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SEMICANONICAL MOVING FRAME OF THE HYPERSURFACE IN A UNIMODULAR 4-DIMENSIONAL AFFINE SPACE

LIBUŠE MARKOVÁ

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Let us attach a frame to each point M in $\mathbf{A}^4$ consisting of this point and of linearly independent vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ such that

$$(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4) = 1.$$

The fundamental equations of the moving frame and the structure equations (i.e. the integrability conditions) are

$$\mathrm{d}m = \omega^i\mathbf{e}_i, \qquad \mathrm{d}\mathbf{e}_i = \omega_i^k\mathbf{e}_k, \tag{1}$$
$$D\omega^i = \omega^k \wedge \omega_k^i, \qquad D\omega_i^k = \omega_i^j \wedge \omega_j^k, \qquad \omega_i^i = 0.$$

We investigate in $\mathbf{A}^4$ the surface $\mathbf{P}_3$ given by the differential equation $\omega = 0$. Let us establish the moving frame $(M, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ such that $M \in \mathbf{P}_3$ and the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form the basis of the tangential hyperplane of $\mathbf{P}_3$ in the point M. Choosing $\omega = \omega^4$, our surface is given by

$$\omega^4 = 0. \tag{2}$$

After exterior differentiation of (2) we get

$$\omega^\alpha \wedge \omega_\alpha^4 = 0, \qquad \alpha = 1, 2, 3, \tag{3}$$

which according to Cartan's lemma results in

$$\omega_\alpha^4 = R_{\alpha\beta}^4\omega^\beta, \qquad R_{\alpha\beta}^4 = R_{\beta\alpha}^4 \tag{4}$$

Let us next assume that on the surface $\mathbf{P}_3$ the system of subvarieties is given by

$$\omega^1 = \omega^2 = 0, \qquad \omega^1 = \omega^3 = 0, \qquad \omega^3 = \omega^2 = 0, \tag{5}$$

where the forms ω^1, ω^2, ω^3 are linearly independent. By affixing the moving frame to the tissue (5) such-wise that the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the tangential vectors at the point M in respect of curves of the corresponding tissue (5), we obtain

$$\delta\mathbf{e}_\alpha \parallel \mathbf{e}_\alpha, \qquad \alpha = 1, 2, 3,$$

which gives rise to following conditions

$$\pi_\alpha^4 = 0, \qquad \pi_\beta^\alpha = 0, \quad \text{for} \quad \alpha \neq \beta, \tag{6}$$

Like usually we denote by δ the differentiation so that the principal parametres of the moving frame are constants, and by π_k^i the form arising from ω_k^i through putting instead of principal parametres arbitrary constants in it. Hereupon we may write

$$\omega_\beta^\alpha = R_{\beta\gamma}^\alpha \omega^\gamma \tag{7}$$

In respect of (3) and (5) the variations of $R_{\alpha\beta}^4$ and $R_{\beta\gamma}^\alpha$ may be obtained in the following form

$$\begin{aligned} \delta R_{\alpha\gamma}^4 &= -R_{\alpha\gamma}^4 \pi_4^4 + R_{\alpha\beta}^4 \pi_\gamma^\beta + R_{\beta\gamma}^4 \pi_\alpha^\beta \\ \delta R_{\beta\gamma}^\alpha &= R_{\beta\gamma}^\alpha \pi_\beta^\beta - R_{\beta\gamma}^4 \pi_4^\alpha + R_{\beta\delta}^\alpha \pi_\gamma^\delta - R_{\beta\gamma}^\varepsilon \pi_\varepsilon^\alpha \end{aligned} \tag{8}$$

(do not sum up according to index β!)

Taking (6) into consideration and after itemizing (8) we get

$$\begin{gathered} R_{11}^4 = R_{11}^4(2\pi_1^1 - \pi_4^4); \qquad R_{22}^4 = R_{22}^4(2\pi_2^2 - \pi_4^4); \\ R_{33}^4 = R_{33}^4(2\pi_3^3 - \pi_4^4); \end{gathered} \tag{9}$$

$$\begin{aligned} R_{22}^1 &= R_{22}^1(2\pi_2^2 - \pi_1^1) - R_{22}^4\pi_4^1; \qquad & R_{33}^1 &= R_{33}^1(2\pi_3^3 - \pi_1^1) - R_{33}^4\pi_4^1; \\ R_{11}^2 &= R_{11}^2(2\pi_1^1 - \pi_2^2) - R_{11}^4\pi_4^2; \qquad & R_{33}^2 &= R_{33}^2(2\pi_3^3 - \pi_2^2) - R_{33}^4\pi_4^2; \\ R_{11}^3 &= R_{11}^3(2\pi_1^1 - \pi_3^3) - R_{11}^4\pi_4^3; \qquad & R_{22}^3 &= R_{22}^3(2\pi_2^2 - \pi_3^3) - R_{22}^4\pi_4^3; \end{aligned}$$

whence we see that the following specialization

$$\begin{gathered} R_{11}^3 = R_{22}^1 = R_{33}^2 = 0, \\ R_{11}^2 = R_{11}^4 \neq 0, \qquad R_{22}^3 = R_{22}^4 \neq 0, \qquad R_{33}^1 = R_{33}^4 \neq 0, \end{gathered} \tag{10}$$

is possible, which gives

$$\begin{gathered} \pi_4^1 = \pi_4^2 = \pi_4^3 = 0, \\ \pi_4^4 = \pi_3^3, \qquad \pi_4^4 = \pi_2^2, \qquad \pi_4^4 = \pi_1^1 \end{gathered}$$

and since it holds

$$\pi_1^1 + \pi_2^2 + \pi_3^3 + \pi_4^4 = 0,$$

the proceeding of this specialization is concluded.

Geometrical meaning of performed specializations

First let us set the first and second differential of the radius vector of the point $M \in \mathbf{P}_3$:

$$\begin{gathered} \mathrm{d}m = \omega^\alpha \mathbf{e}_\alpha, \\ \mathrm{d}^2 m = \mathbf{e}_\beta(\mathrm{d}\omega^\beta + \omega^\alpha \omega_\alpha^\beta) + \mathbf{e}_4 \omega^\alpha \omega_\alpha^4. \qquad \alpha, \beta = 1, 2, 3 \end{gathered} \tag{11}$$

From (11) we may derive that the osculating planes of the coordinate curves are determined by vectors

$$\mathbf{e}_1,\quad R^2_{11}\mathbf{e}_2 + R^3_{11}\mathbf{e}_3 + R^4_{11}\mathbf{e}_4,$$
$$\mathbf{e}_2,\quad R^1_{22}\mathbf{e}_1 + R^3_{22}\mathbf{e}_3 + R^4_{22}\mathbf{e}_4,$$
$$\mathbf{e}_3,\quad R^1_{33}\mathbf{e}_1 + R^2_{33}\mathbf{e}_2 + R^4_{33}\mathbf{e}_4.$$

The expression (10) is geometrically characterized in such a way that the osculating planes of coordinate curves are

$$\begin{aligned} &(M,\ \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_4) && \text{for the curve} && \omega^2 = \omega^3 = 0 \\ &(M,\ \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4) && \text{for the curve} && \omega^1 = \omega^3 = 0 \\ &(M,\ \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_4) && \text{for the curve} && \omega^1 = \omega^2 = 0. \end{aligned} \tag{12}$$

The plane (12_1) and the vector $\mathbf{e}_2$ (i.e. the tangent vector of the curve $\omega^1 = \omega^3 = 0$) are determining the hyperplane α. Analogous the plane (12_2) or (12_3), and the vector $\mathbf{e}_3$ or $\mathbf{e}_1$ are determining the hyperplane β or γ. The hyperplanes α, β, γ, have a common straight-line, determining the direction of the vector $\mathbf{e}_3$ in case that (10) is realized.

When we guide through the plane (12_1) or (12_2) or (12_3) hyperplane containing the direction of $\mathbf{e}_3$ or $\mathbf{e}_1$ or $\mathbf{e}_2$, we get the tripple of hyperplanes intersecting themselves in the straight-line, the direction of which is

$$\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4.$$

Let be the hyperplane

$$(X - E_\beta, \mathbf{e}_\alpha, \mathbf{e}_\gamma, \mathrm{d}\mathbf{e}_\alpha) = 0, \tag{13}$$

where $E_\beta = M - \mathbf{e}_\beta$, for the following series of values

$$\begin{aligned} &\alpha = 1, && \beta = 2, && \gamma = 3, \\ &\alpha = 2, && \beta = 3, && \gamma = 1, \\ &\alpha = 3, && \beta = 1, && \gamma = 2. \end{aligned}$$

Let us look for the point of intersection of this hyperplane and of the straight-line

$$X = M + t\mathbf{e}_4, \tag{14}$$

in the motion

$$\omega^\beta = \omega^\gamma = 0.$$

For the coordinates of this point of intersection we obtain

$$tR^\beta_{\alpha\alpha} + R^4_{\alpha\alpha} = 0.$$

The specialization (10) is then geometrically determined such that the sought point of intersection of the straight-line (14) and the hyperplane (13) is

$$X = M - \mathbf{e}_4.$$

As the specialization of the moving frame after the second step is concluded, it is clear that we cannot get other geometric objects but those, the order of which is greater than two. For such a moving frame there holds the following system of differential equations

$$
\begin{gathered}
\omega_\alpha^4 = R_{\alpha\beta}^4 \omega^\beta, \qquad R_{\alpha\beta}^4 = R_{\beta\alpha}^4, \\
\omega_\beta^\alpha = R_{\beta\gamma}^\alpha \omega^\gamma, \\
R_{11}^3 = R_{22}^1 = R_{33}^2 = 0, \qquad R_{11}^2 = R_{11}^4, \qquad R_{22}^3 = R_{22}^4, \qquad R_{33}^1 = R_{33}^4, \\
\alpha, \beta = 1, 2, 3.
\end{gathered} \tag{15}
$$

From the following exterior system

$$
\begin{gathered}
(\mathrm{d}R_{\alpha\gamma}^4 + R_{\alpha\gamma}^4 \omega_4^4 - R_{\beta\gamma}^4 \omega_\alpha^\beta - R_{\alpha\beta}^4 \omega_\gamma^\beta) \wedge \omega^\gamma = 0, \\
(\mathrm{d}R_{\beta\gamma}^\alpha + R_{\beta\delta}^4 \omega_4^\alpha - R_{\beta\gamma}^\alpha \omega_\delta^\gamma - R_{\beta\delta}^\alpha \omega^\beta + R_{\beta\delta}^\varepsilon \omega_\varepsilon^\alpha) \wedge \omega^\delta = 0, \qquad \varepsilon \neq \beta
\end{gathered}
$$

of equations (15) it can be derived that the solution of system (15) is dependent on the seven arbitrary functions of three variables.

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Shrnutí

POLOKANONICKÝ REPER NADPLOCHY V EKVIAFINNÍM ČTYŘROZMĚRNÉM PROSTORU

LIBUŠE MARKOVÁ

V článku je zkonstruován polokanonický reper nadplochy v čtyřrozměrném ekviafinním prostoru a je dána jeho geometrická charakteristika. Je-li na nadploše daná libovolná trojtkáň, pak vektory reperu $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ jsou tečnými vektory ke křivkám této trojtkáně a vektor $\mathbf{e}_4$ je ve směru průsečnice tří nadrovin, které se získají následujícím způsobem. Vezmeme oskulační rovinu jedné z křivek — rovinu $(M, \mathbf{e}_\alpha, \mathbf{e}_\beta + \mathbf{e}_4)$. Potom hledaná nadrovina prochází touto rovinou rovnoběžně s tečným vektorem druhé křivky trojtkáně $\mathbf{e}_\beta$.

Резюме

ПОЛУКАНОНИЧЕСКИЙ РЕПЕР ГИПЕРПОВЕРХНОСТИ В ЭКВИАФФИННОМ ЧЕТЫРЕХМЕРНОМ ПРОСТРАНСТВЕ

ЛИБУШЕ МАРКОВА

В статье конструируется полуканонический репер гиперповерхности в четырехмерном эквиаффинном пространстве и приводится его геометрическая характеристика. Если на поверхности дана произвольная три-ткань, то векторы репера $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ являются касательными к кривым этой три-ткани и вектор $\mathbf{e}_4$ направлен по прямой, которая является пересечением трех гиперплоскостей, которые строятся следующим образом. Возмём соприкасающуюся плоскость одной из кривых — плоскость (M, $\mathbf{e}_\alpha$, $\mathbf{e}_\beta + \mathbf{e}_4$). Потом гиперплоскость проходит через эту плоскость параллельно касательному вектору другой кривой три-ткани $\mathbf{e}_\beta$.