1977 — ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM — TOM 53

Katedra matematické analýzy a numerické matematiky přírodovědecké fakulty
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A CRITERION FOR DETERMINING THE 2^{nd} ORDER LINEAR DIFFERENTIAL EQUATIONS POSSESSING THE CENTRAL DISPERSION WITH THE INDEX n EQUAL TO $t + \pi$

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(Received on September 7th, 1975)

In this paper we shall concern ourselves with a differential equation

$$y'' = q(t)\,y, \qquad q \in C_j^0, \qquad j = (-\infty, \infty). \tag{q}$$

Throughout, the equation (q) will be understood to be oscillatory on both sides on j (which implies every nontrivial solution of (q) with infinitely many zeros on each of the intervals $(-\infty, a)$ and (b, ∞), $a \in j$, $b \in j$).

We now recall some definitions and results adopted from the monograph [1] that will be of need below. Trivial solutions of (q) will be always from our considerations eliminated.

Let n be a positive integer, $x \in j$ and y be a solution of (q) such that $y(x) = 0$. If $\varphi_n(x)$ denotes the n-th zero of the y lying to the right of x, then φ_n is called the 1st kind central dispersion (from now on only the central dispersion) with the index n of (q). Instead of φ_1 we write φ which is called the basic dispersion (of the 1st kind) of (q).

Let (u, v) be a basis of (q) and w its Wronskian ($w = uv' - u'v$). Then $r(t) := \sqrt{u^2(t) + v^2(t)}$, $t \in j$, is called the (first) amplitude of the basis (u, v) and every function α, $\alpha \in C_j^0$, satisfying the equation $\operatorname{tg} \alpha(t) = \dfrac{u(t)}{v(t)}$ wherever $v(t) \neq 0$ is called the (first) phase of the basis (u, v). Let us say that α is a phase of (q) if there is a basis (u, v) of (q) possessing the function α as a phase. If α is a phase and r the amplitude of the basis (u, v) with the Wronskian equal to w then $\alpha'(t) = \dfrac{w}{r^2(t)}$, $t \in j$.

Let φ be the basic dispersion and α a phase of (q). Then

(i) $$\alpha \in C_j^3, \; \alpha'(t) \neq 0 \text{ on } j,$$

(ii) $$-\frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 - \alpha'^2(t) = q(t), \qquad t \in j,$$

(iii) $$\varphi_n(t) = \underbrace{\varphi \circ \varphi \circ \ldots \circ \varphi}_{n}(t), \; \varphi \in C_j^3, \; \varphi'(t) \neq 0 \text{ on } j,$$

(iv) $$\alpha \circ \varphi_n(t) = \alpha(t) + n\pi \operatorname{sgn} \alpha', \; t \in j,$$

(v) α_1 is a phase of (q) if and only if there are the numbers $a_{11}, a_{12}, a_{21}, a_{22}$, $\det(a_{ik}) \neq 0$ such that

$$\operatorname{tg} \alpha_1(t) = \frac{a_{11} \operatorname{tg} \alpha(t) + a_{12}}{a_{21} \operatorname{tg} \alpha(t) + a_{22}}$$

for all t for which both sides of the last formula are meaningful,

(vi) If α is a phase of the basis (u, v) of (q), $w = uv' - u'v$ then r, $r(t) := \sqrt{-\frac{w}{\alpha'(t)}}$, $t \in j$, is a solution of the equation

$$r'' = q(t)\, r + \frac{w^2}{r^3}.$$

Theorem. *Let $q \in C_j^1$, $q'(t) \not\equiv 0$. Let next α be a phase and φ the basic dispersion of* (q). *Then there exists a positive integer n such that $\varphi_n(t) = t + \pi$ if and only if it holds:*

$$q(t + \pi) = q(t), \qquad t \in j, \tag{1}$$

$$\int_t^{t+\pi} \frac{q'(s)}{\alpha'(s)}\, ds = 0, \qquad \int_t^{t+\pi} \frac{q'(s)}{\alpha'(s)} \sin^2 \alpha(s)\, ds = 0, \qquad t \in j. \tag{2}$$

Proof: a) Let for a positive integer n $\varphi_n(t) = t + \pi$, $t \in j$. Let α be a phase of the basis (u, v) of (q) whose Wronskian is equal to w. Following (iv) we have

$$\alpha(t + \pi) = \alpha(t) + n\pi \operatorname{sgn} \alpha', \qquad t \in j, \tag{3}$$

and according to (vi)

$$r''(t) = q(t)\, r(t) + \frac{w^2}{r^3(t)}, \qquad t \in j, \tag{4}$$

for $r(t) := \sqrt{-\frac{w}{\alpha'(t)}}$, $t \in j$.

From the formula $q(t) = -\frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 - \alpha'^2(t)$ and $\alpha'(t + \pi) = \alpha'(t)$ that follows from (3), we get (1).

On multiplying out both sides of (4) by $2r'$ we get after an elementary modifica tion the equality

$$(r'^2(t))' = q(t)(r^2(t))' - \left(\frac{w^2}{r^2(t)}\right)', \qquad t \in j$$

and integrating this from t to $t + \pi$ we have

$$r'^2(t+\pi) - r'^2(t) = \int_t^{t+\pi} q(s)(r^2(s))' \, ds - w^2\left(\frac{1}{r^2(t+\pi)} - \frac{1}{r^2(t)}\right), \qquad t \in j. \quad (5)$$

The functions qr^2, r, r' are periodic with the period π which follows from (1), (3) and from the definition of the function r. Next we have

$$\int_t^{t+\pi} q(s)(r^2(s))' \, ds = - \int_t^{t+\pi} q'(s) r^2(s) \, ds, \qquad t \in j,$$

and with respect to (5) $\left(r^2 = -\frac{w}{\alpha'}\right)$, also

$$\int_t^{t+\pi} \frac{q'(s)}{\alpha'(s)} \, ds = 0, \qquad t \in j. \tag{6}$$

Let us note that α in (6) is an arbitrary phase of (q). We will utilize this fact to the proof of (2). Let $x \neq 0$ and $\alpha_x \in C_j^0$ a function such that $\operatorname{tg} \alpha_x(t) = x^2 \operatorname{tg} \alpha(t)$ for all t for which $\operatorname{tg} \alpha(t)$ has been defined. Then α_x is a phase of (q) as follows from (vi) $\left(a_{11} = \frac{1}{a_{22}} = x,\ a_{12} = a_{21} = 0\right)$. Next we have

$$\alpha_x'(t) = \frac{x^2}{\cos^2 \alpha(t) + x^4 \sin^2 \alpha(t)} \alpha'(t), \qquad t \in j, x \neq 0.$$

Since

$$0 = \int_t^{t+\pi} \frac{q'(s)}{\alpha'(s)} \, ds = \frac{1}{x^2} \int_t^{t+\pi} \frac{q'(s)}{\alpha'(s)} (\cos^2 \alpha(s) + x^4 \sin^2 \alpha((s))) \, ds$$

for every $x \neq 0$ and hence also

$$\int_t^{t+\pi} \frac{q'(s)}{\alpha'(s)} \cos^2 \alpha(s) \, ds = -x^4 \int_t^{t+\pi} \frac{q'(s)}{\alpha'(s)} \sin^2 \alpha(s) \, ds,$$

it is necessarily

$$\int_t^{t+\pi} \frac{q'(s)}{\alpha'(s)} \sin^2 \alpha(s) \, ds = 0, \qquad \int_t^{t+\pi} \frac{q'(s)}{\alpha'(s)} \cos^2 \alpha(s) \, ds = 0, \qquad t \in j.$$

By this we have proved statement of the Theorem in one direction.

b) Let the phase α and q satisfy the assumptions (1), (2), $q \in C_j^1$ and $q' \not\equiv 0$. The function q is periodic with the period π and therefore exists (uniquely) a phase ε of the equation $y'' = -y : \alpha(t + \pi) = \varepsilon \circ \alpha(t)$ ([2], § 3.8). By the assumption $q \in C_j^1$, $q' \not\equiv 0$ thus there exists an interval (λ, μ), where $q'(t) \neq 0$. By differentiating (2) we obtain $\frac{q'(t+\pi)}{\alpha'(t+\pi)} = \frac{q'(t)}{\alpha'(t)}$, $\frac{q'(t+\pi)}{\alpha'(t+\pi)} \sin^2 \alpha(t + \pi) = \frac{q'(t)}{\alpha'(t)} \sin^2 \alpha(t)$ $(t \in j)$ and making use of (1) we get $\alpha'(t + \pi) = \alpha'(t)$ and $\sin^2 \alpha(t + \pi) = \sin^2 \alpha(t)$ for $t \in (\lambda, \mu)$. Therefore $\alpha(t + \pi) = \alpha(t) + c$, where $c(\neq 0)$ is a constant, $\operatorname{sgn} c = \operatorname{sgn} \alpha'$ and from $\sin^2(\alpha(t) + c) = \sin^2 \alpha(t)$ then follows $c = n\pi \operatorname{sgn} \alpha'$ (n is a positive integer). So, we have proved $\alpha(t + \pi) = \alpha(t) + n\pi \operatorname{sgn} \alpha'$, $t \in (\lambda, \mu)$. From the last equality and from $\alpha(t + \pi) = \varepsilon \circ \alpha(t)$ we get $\varepsilon(t) = t + n\pi \operatorname{sgn} \alpha'$ for t from the open interval with the end points $\alpha(\lambda)$, $\alpha(\mu)$. By the Theorem in [1] p. 209 there is the phase ε uniguely determined by the values of ε, ε', ε'' at a point $t_0(\in j)$. Therefore $\varepsilon(t) = t + n\pi \operatorname{sgn} \alpha'$ even for $t \in j$ and from $\alpha(t + \pi) = \alpha(t) + n\pi \operatorname{sgn} \alpha'$ and (iv) we have $\varphi_n(t) = t + \pi$. This completes the proof.

Remark 1. There is $q' \not\equiv 0$ on j in the assumptions of the Theorem. If q is a constant ($= k$), then we can easily see $\varphi_n(t) = t + \pi (t \in j)$ for a positive integer n if and only if $k = -n^2$.

Remark 2. The integral conditions (2) may be formulated in terms of q. Then these are more complicated.

Remark 3 A general form of the carrier q of (q) having the basic dispersion equal to $t + \pi$ has been found in [1] and [3].

References

[1] *Borůvka, O.*: Linear Differential Transformations of the Second Order. The English Universities Press, London, (1971).

[2] *Borůvka, O.*: Sur la périodicité de la distance des zéros intégrales de léquation differéntielle $y'' = q(t) y$. Tensor, N. S. Vol. 26 (1972), 121—128.

[3] *Neuman, F.*: Linear differential equations of the second order and their applications. Rediconti di Mat. 4 (1971), 559—617.

Shrnutí

KRITERIUM PRO URČENÍ LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC 2. ŘÁDU, KTERÉ MAJÍ CENTRÁLNÍ DISPERSI S INDEXEM n ROVNU $t + \pi$

Svatoslav Staněk

V práci jsou uvedeny nutné a postačující podmínky, aby funkce $t + \pi$ byla rovna centrální dispersi s indexem n diferenciální rovnice (q): $y'' = q(t) y$, $q \in C^0_{(-\infty, \infty)}$. Tyto podmínky jsou vyjádřeny pomocí funkce q a první fáze diferenciální rovnice (q).

Резюме

КРИТЕРИУМ ДЛЯ ОПРЕДЕЛЕНИЯ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ КОТОРЫЕ ОБЛАДАЮТ ЦЕНТРАЛЬНОЙ ДИСПЕРСИЕЙ С ИНДЕКСОМ n РОВНОЙ $t + \pi$

Сватослав Станек

В работе иследуются линейные дифференциальные уравнения второго порядка вида (q) : $y'' = q(t)y$, $q \in C^0_{(-\infty, \infty)}$. Указаны необходимые и достаточные условия при выполнении которых центральная дисперсия с индексом n дифференциального уравнения (q) ровна $t + \pi$. Эти условия представлены при помощи функции q и первой фазы дифференциального уравнения (q).