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# Homogeneous geodesics in a three-dimensional Lie group

Rosa Anna Marinosci

*Dedicated to Professor Oldřich Kowalski on the occasion of his 65th birthday*

*Abstract.* O. Kowalski and J. Szenthe [KS] proved that every homogeneous Riemannian manifold admits at least one homogeneous geodesic, i.e. one geodesic which is an orbit of a one-parameter group of isometries. In [KNV] the related two problems were studied and a negative answer was given to both ones: (1) Let $M = K/H$ be a homogeneous Riemannian manifold where $K$ is the largest connected group of isometries and $\dim M \geq 3$. Does $M$ always admit more than one homogeneous geodesic? (2) Suppose that $M = K/H$ admits $m = \dim M$ linearly independent homogeneous geodesics through the origin $o$. Does it admit $m$ mutually orthogonal homogeneous geodesics? In this paper the author continues this study in a three-dimensional connected Lie group $G$ equipped with a left invariant Riemannian metric and investigates the set of all homogeneous geodesics.

*Keywords:* Riemannian manifold, homogeneous space, geodesics as orbits

*Classification:* 53C20, 53C22, 53C30

## 1. Introduction

Let $(M, g)$ be a homogeneous Riemannian manifold, i.e., a connected Riemannian manifold on which the largest connected group $K$ of isometries acts transitively. Then $M$ can be interpreted as a homogeneous space $(K/H, g)$ where $H$ is the isotropy group at a fixed point $o$ of $M$. In this situation the Lie algebra $\underline{\boldsymbol{k}}$ of $K$ has an ad$(H)$-invariant direct sum decomposition (= reductive decomposition) $\underline{\boldsymbol{k}} = \boldsymbol{m} \oplus \underline{\boldsymbol{h}}$, where $\boldsymbol{m} \subset \underline{\boldsymbol{k}}$ is a linear subspace of $\underline{\boldsymbol{k}}$ and $\underline{\boldsymbol{h}}$ is the Lie algebra of $H$ ([KoNo]). In general such decomposition is not unique. The ad$(H)$-invariant subspace $\boldsymbol{m}$ can be naturally identified with the tangent space $T_o(M)$ via the projection $p : K \to K/H$.

A geodesic $\gamma(t)$ through the origin $o$ of $M = K/H$ is called *homogeneous* if it is an orbit of a one-parameter subgroup of $K$, that is

$$\gamma(t) = \exp(tZ)(o), \quad t \in R, \tag{1}$$

where $Z$ is a nonzero vector of $\underline{\boldsymbol{k}}$.

A homogeneous Riemannian manifold is called a g.o. space if all geodesics are homogeneous with respect to the largest connected group of isometries. All naturally reductive spaces ([KoNo]) are g.o. spaces, but the converse does not hold. In [Kp] A. Kaplan proved the existence of g.o. spaces that are in no way naturally reductive; the examples of A.Kaplan are generalized Heisenberg groups with two-dimensional center. O. Kowalski and L. Vanhecke made an explicit classification of all naturally reductive spaces up to dimension five ([KPV]). In [KV] they gave a classification of all g.o. spaces, which are in no way naturally reductive, up to dimension six.

About the existence of homogeneous geodesics in a general homogeneous Riemannian manifold, we have, at first, a result due to V.V. Kajzer who proved that a Lie group endowed with a left-invariant metric admits at least one homogeneous geodesic ([Ka]). More recently O. Kowalski and J. Szenthe extended this result to all homogeneous Riemannian manifolds ([KS]).

Hence the study of the set of all homogeneous geodesics of a general homogeneous Riemannian manifold arises as a natural problem. In [KNV] O. Kowalski, S. Nikčević and Z. Vlášek started this study by considering the following problems:

(1) Let $M = K/H$ be a homogeneous Riemannian manifold where $K$ is the largest connected group of isometries and $\dim M \geq 3$. Does $M$ always admit more than one homogeneous geodesic?

(2) Suppose that $M = K/H$ admits $m = \dim M$ linearly independent homogeneous geodesics through the origin $o$. Does it admit $m$ mutually orthogonal homogeneous geodesics?

They gave a negative answer to both ones by considering the case of a *three-dimensional non-unimodular Lie group* $G = K/H$ endowed with a left-invariant Riemannian metric $g$ and with distinct Ricci principal curvatures.

In the present paper the author extends the study for the case of a three-dimensional non-unimodular Lie group whose principal Ricci curvatures are not all distinct. Then she studies homogeneous geodesics in a three-dimensional unimodular Lie group. The main results are resumed in Theorems 3.1 and 3.2.

## 2. Preliminaries concerning homogeneous geodesics in homogeneous Riemannian manifolds

As in the introduction, let $(M = K/H, g)$ be a homogeneous Riemannian manifold with a fixed origin $o$. Let $\underline{\boldsymbol{k}}$ and $\underline{\boldsymbol{h}}$ be the Lie algebras of $K$ and $H$ respectively and let

$$\underline{\boldsymbol{k}} = \boldsymbol{m} \oplus \underline{\boldsymbol{h}} \tag{2}$$

be a reductive decomposition; the canonical projection $p : K \to K/H$ induces an isomorphism between the subspace $\boldsymbol{m}$ and the tangent space $T_o(M)$ and consequently the scalar product $g_o$ on $T_o(M)$ induces a scalar product $B$ on $\boldsymbol{m}$ which is $\mathrm{Ad}(H)$-invariant.

**Definition 2.1.** *A nonzero vector $Z \in \underline{\boldsymbol{k}}$ is called a geodesic vector if the curve* (1) *is a geodesic.*

In the next section we shall use the following lemma which gives a characterization of geodesic vectors ([G], [KN], [KV]).

**Lemma 2.2.** *A nonzero vector $Z \in \underline{\boldsymbol{k}}$ is a geodesic vector if and only if*

$$B([Z,W]_{\boldsymbol{m}}, Z_{\boldsymbol{m}}) = 0 \tag{3}$$

*for all $W \in \boldsymbol{m}$ (the subscript $\boldsymbol{m}$ denotes the projection into $\boldsymbol{m}$).*

Now if we want to find all homogeneous geodesics of the homogeneous Riemannian manifold $(M = K/H, g)$, we have to calculate all geodesic vectors of the Lie algebra $\underline{\boldsymbol{k}}$. For this purpose we shall use the technique presented in [KNV]: at first we calculate the connected component $K$ of the full isometry group $I(M)$, or at least the corresponding Lie algebra $\underline{\boldsymbol{k}}$. Then we find a decomposition of the form (2) and look for the geodesic vectors in the form

$$Z = \sum_{i=1}^{r} x_i e_i + \sum_{j=1}^{s} a_j A_j, \tag{4}$$

where $\{e_i\}_{i=1,2,\dots,r}$ is a convenient basis of $\boldsymbol{m}$ and $\{A_j\}_{j=1,2,\dots,s}$ is a basis of $\underline{\boldsymbol{h}}$.

The condition (3) produces a system of $r$ quadratic equations for the variables $x_i$ and $a_j$ when we write condition (3) taking $W = e_i$, $i = 1, 2, \dots, r$. Then we see for which values of $x_1, x_2, \dots, x_r$ and $a_1, a_2, \dots, a_s$ this system is satisfied. The geodesic vectors correspond to those solutions for which $x_1, x_2, \dots, x_r$ are not all equal to zero.

A finite family of geodesics through the origin $o$ is said to be linearly independent if the corresponding initial tangent vectors are linearly independent. Then the following proposition holds ([KNV]):

**Proposition 2.3.** *A finite family $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$ of homogeneous geodesics through $o \in M$ is orthogonal or linearly independent, respectively, if the $\boldsymbol{m}$-components of the corresponding geodesic vectors are orthogonal, or linearly independent, respectively.*

## 3. Homogeneous geodesics in three-dimensional Lie groups

Let $G$ be a three-dimensional connected Lie group endowed with a left invariant metric $g$ and let $\nabla$ be its Riemannian connection with Ricci tensor $\varrho$. Write $G$ in the form $G = K/H$, where $K$ is the largest connected group of isometries of $(G, g)$ and consider the reductive decomposition

$$\underline{\boldsymbol{k}} = \boldsymbol{m} \oplus \underline{\boldsymbol{h}}\ , \tag{5}$$

where $\underline{\boldsymbol{k}}$ is the Lie algebra of the Lie group $K$, $\underline{\boldsymbol{h}}$ is the Lie algebra of the Lie group $H$ and $\boldsymbol{m}$ is a real vector space isomorphic to the tangent space $T_{\mathrm{e}}(G)$ (e = identity of $G$) or equivalently to the Lie algebra $\underline{\boldsymbol{g}}$ of $G$. Because $G = K/H$ itself is a group space, it admits a canonical connection $\widetilde{\nabla}$ with the torsion tensor $\widetilde{T}(X,Y) = -[X,Y]$ and curvature tensor $\widetilde{R} = 0$ ([KoNo]). The tensor $D = \nabla - \widetilde{\nabla}$ satisfies ([Kw]):

$$(6) \qquad 2g(D_Y X, Z) = g(\widetilde{T}(X,Y),Z) + g(\widetilde{T}(X,Z),Y) + g(\widetilde{T}(Y,Z),X).$$

The Lie algebra $\underline{\boldsymbol{h}}$ consists of all skew-symmetric endomorphisms $A$ of $\underline{\boldsymbol{g}}$ such that $A(g) = 0$, $A(R) = 0$, $A(D^n R) = 0$ for $n = 1, 2, \ldots$, where $R$ is the Riemannian curvature (note that since $G$ is three-dimensional $A(R) = 0$ is equivalent to $A(\varrho) = 0$ and $A(D^n R) = 0$ is equivalent to $A(D^n \varrho) = 0$).

The algebra $\underline{\boldsymbol{h}}$ contains as its subalgebra the Lie algebra $\underline{\boldsymbol{d}}$ of all skew-symmetric derivations of $\underline{\boldsymbol{g}}$.

We want to describe all geodesic vectors of $(G, g)$, which are contained in $\underline{\boldsymbol{k}}$ according to the definition. For this purpose we shall distinguish two cases:

(I) $G$ is an unimodular Lie group;

(II) $G$ is a non-unimodular Lie group.

**CASE (I): $G$ unimodular.**

According to a result due to J. Milnor (see [M, Theorem 4.3, p. 305]) there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\underline{\boldsymbol{g}}$ such that

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2.$$

The basis $\{e_1, e_2, e_3\}$ diagonalizes the Ricci tensor $\varrho$ and the principal Ricci curvatures are given by

$$\varrho_1 = 2\mu_2\mu_3, \quad \varrho_2 = 2\mu_1\mu_3, \quad \varrho_3 = 2\mu_1\mu_2,$$

where

$$\mu_i = (1/2)(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i,$$

for each $i = 1, 2, 3$.

We note, by using Lemma 2.2, that $e_1, e_2, e_3$ are geodesic vectors.

Now we must calculate the Lie algebra $\underline{\boldsymbol{h}}$ of $H$.

A skew-symmetric endomorphism $A : \underline{\boldsymbol{g}} \to \underline{\boldsymbol{g}}$ of the Lie algebra $\underline{\boldsymbol{g}}$ is of the form:

$$A(e_1) = ae_2 + be_3, \quad A(e_2) = -ae_1 + ce_3, \quad A(e_3) = -be_1 - ce_2.$$

The condition $A(\varrho) = 0$ gives in particular

$$\varrho(A(e_i), e_j) + \varrho(e_i, A(e_j)) = 0$$

for each $i, j = 1, 2, 3$; so we get

$$a(\varrho_2 - \varrho_1) = 0, \quad b(\varrho_1 - \varrho_3) = 0, \quad c(\varrho_2 - \varrho_3) = 0. \tag{7}$$

From now on, let us suppose that *all $\lambda_i$ are distinct*. Then *all $\mu_i$ are distinct*, as well.

If $\mu_1\mu_2\mu_3 \neq 0$, then $\varrho_1\varrho_2\varrho_3 \neq 0$ and $\varrho_i$ are all distinct; consequently from (7) we get $a = b = c = 0$ and $\underline{\boldsymbol{h}} = \{\mathbf{0}\}$, hence all geodesic vectors are contained in the Lie algebra $\underline{\boldsymbol{g}}$.

Suppose $\mu_1\mu_2\mu_3 = 0$; without loss of generality let $\mu_1 = 0$.

Condition $\mu_1 = 0$ implies $\varrho_2 = \varrho_3 = 0$; we note that $\varrho_1 \neq 0$ because $\lambda_i$ are all distinct, consequently from (7) we get $a = b = 0$ and the endomorphism $A$ is of the form

$$A(e_1) = 0, \quad A(e_2) = ce_3, \quad A(e_3) = -ce_2.$$

The endomorphim $A$ is not a derivation of the Lie algebra $\underline{\boldsymbol{g}}$ in general; in fact condition $A([e_1, e_2]) = [A(e_1), e_2] + [e_1, A(e_2)]$ is satisfied if and only if $c = 0$. Now each endomorphism $A \in \underline{\boldsymbol{h}}$ satisfies the condition $A(D\varrho) = 0$. An easy calculation gives for $D$ the following expression:

$$\begin{aligned}
&D_{e_1}e_1 = 0, \quad D_{e_1}e_2 = -\lambda_3 e_3, \quad D_{e_1}e_3 = \lambda_2 e_2,\\
&D_{e_2}e_1 = 0, \quad D_{e_2}e_2 = 0, \qquad\quad\; D_{e_2}e_3 = -\lambda_2 e_1,\\
&D_{e_3}e_1 = 0, \quad D_{e_3}e_2 = \lambda_3 e_1, \qquad D_{e_3}e_3 = 0.
\end{aligned}$$

$D\varrho$ and $A(D\varrho)$ are defined by

$$D\varrho(X, Y, Z) = -\varrho(D_X Z, Y) - \varrho(X, D_Y Z),$$
$$A(D\varrho)(X, Y; Z) = -D\varrho(A(X), Y, Z) - D\varrho(X, A(Y), Z) - D\varrho(X, Y, A(Z));$$

in particular we see that $A(D\varrho)(e_1, e_2; e_2) = 0$ implies $c = 0$; consequently the Lie algebra $\underline{\boldsymbol{h}}$ is equal to zero, hence all geodesic vectors can be found in $\underline{\boldsymbol{g}}$.

By using Lemma 2.2 a vector $X = x_1e_1 + x_2e_2 + x_3e_3$ of $\underline{\boldsymbol{g}}$ is a geodesic vector if and only if $g([x_1e_1 + x_2e_2 + x_3e_3, e_i], x_1e_1 + x_2e_2 + x_3e_3) = 0$ for each $i = 1, 2, 3$.

So we get:

$$\begin{aligned}
(-\lambda_3 + \lambda_2)x_3x_2 &= 0,\\
(\lambda_3 - \lambda_1)x_3x_1 &= 0,\\
(\lambda_1 - \lambda_2)x_1x_2 &= 0
\end{aligned}$$

or equivalently (because $\lambda_i$ are all distinct):

$$\begin{aligned}
x_2x_3 &= 0,\\
x_1x_3 &= 0,\\
x_1x_2 &= 0.
\end{aligned}$$

We conclude that all geodesic vectors $X$ are those from the set $\mathrm{span}\{e_1\} \cup \mathrm{span}\{e_2\} \cup \mathrm{span}\{e_3\}$.

The above study allows us to announce the following theorem:

**Theorem 3.1.** *In a three-dimensional, connected and unimodular Lie group $G$ endowed with a left invariant metric $g$, there always exist three mutually orthogonal homogeneous geodesics through each point. Moreover, if all $\lambda_i$ are distinct, there are no other homogeneous geodesics.*

**Remark.** If $\lambda_i$ are not all distinct, we can suppose $\lambda_2 = \lambda_3$ without loss of generality. If $\lambda_1 = \lambda_2 = \lambda_3$ we have $\varrho_1 = \varrho_2 = \varrho_3$ and the space is Riemannian symmetric. Suppose now $\lambda_1 \neq \lambda_2 = \lambda_3$, then $\mu_1 \neq \mu_2 = \mu_3$. If $\mu_2 = \mu_3 = 0$ then $\varrho_1 = \varrho_2 = \varrho_3 = 0$ and the space is Riemannian symmetric. Thus suppose $\mu_2 = \mu_3 \neq 0$, then we have $\varrho_1 \neq \varrho_2 = \varrho_3$ and from (7) $a = b = 0$. The endomorphism $A$ takes on the form

$$A(e_1) = 0, \quad A(e_2) = ce_3, \quad A(e_3) = -ce_2.$$

In this case, the endomorphism $A$ is a derivation of the Lie algebra $\underline{\boldsymbol{g}}$. We see that the algebras $\underline{\boldsymbol{h}}$ and $\underline{\boldsymbol{d}}$ coincide, and $\underline{\boldsymbol{h}}$ is spanned by the endomorphim

$$A'(e_1) = 0, \quad A'(e_2) = e_3, \quad A'(e_3) = -e_2.$$

A vector $X = x_1e_1 + x_2e_2 + x_3e_3 + cA'$ is a geodesic vector if and only if $g([x_1e_1 + x_2e_2 + x_3e_3 + cA', e_i], x_1e_1 + x_2e_2 + x_3e_3) = 0$ for each $i = 1, 2, 3$.

So we get:

$$\begin{aligned} (-\lambda_3 + \lambda_2)x_3x_2 &= 0, \\ (\lambda_3 - \lambda_1)x_3x_1 + cx_3 &= 0, \\ (\lambda_1 - \lambda_2)x_1x_2 - cx_2 &= 0. \end{aligned}$$

Since $\lambda_2 = \lambda_3$ we see from the above system that for every choice of $x_1,\ x_2,\ x_3$ the vector $X = x_1e_1 + x_2e_2 + x_3e_3 + (\lambda_1 - \lambda_2)x_1A'$ is a geodesic vector, hence $G = K/H$ is a geodesic orbit space or equivalently a naturally reductive space (because in dimension three the two classes coincide) ([KPV]).

**CASE (II): $G$ non-unimodular.**

According to a result due to J. Milnor (see [M, Lemma 4.10, p. 309]) there exists an orthogonal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\underline{\boldsymbol{g}}$ such that

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = \gamma e_2 + \delta e_3,$$

where $\alpha$, $\beta$, $\gamma$, $\delta$ are real numbers such that $\alpha + \delta = 2$ and $\alpha\gamma + \beta\delta = 0$.

The above basis diagonalizes the Ricci form and the principal Ricci curvatures are given by

$$\begin{aligned} \varrho_1 &= -\alpha^2 - \delta^2 - (\beta + \gamma)^2, \\ \varrho_2 &= -\alpha(\alpha + \delta) + (\gamma^2 - \beta^2)/2, \\ \varrho_3 &= -\delta(\alpha + \delta) + (\beta^2 - \gamma^2)/2. \end{aligned}$$

Putting
$$\alpha = 1+\xi, \quad \delta = 1-\xi, \quad \beta = (1+\xi)\eta, \quad \gamma = -(1-\xi)\eta,$$
the principal curvatures take the form
$$\begin{aligned} \varrho_1 &= -2(1+\xi^2(1+\eta^2)),\\ \varrho_2 &= -2(1+\xi(1+\eta^2)),\\ \varrho_3 &= -2(1-\xi(1+\eta^2)). \end{aligned}$$
We note, by using Lemma 2.2, that $e_1$ is a geodesic vector.

A skew-symmetric endomorphism $A : \underline{\boldsymbol{g}} \to \underline{\boldsymbol{g}}$ of the Lie algebra $\underline{\boldsymbol{g}}$ is of the form:
$$A(e_1) = ae_2 + be_3, \quad A(e_2) = -ae_1 + ce_3, \quad A(e_3) = -be_1 - ce_2.$$
The condition $A(\varrho) = 0$ gives in particular
$$\varrho(A(e_i), e_j) + \varrho(e_i, A(e_j)) = 0$$
for each $i, j = 1, 2, 3$; so we get
$$a(\varrho_2 - \varrho_1) = 0, \quad b(\varrho_1 - \varrho_3) = 0, \quad c(\varrho_2 - \varrho_3) = 0. \tag{8}$$

The case $\varrho_1$, $\varrho_2$, $\varrho_3$ all distinct has been studied in [KNV] by O. Kowalski, S. Nikčević and Z. Vlášek. They proved the following theorem:

**Theorem A.** *Let $\alpha$, $\beta$, $\gamma$, $\delta$ be such that all Ricci principal curvatures are distinct. Denote $D = (\beta+\gamma)^2 - 4\alpha\delta$. Then up to a parametrization, the space $(G, g)$ admits*

(a) *just one homogeneous geodesic through a point, if $D < 0$,*

(b) *just two homogeneous geodesics through a point, if $D = 0$; they are mutually orthogonal,*

(c) *just three homogeneous geodesics through a point, if $D > 0$; they are linearly independent but never mutually orthogonal.*

We remark that the case $\varrho_2 = \varrho_3 \neq \varrho_1$ does not happen (in fact $\varrho_2 = \varrho_3 \Leftrightarrow \xi(1+\eta_2) = 0 \Leftrightarrow \xi = 0 \Leftrightarrow \varrho_1 = \varrho_2 = \varrho_3$).

Suppose $\varrho_1 = \varrho_2 \neq \varrho_3$. In this case we have $\xi = 1$ and the Ricci curvatures assume the form:
$$\begin{aligned} \varrho_1 &= -2(2+\eta^2),\\ \varrho_2 &= -2(2+\eta^2),\\ \varrho_3 &= -2\eta^2. \end{aligned}$$

From (8) we get $b = c = 0$, so the endomorphism $A$ takes on the form:

$$A(e_1) = ae_2, \quad A(e_2) = -ae_1, \quad A(e_3) = 0.$$

Now $A$ is not (in general) a derivation of the Lie algebra $\underline{\boldsymbol{g}}$, in fact we have

$$\begin{gathered} A([e_1, e_2]) = [A(e_1), e_2] + [e_1, A(e_2)] \Leftrightarrow \\ A(\alpha e_2 + \beta e_3) = [ae_2, e_2] + [e_1, -ae_1] \Leftrightarrow \\ \alpha a e_1 = 0 \Leftrightarrow \\ \alpha a = 0 \Leftrightarrow a = 0 \end{gathered}$$

because $\alpha = \xi + 1 = 2$.

We must check for which values of "$a$" the endomorphism $A$ satisfies the condition $A(D\varrho) = 0$. An easy calculation gives for the tensor $D$ the following expression

$$\begin{array}{lll} D_{e_1}e_1 = 0, & D_{e_1}e_2 = -2e_2 - \eta e_3, & D_{e_1}e_3 = -e_2, \\ D_{e_2}e_1 = \eta e_3, & D_{e_2}e_2 = 0, & D_{e_2}e_3 = -\eta e_1, \\ D_{e_3}e_1 = -\eta e_2, & D_{e_3}e_2 = \eta e_1, & D_{e_3}e_3 = 0. \end{array}$$

Note that $A(D\varrho)(e_1, e_2, e_1) = 0$ implies $a = 0$; in fact

$$\begin{aligned} 0 &= A(D\varrho)(e_1, e_2, e_1) \\ &= -(D\varrho)(Ae_1, e_2, e_1) - (D\varrho)(e_1, Ae_2, e_1) - (D\varrho)(e_1, e_2, Ae_1) \\ &= -(D\varrho)(ae_2, e_2, e_1) - (D\varrho)(e_1, -ae_1, e_1) - (D\varrho)(e_1, e_2, ae_2) \\ &= \varrho(D_{ae_2}e_1, e_2) + \varrho(ae_2, D_{e_2}e_1) + \varrho(D_{e_1}ae_2, e_2) + \varrho(e_1, D_{e_2}ae_2) \\ &= -a2\varrho_2 \Leftrightarrow a = 0 \quad (\text{because } \varrho_2 = -2(2 + \eta^2) \neq 0). \end{aligned}$$

We conclude that $\underline{\boldsymbol{h}} = \{0\}$ and all geodesic vectors are contained in $\underline{\boldsymbol{g}}$. A vector $X = x_1e_1 + x_2e_2 + x_3e_3$ of $\underline{\boldsymbol{g}}$ is a geodesic vector if and only if $g([x_1e_1 + x_2e_2 + x_3e_3, e_i], x_1e_1 + x_2e_2 + x_3e_3) = 0$ for each $i = 1, 2, 3$. This condition leads to the system of equations

$$x_2(x_2 + \eta x_3) = 0, \quad x_1(x_2 + \eta x_3) = 0.$$

So, a vector $X$ of $\underline{\boldsymbol{g}}$ is a geodesic vector if and only if:

- $X \in \text{span}(e_1, e_3)$ for $\eta = 0$.
- $X \in \text{span}(e_1) \cup \text{span}(e_3) \cup \text{span}(\eta e_2 - e_3)$ for $\eta \neq 0$.

Making an analogous study for the case $\varrho_1 = \varrho_3 \neq \varrho_2$ we obtain the following system of equations:

$$x_3(\eta x_2 - x_3) = 0, \quad x_1(x_3 - \eta x_2) = 0.$$

So, a vector $X$ of $\underline{\boldsymbol{g}}$ is a geodesic vector if and only if

- $X \in \operatorname{span}(e_1, e_2)$ for $\eta = 0$.
- $X \in \operatorname{span}(e_1) \cup \operatorname{span}(e_2) \cup \operatorname{span}(e_2 + \eta e_3)$ for $\eta \neq 0$.

As a consequence we can state the following theorem:

**Theorem 3.2.** *Let $G$ be a three-dimensional connected non-unimodular Lie group endowed with a left invariant metric $g$ and with two distinct principal curvatures. If $\eta \neq 0$, then there exist always three linearly independent homogeneous geodesics through each point which are never mutually orthogonal. Moreover, there are no other homogeneous geodesics. If $\eta = 0$, then the geodesic vectors form a two-dimensional subspace of the Lie algebra $\underline{\boldsymbol{g}}$ of $G$, i.e., there are infinitely many homogeneous geodesics through each point but every three of them are linearly dependent.*

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