)],$$
$$E_x^{(5)}(x, y) = -120E(x, y)\,[6C^5(x, y) - 20C^3(x, y)\, S^2(x, y) + 6C(x, y)\, S^4(x, y)],$$

etc., i.e.

$$E_x^{(3)}(x, y) = 3!\,E(x, y)\left[-\binom{4}{3}C^3(x, y) + \binom{4}{1}C(x, y)\,S^2(x, y)\right],$$

$$E_x^{(5)}(x, y) = 5!\,E(x, y)\left[-\binom{6}{5}C^5(x, y) + \binom{6}{3}C^3(x, y)\,S^2(x, y) - \right.$$

$$\left. - \binom{6}{1}C(x, y)\,S^4(x, y)\right],$$

etc.

All formulas (3.a)–(3.d) can be proved by complete induction (see an analogic proving procedure as in the proof of the foregoing Theorem).

Theorem 7.

For $\forall\,[x > 0,\ y \in (-\infty, +\infty)] \wedge$

a) $\forall\,[n \in N]$:

(1) $$\int_0^{+\infty} \exp(-xt)\cos^{2n}(yt)\,\mathrm{d}t =$$

$$= \frac{1}{2^{2n-1}}\left\{\frac{1}{2}\binom{2n}{n}C(x, 0) + \sum_{i=0}^{n-1}\binom{2n}{i}C(x, 2(n-i)\,y)\right\},$$

(2) $$\int_0^{+\infty} \exp(-xt)\sin^{2n}(yt)\,\mathrm{d}t =$$

$$= \frac{1}{2^{2n-1}}\left\{\frac{1}{2}\binom{2n}{n}C(x, 0) + \sum_{i=0}^{n-1}(-1)^{n-i}\binom{2n}{i}C(x, 2(n-i)\,y)\right\}.$$

b) $\forall\,[n \in N \cup \{0\}]$:

(3) $$\int_0^{+\infty} \exp(-xt)\cos^{2n+1}(yt)\,\mathrm{d}t = \frac{1}{2^{2n}}\sum_{i=0}^{n}\binom{2n+1}{i}C(x, [2(n-i)+1]\,y).$$

(4) $$\int_0^{+\infty} \exp(-xt)\sin^{2n+1}(yt)\,\mathrm{d}t = \frac{1}{2^{2n}}\sum_{i=0}^{n}(-1)^{n-i}\binom{2n+1}{i}S(x, [2(n-i)+1]\,y).$$

Proof: For $\forall\,[\alpha \in R] \wedge$

1. $\forall\,[n \in N]$:

$$\cos^{2n}\alpha = \frac{1}{2^{2n-1}}\left[\frac{1}{2}\binom{2n}{n} + \sum_{i=0}^{n-1}\binom{2n}{i}\cos 2(n-i)\,\alpha\right],$$

$$\sin^{2n}\alpha = \frac{1}{2^{2n-1}}\left[\frac{1}{2}\binom{2n}{n} + \sum_{i=0}^{n-1}(-1)^{n-i}\binom{2n}{i}\cos 2(n-i)\,\alpha\right].$$

2. $\forall\,[n \in N \cup \{0\}]$:

$$\cos^{2n+1}\alpha = \frac{1}{2^{2n}}\sum_{i=0}^{n}\binom{2n+1}{i}\cos\,[2(n-i)+1]\,\alpha,$$

$$\sin^{2n+1}\alpha = \frac{1}{2^{2n}}\sum_{i=0}^{n}(-1)^{n-i}\binom{2n+1}{i}\sin\,[2(n-i)+1]\,\alpha.$$

Using the first of the formulas sub 1. we prove the relation (1) (the remaining relations (2), (3) and (4) in an analogic form):

for $\forall\, [y \in (-\infty, +\infty)]\ \forall\, [t \in \langle 0, +\infty)]\ \forall\, [n \in \boldsymbol{N}]$:

$$\cos^{2n}(yt) = \frac{1}{2^{2n-1}} \left\{ \frac{1}{2} \binom{2n}{n} + \sum_{i=0}^{n-1} \binom{2n}{i} \cos 2(n-i)\, yt \right\},$$

for $\forall\, [x \in (0, +\infty)]$:

$$\exp(-xt) \cos^{2n}(yt) =$$

$$= \frac{1}{2^{2n-1}} \left\{ \frac{1}{2} \binom{2n}{n} \exp(-xt) + \sum_{i=0}^{n-1} \binom{2n}{i} \exp(-xt) \cos 2(n-i)\, yt \right\}$$

and using integration (with respect to t) on the interval $\langle 0, +\infty)$:

$$\int_0^{+\infty} \exp(-xt) \cos^{2n}(yt)\, \mathrm{d}t =$$

$$= \frac{1}{2^{2n-1}} \left\{ \frac{1}{2} \binom{2n}{n} \int_0^{+\infty} \exp(-xt)\, \mathrm{d}t + \sum_{i=0}^{n-1} \binom{2n}{i} \int_0^{+\infty} \exp(-xt) \cos 2(n-i)\, yt\, \mathrm{d}t \right\} =$$

$$= \frac{1}{2^{2n-1}} \left\{ \frac{1}{2} \binom{2n}{n} C(x, 0) + \sum_{i=0}^{n-1} \binom{2n}{i} C(x, 2(n-i)\, y) \right\}, \quad \text{q.e.d.}$$

THE IMAGE OF A CURVE IN THE INTEGRAL TRANSFORMATION $C(x, y)$ OR $S(x, y)$

Let a curve $\boldsymbol{K}$ be given by parametric equations

$$x = \varphi(\tau),$$
$$y = \psi(\tau),$$

where φ, ψ are continuous functions of the parameter τ, defined for $\forall\, [\tau \in \boldsymbol{M}_\tau \subset \boldsymbol{R}]$ so that $\varphi(\tau) > 0$, $\psi(\tau) \in (-\infty, +\infty)$.

Then for the one-parametric integrals $C = C(\tau)$, $S = S(\tau)$ holds

$$C[\varphi(\tau), \psi(\tau)] = \frac{\varphi(\tau)}{\varphi^2(\tau) + \psi^2(\tau)},$$

$$S[\varphi(\tau), \psi(\tau)] = \frac{\psi(\tau)}{\varphi^2(\tau) + \psi^2(\tau)}.$$

Specially, if also $\psi(\tau) > 0$, then

$$C[\varphi(\tau), \psi(\tau)] = S[\psi(\tau), \varphi(\tau)],$$
$$S[\varphi(\tau), \psi(\tau)] = C[\psi(\tau), \varphi(\tau)].$$

In particular, if

a) $\boldsymbol{K}\{y = f(x),\ x \in \boldsymbol{D}_f \subset (0, +\infty)\}$, then

$$C[x, f(x)] = \frac{x}{x^2 + f^2(x)},$$

$$S[x, f(x)] = \frac{f(x)}{x^2 + f^2(x)},$$

b) $\boldsymbol{K}\{x = g(y) > 0,\ y \in \boldsymbol{D}_g \subset (-\infty, +\infty)\}$, then

$$C[g(y), y] = \frac{g(y)}{g^2(y) + y^2},$$

$$S[g(y), y] = \frac{y}{g^2(y) + y^2}.$$

For instance

1. for $\forall \left[\varrho = \varrho(\tau) > 0,\ \tau \in \left(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi \right),\ k = 0, \pm 1, \ldots \right]$:

$$C[\varrho(\tau) \cos \tau, \varrho(\tau) \sin \tau] = \frac{1}{\varrho(\tau)} \cos \tau,$$

$$S[\varrho(\tau) \cos \tau, \varrho(\tau) \sin \tau] = \frac{1}{\varrho(\tau)} \sin \tau.$$

2. for $\forall\ [\varrho = \varrho(\tau) > 0,\ \tau \in (2k\pi, (2k + 1)\pi),\ k = 0, \pm 1, \ldots]$:

$$C[\varrho(\tau) \sin \tau, \varrho(\tau) \cos \tau] = \frac{1}{\varrho(\tau)} \sin \tau,$$

$$S[\varrho(\tau) \sin \tau, \varrho(\tau) \cos \tau] = \frac{1}{\varrho(\tau)} \cos \tau.$$

Corollary:

ad 1. $C^2[\varrho(\tau) \cos \tau, \varrho(\tau) \sin \tau] + S^2[\varrho(\tau) \cos \tau, \varrho(\tau) \sin \tau] = \dfrac{1}{\varrho^2(\tau)}.$

ad 2. $C^2[\varrho(\tau) \sin \tau, \varrho(\tau) \cos \tau] + S^2[\varrho(\tau) \sin \tau, \varrho(\tau) \cos \tau] = \dfrac{1}{\varrho^2(\tau)},$

so that for $\varrho = 1$:

ad 1. $C^2(\cos \tau, \sin \tau) + S^2(\cos \tau, \sin \tau) = 1,$
ad 2. $C^2(\sin \tau, \cos \tau) + S^2(\sin \tau, \cos \tau) = 1.$

SOUHRN

POZNÁMKA K JISTÉ DVOJICI PARAMETRICKÝCH INTEGRÁLŮ

VLADIMÍR VLČEK

V práci se studují vlastnosti složek komplexního zobrazení $\int_0^{+\infty} \exp(-\bar{z}t)\,\mathrm{d}t$, $\operatorname{Re} z > 0$ (inverze vzhledem k jednotkové kružnici v pravé komplexní polorovině), tj. dvojice nevlastních reálných dvouparametrických integrálů tvaru

$$\int_0^{+\infty} \exp(-xt) \cos(yt)\,\mathrm{d}t \quad \text{a} \quad \int_0^{+\infty} \exp(-xt) \sin(yt)\,\mathrm{d}t,$$

označených (pořadě) $C(x, y)$ a $S(x, y)$, navíc v souvislosti s (reálným) dvouparametrickým integrálem

$$E(x, y) = \int_0^{+\infty} \exp(-|z|^2 t)\,\mathrm{d}t, \qquad z \neq 0,$$

které současně všechny stejnoměrně konvergují v polorovině $\{x \geqq \delta > 0,\ y \in (-\infty, +\infty)\}$, kde $\delta > 0$ je libovolné číslo.

Vzhledem k tomu, že obě funkce $C(x, y)$, $S(x, y)$ představují obrazy originálů (pořadě) $f(y, t) = \cos(yt)$, $g(y, t) = \sin(yt)$ v Laplaceově integrální transformaci s jádrem $K(x, t) = \exp(-xt)$, $x > 0$, $t \in \langle 0, +\infty)$, je ve větě 1 ukázáno, nakolik tyto obrazy obrážejí známé vlastnosti svých předmětů a vzájemné vztahy mezi nimi (zvláště část III této věty).

V dalších větách (věty 2, 3 a 4) jsou uvedeny vzájemné závislosti mezi parciálními derivacemi libovolných řádů obou funkcí $C(x, y)$ a $S(x, y)$ (jejichž existence a spojitost je zaručena analytičností funkce z^{-1}, $z \neq 0$); ve větě 5 vyjádření těchto derivací (libovolného sudého, resp. lichého řádu) a navíc analogických derivací funkce $E(x, y)$ (ve větě 6) pomocí jisté lineární kombinace součinů příslušných mocnin funkcí $C(x, y)$ a $S(x, y)$.

V závěrečné části práce je ukázáno užití obou integrálních transformací $C(x, y)$ a $S(x, y)$ k transformaci rovinných křivek.

РЕЗЮМЕ

ЗАМЕТКА К ПАРЕ ПАРАМЕТРИЧЕСКИХ ИНТЕГРАЛОВ ОПРЕДЕЛЕННОГО ТИПА

ВЛАДИМИР ВЛЧЕК

В работе изучаются свойства составных частей комплексного отображения $\int_0^{+\infty} \exp(-\bar{z}t)\,\mathrm{d}t$, $\mathrm{Re}\,z > 0$ (инверзии относительно единичной окружности в правой комплексной полуплоскости), это значит пары несобственных вещественных двухпараметрических интегралов вида

$$\int_0^{+\infty} \exp(-xt)\cos(yt)\,\mathrm{d}t \qquad \text{и} \qquad \int_0^{+\infty} \exp(-xt)\sin(yt)\,\mathrm{d}t,$$

означенных (по очереди) $C(x, y)$ и $S(x, y)$, более в связности с (вещественным) двухпараметрическим интегралом

$$E(x, y) = \int_0^{+\infty} \exp(-|z|^2 t)\,\mathrm{d}t, \qquad z \neq 0,$$

вси которые одновременно равномерно сходятся в полуплоскости $\{x \geqq \delta > 0, y \in (-\infty, +\infty)\}$, где δ любое положительное число.

Ввиду того, что обе функции $C(x, y)$, $S(x, y)$ представляют собой образы оригиналов (по очереди) $f(y, t) = \cos(yt)$, $g(y, t) = \sin(yt)$ в интегральном преобразовании Лапласа с ядром $K(x, t) = \exp(-xt)$, $x > 0$, $t \in (-\infty, +\infty)$, в теореме 1 показано, насколько эти образы отображают знакомые свойства своих предметов и взаимные соотношения между ними (главным образом часть III этой теоремы).

В следующих теоремах (теоремы 2, 3 и 4) появляются взаимные зависимости между частными производными любых порядков от обеих функций $C(x, y)$ и $S(x, y)$ [существование и непрерывность которых гарантирована аналитичностью функции z^{-1}, $z \neq 0$]; в теореме 5 выражение этих производных (любого парного или не парного порядка) и также аналогических производных функции $E(x, y)$ [в теореме 6] при помощи определенной линейной комбинации произведений надлежащих степеней функций $C(x, y)$ и $S(x, y)$.

В заключительной части этой работы показано употребление обеих интегральных преобразований $C(x, y)$ и $S(x, y)$ при преобразовании плоских кривых.