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Svatoslav Staněk

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Vedoucí katedry: Prof. RNDr. Miroslav Laitoch, CSc.

INTEGRAL PROPERTIES OF COEFFICIENTS OF THE 2nd ORDER LINEAR DIFFERENTIAL EQUATIONS HAVING THE SAME DISTRIBUTION OF ZEROS OF SOLUTIONS

SVATOSLAV STANĚK

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In [3] the author sets the problem to find sufficient and/or necessary conditions to functions q, $\bar{q}$ for differential equations (q): $y'' = q(t)\,y$, ($\bar{\mathrm{q}}$): $y'' = \bar{q}(t)\,y$ to have the same distribution of zeros of solutions, in other words, the same dispersion. This paper gives some necessary conditions expressed in an integral form. The reader is referred to [3] for the list of necessary and sufficient conditions known as yet and for the list of references.

1. Let (q) denote a differential equation

$$y'' = q(t)\,y, \qquad q \in C^0_{\mathrm{R}}, \quad \mathrm{R} = (-\infty, \infty), \tag{q}$$

oscillatory on R which means that every non-trivial solution of (q) has infinitely many zeros on every interval of the type $(-\infty, a\rangle$, $\langle b, \infty)$.

For better understanding we now introduce some definitions and results from the theory of phases and from the theory of dispersions stated in [1].

Let $x \in \mathrm{R}$ and y be a non-trivial solution of (q) and $y(x) = 0$. If $\varphi(x)$ is the first zero of y lying to the right of x, then φ is called the basic central dispersion of the 1st kind (briefly dispersion) of (q). By assumption (q) is oscillatory on R and consequently its dispersion φ is defined on the whole interval R, and the dispersion φ of (q) satisfies:

$$\varphi(t) > t, \qquad \varphi'(t) > 0 \qquad \text{for } t \in \mathrm{R},\ \varphi \in C^3_{\mathrm{R}}.$$

Let (u, v) be a basis of (q). A function $\alpha \in C^0_{\mathrm{R}}$ defined by

$$\operatorname{tg} \alpha(t) := \frac{u(t)}{v(t)} \qquad \text{for all} \qquad t \in \mathrm{R} - \{t;\ t \in \mathrm{R},\ v(t) = 0\},$$

is called a (first) phase of the basis (u, v) of (q). Suppose the function α is a (first) phase of (q) if there exists such a basis (u, v) of this differential equation possessing the function α as its phase. Each phase α of (q) satisfies:

$$\alpha \in C^3_{\mathrm{R}}, \qquad \alpha'(t) \neq 0 \qquad \text{for } t \in \mathrm{R}, \tag{1}$$

$$q(t) = -\{\alpha, t\} - \alpha'^2(t), \qquad \text{where} \qquad \{\alpha, t\} = \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} - \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2, \tag{2}$$

$$\lim_{t \to \sigma\infty} \alpha(t) = \sigma \operatorname{sgn} \alpha' \, \infty \qquad (\sigma = \pm 1).$$

Between any phase α and the dispersion φ of the same differential equation the following Abel equation holds

$$\alpha \circ \varphi(t) = \alpha(t) + \pi \operatorname{sgn} \alpha', \qquad t \in \mathrm{R}. \tag{3}$$

2. First let us prove the following

Lemma 1. *Let* (q), $(\bar{\mathrm{q}})$ *be oscillatory on* R *and* α *be a phase of* (q). *Then* (q) *and* $(\bar{\mathrm{q}})$ *have the same dispersion if and only if there exists a function g such that the differential equation* (g): $y'' = g(t)\,y$ *has the dispersion* $t + \pi$ *and*

$$\bar{q}(t) = q(t) + (1 + g \circ \alpha(t))\,\alpha'^2(t), \qquad t \in \mathrm{R}. \tag{4}$$

Proof: Lemma 1 follows directly from the theorems given in [1], pages 147 and 148.

Remark 1. If (g) has the dispersion $t + \pi$, then it follows from (3) and (2) that g is a periodic function on R with period π.

Theorem 1. Let (q), $(\bar{\mathrm{q}})$ *be oscillatory on* R *having the same dispersion,* $q \neq \bar{q}$. *Then the improper integrals*

$$\int_{-\infty}^{0} \sqrt{|\bar{q}(t) - q(t)|}\,\mathrm{d}t, \qquad \int_{0}^{\infty} \sqrt{|\bar{q}(t) - q(t)|}\,\mathrm{d}t$$

are divergent.

Proof: Let α be a phase of (q), $\operatorname{sgn} \alpha' = 1$. By assumption (q) and $(\bar{\mathrm{q}})$ have the same dispersion, which by Lemma 1 is true if and only if there exists such a function g where (g) has the dispersion $t + \pi$ and where the formula (4) holds, and consequently also

$$\sqrt{|\bar{q}(t) - q(t)|} = \alpha'(t)\sqrt{|1 + g \circ \alpha(t)|}, \qquad t \in \mathrm{R}. \tag{5}$$

We shall now show that the improper integral $\int_{-\infty}^{0} \sqrt{|\bar{q}(t) - q(t)|}\,\mathrm{d}t$ is divergent. Completely the divergence of the improper integral $\int_{-\infty}^{0} \sqrt{|\bar{q}(t) - q(t)|}\,\mathrm{d}t$ can be proved analogous.

Let a be a number. Integrating (5) from $t(\leqq a)$ to a and using the equatity

$$\int_t^a \alpha'(s)\sqrt{|1 + g\circ\alpha(s)|}\,ds = \int_{\alpha(t)}^{\alpha(a)} \sqrt{|1 + g(s)|}\,ds,$$

we obtain

$$\int_t^a \sqrt{|\bar{q}(s) - q(s)|}\,ds = \int_{\alpha(t)}^{\alpha(a)} \sqrt{|1 + g(s)|}\,ds.$$

According to Remark 1, the function g is a periodic function with period π, $\lim_{t\to-\infty}\alpha(t) = -\infty$ and therefore $\int_{-\infty}^{0} \sqrt{|\bar{q}(t) - q(t)|}\,dt$ converges if and only if $g(t) \equiv -1$. Then, of course, with respect to (4) we have $q = \bar{q}$, which contradicts our assumption $q \neq \bar{q}$. Consequently $\int_{-\infty}^{0} \sqrt{|\bar{q}(t) - q(t)|}\,dt$ is divergent.

Theorem 2. *Let* (q), ($\bar{\text{q}}$) *be oscillatory on* R *having the same dispersion,* $q \neq \bar{q}$. *Let* α *be a phase of* (q), $\sigma = \operatorname{sgn}\alpha'$.

Then

$$\int_{-\infty}^{0} \frac{\bar{q}(t) - q(t)}{\alpha'(t)}\,dt = \sigma\infty, \qquad \int_{0}^{\infty} \frac{\bar{q}(t) - q(t)}{\alpha'(t)}\,dt = \sigma\infty.$$

Proof: Let α be a phase of (q) having the same dispersion as ($\bar{\text{q}}$), $q \neq \bar{q}$. Then by Lemma 1 there exists a function g such that (g) has the dispersion $t + \pi$ and the formula (4) and consequently also the formula

$$\frac{\bar{q}(t) - q(t)}{\alpha'(t)} = (1 + g\circ\alpha(t))\,\alpha'(t), \qquad t \in \mathrm{R} \tag{6}$$

hold.

Let a be a number. Suppose first $\sigma = 1$. Integrating (6) from a to $t(\geqq a)$ and using the equality

$$\int_a^t (1 + g\circ\alpha(s))\,\alpha'(s)\,ds = \int_{\alpha(a)}^{\alpha(t)} (1 + g(s))\,ds,$$

we obtain

$$\int_a^t \frac{\bar{q}(s) - q(s)}{\alpha'(s)}\,ds = \int_{\alpha(a)}^{\alpha(t)} (1 + g(s))\,ds.$$

By Remark 1 g is a periodic function with period π and by Theorem 7.1 from [2] page 590 $\int_0^\pi (1 + g(t))\,dt \geqq 0$ whereby $\int_0^\pi (1 + g(t))\,dt = 0$ if and only if $g(t) \equiv -1$. Respecting $\lim_{t\to\infty}\alpha(t) = \infty$ we have $\int_a^\infty \frac{\bar{q}(t) - q(t)}{\alpha'(t)}\,dt$ convergent if and only if $\int_0^\pi (1 + g(t))\,dt = 0$. Then, naturally, $g(t) \equiv -1$ and we obtain from (4) $q = \bar{q}$

contradicting our assumption $q \neq \bar{q}$. Consequently $\int_0^{\pi} (1 + g(t))\, dt = k > 0$ and $\int_0^{\infty} \frac{\bar{q}(t) - q(t)}{\alpha'(t)}\, dt = \infty$.

Likewise it can be shown that $\int_{-\infty}^{0} \frac{\bar{q}(t) - q(t)}{\alpha'(t)}\, dt = \infty$.

Not let $\sigma = -1$. From the theory of phases then follows that $-\alpha$ is a phase of (q) as well. Since $\operatorname{sgn}(-\alpha)' = -\operatorname{sgn}\alpha' = 1$, it is possible in a manner completely analogous to that of the first part of the proof—only that we consider $-\alpha$ instead of α— to come to $\int_0^{\infty} \frac{\bar{q}(t) - q(t)}{-\alpha'(t)}\, dt = \infty$, $\int_{-\infty}^{0} \frac{\bar{q}(t) - q(t)}{-\alpha'(t)}\, dt = \infty$ and thus to $\int_0^{\infty} \frac{\bar{q}(t) - q(t)}{\alpha'(t)}\, dt = -\infty$, $\int_{-\infty}^{0} \frac{\bar{q}(t) - q(t)}{\alpha'(t)}\, dt = -\infty$. This completes the proof of the Theorem.

Corollary. *Let* (q) *have the dispersion* $t + \pi$, $q \neq -1$. *Then the improper integrals*

$$\int_{-\infty}^{0} \sqrt{|1 + q(t)|}\, dt, \qquad \int_0^{\infty} \sqrt{|1 + q(t)|}\, dt$$

are divergent and for every phase ε *of* (-1): $y'' = -y$

$$\int_{-\infty}^{0} \frac{1 + q(t)}{\varepsilon'(t)}\, dt = \operatorname{sgn}\varepsilon'\infty, \qquad \int_0^{\infty} \frac{1 + q(t)}{\varepsilon'(t)}\, dt = \operatorname{sgn}\varepsilon'\infty.$$

Proof: The above Corollary follows directly from Theorem 2 where now -1 and q instead of q and $\bar{q}$ is considered.

Remark 2. There can be investigated equations of the type (q) as well, where $q \in C_I^{\circ}$, $I = \langle a, \infty)$ are oscillatory on I which means that every non-trivial solution of this differential equation has infinitely many zeros on interval I. It can be shown, too, if (q), $(\bar{q})$, $q \neq \bar{q}$ have the same dispersion then $\int_a^{\infty} \sqrt{|\bar{q}(t) - q(t)|}\, dt$ diverges and if α is a phase of (q), $\sigma = \operatorname{sgn}\alpha'$, then $\int_a^{\infty} \frac{\bar{q}(t) - q(t)}{\alpha'(t)}\, dt = \sigma\infty$.

References

[1] *Borůvka, O.*: Linear Differential Transformations of the Second Order. The English Universities Press Ltd., 1971.

[2] *Neuman, F.*: Linear differential equations of the second order and their applications, Rendiconti di Mat. 4 (1971), 559—617.

[3] *Neuman, F.*: Distribution of zeros of solutions of $y'' = q(t)y$ in relation to their behaviour in large, Acta Math. Acad. Scien. Hungaricae 8 (1973), 177—185.

Shrnutí

INTEGRÁLNÍ VLASTNOSTI KOEFICIENTŮ LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC 2. ŘÁDU SE STEJNÝM ROZLOŽENÍM KOŘENŮ ŘEŠENÍ

Svatoslav Staněk

V práci jsou uvedeny dvě nutné podmínky, aby diferenciální rovnice (q): $y'' = q(t)\,y$ a ($\bar{q}$): $y'' = \bar{q}(t)\,y$ měly stejné rozložení kořenů řešení, jinými slovy, aby měly stejnou dispersi. Jedna podmínka je vyjádřena přímo pomocí funkcí q a $\bar{q}$, druhá podmínka používá ještě navíc první fázi diferenciální rovnice (q). Podmínky jsou vyjádřeny v integrálním tvaru.

Резюме

ИНТЕГРАЛЬНЫЕ СВОЙСТВА КОЭФФИЦИЕНТОВ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВТОРОГО ПОРЯДКА С ТЕМ ЖЕ РОЗЛОЖЕНИЕМ КОРНЕЙ РЕШЕНИЙ

Сватослав Станек

В работе приведены два необходимых условия при выполнении которых дифференциальные уравнения (q) : $y'' = q(t)y$ и ($\bar{q}$) : $y'' = \bar{q}(t)y$ имеют одинаковое розложение корней решений, другими словами, имеют ту же дисперсию. Первое условие выражается прямо при помощи функций q и $\bar{q}$, второе условие использует кроме того нервую фазу дифференциального уравнения (q). Условия представлены в интегральной форме.