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Locally compact linearly Lindelöf spaces

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Abstract. There is a locally compact Hausdorff space which is linearly Lindelöf and not Lindelöf. This answers a question of Arhangel'skii and Buzyakova.

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This note is devoted to the proof of:

Theorem 1. *There is a compact Hausdorff space X and a point p in X such that:*

- (1) $\chi(p, X) > \omega$;
- (2) for all regular $\kappa > \omega$, no κ -sequence of points distinct from p converges to p .

As usual, $\chi(p, X)$, the *character* of p in X , is the least size of a local base at p . Regarding (2), if $\vec{q} = \langle q_\alpha : \alpha < \kappa \rangle$ is a κ -sequence, we say $\vec{q} \rightarrow p$ iff whenever U is a neighborhood of p , $\exists \alpha \forall \beta > \alpha [q_\beta \in U]$. Then, (2) asserts that $\vec{q} \not\rightarrow p$ whenever $\kappa > \omega$ is regular and all the $q_\alpha \neq p$. Observe that if $\chi(p, X) = \omega$, then (2) holds trivially.

Theorem 1 answers Question 1 of Arhangel'skii and Buzyakova [1]. They point out that given such an X, p , the space $X \setminus \{p\}$ is linearly Lindelöf (by (2)), not Lindelöf (by (1)), and locally compact.

Note that in any compact Hausdorff space X , if the point x is not isolated, then there is a sequence of type $\text{cf}(\chi(x, X))$ converging to x . Thus, the X, p in Theorem 1 must satisfy $\text{cf}(\chi(p, X)) = \omega$. In our example, $\chi(p, X)$ will be \beth_ω .

Our X will be constructed as an inverse limit. We begin by reviewing some basic terminology:

Definition 2. *An inverse system is a sequence $\langle X_n, \pi_n^{n+1} : n \in \omega \rangle$, where each X_n is a compact Hausdorff space, and each π_n^{n+1} is a continuous map from X_{n+1} onto X_n .*

Such an inverse systems yields a compact Hausdorff space,

$$X_\omega = \lim_n X_n = \{x \in \prod_n X_n : \forall n [x_n = \pi_n^{n+1}(x_{n+1})]\}.$$

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It also yields the obvious maps $\pi_m^\omega : X_\omega \rightarrow X_m$ for $m < \omega$ and $\pi_m^n : X_n \rightarrow X_m$ for $m \leq n < \omega$.

Lemma 3. *Suppose that $\langle X_n, \pi_n^{n+1} : n \in \omega \rangle$, is an inverse system and $p = \langle p_n : n \in \omega \rangle \in X = X_\omega$ satisfies:*

- (A) *each p_n is a weak P_{\beth_n} -point in X_n ;*
- (B) *each $\chi(p_n, X_n) \leq \beth_{n+1}$;*
- (C) *each $(\pi_0^n)^{-1}\{p_0\}$ is nowhere dense in X_n .*

Then X, p satisfies Theorem 1 with $\chi(p, X) = \beth_\omega$.

As usual, $y \in Y$ is a weak P_κ -point iff y is not in the closure of any subset of $Y \setminus \{y\}$ of size less than κ , and y is a P_κ -point iff the intersection of fewer than κ neighborhoods of y is always a neighborhood of y . In a Hausdorff space, every P_κ -point is a weak P_κ -point, but note that in (A), the p_n cannot all be P_{\beth_n} -points, as that would contradict (C). Note that (C) cannot be omitted; it is easy to construct an example satisfying (A) and (B) where each X_n is a LOTS and each π_n^{n+1} collapses an interval around p_{n+1} to the point p_n ; then $\chi(p, X) = \omega$.

PROOF OF LEMMA 3: First, note that one local base at any $x \in X$ consists of all the $(\pi_n^\omega)^{-1}(U)$ such that $n \in \omega$ and U is an open neighborhood of x_n in X_n . It follows that:

- (α) $\chi(p, X_\omega) \leq \sup_n \chi(p_n, X_n) = \beth_\omega$;
- (β) $(\pi_0^\omega)^{-1}\{p_0\}$ is nowhere dense in X_ω .

Now, to verify (2) of Theorem 1, assume that $\vec{q} = \langle q_\alpha : \alpha < \kappa \rangle \rightarrow p$, where $\kappa > \omega$ is regular and all the $q_\alpha \neq p$. The definition of $\vec{q} \rightarrow p$ implies that $\kappa \leq \chi(p, X)$, so fix m with $\kappa < \beth_m$. Now, $q_\alpha \neq p$ implies that $\pi_n^\omega(q_\alpha) \neq p_n = \pi_n^\omega(p)$ for some n , so we can fix $n \geq m$ and an $S \subseteq \kappa$ with $|S| = \kappa$ and $\pi_n^\omega(q_\alpha) \neq p_n$ for all $\alpha \in S$. But then $p_n \in \text{cl}\{\pi_n^\omega(q_\alpha) : \alpha \in S\}$, contradicting (A).

In view of (α), to prove that $\chi(p, X) = \beth_\omega$, it is sufficient to fix $m < \omega$ and prove that $\chi(p, X) \geq \beth_m$. Suppose that \mathcal{B} were a local base at p in X with $|\mathcal{B}| < \beth_m$. Let $F = (\pi_m^\omega)^{-1}\{p_m\}$. By (β), F is nowhere dense in X , so for each $U \in \mathcal{B}$, we can choose $y_U \in U \setminus F$. Then $p \in \text{cl}\{y_U : U \in \mathcal{B}\}$, so $p_m = \pi_m^\omega(p) \in \text{cl}\{\pi_m^\omega(y_U) : U \in \mathcal{B}\}$, contradicting (A). \square

We now need to find an inverse system satisfying the hypotheses of Lemma 3. X_n will be $\beta \beth_n$. In general, $\beta\kappa$ denotes the Čech compactification of a discrete κ ; equivalently, $\beta\kappa$ is the space of ultrafilters on κ ; thus, the remainder, $\kappa^* = \beta\kappa \setminus \kappa$, is the space of non-principal ultrafilters on κ .

The p_n will be good ultrafilters. Following Keisler [5], an ultrafilter x on κ is good (i.e., κ^+ -good) iff given $A_s \in x$ for $s \in [\kappa]^{<\omega}$, there are $B_\alpha \in x$ for $\alpha < \kappa$ such that $\bigcap_{\alpha \in s} B_\alpha \subseteq A_s$ for all non-empty $s \in [\kappa]^{<\omega}$. For every infinite κ , there is a non-principal $x \in \beta\kappa$ such that x is a good ultrafilter (Keisler [5] under GCH and Kunen [7] in ZFC; see also Chang and Keisler [3, Theorem 6.1.4]). The following folklore result about such ultrafilters is proved in [2] and [4]:

Lemma 4. *If x is a good ultrafilter on κ , then x is a weak P_κ -point in $\beta\kappa$.*

Thus, fixing $p_n \in \beta\mathfrak{I}_n$ to be good will handle (A) of Lemma 3, but to get $p = \langle p_n : n \in \omega \rangle$ to really define a point in $X = X_\omega$, we need to choose the $\pi_n^{n+1} : \beta\mathfrak{I}_{n+1} \rightarrow \beta\mathfrak{I}_n$ such that each $p_n = \pi_n^{n+1}(p_{n+1})$. In fact, π_n^{n+1} will be $\beta(\Pi_n^{n+1})$, where $\Pi_n^{n+1} : \mathfrak{I}_{n+1} \rightarrow \mathfrak{I}_n$. Here, as usual, if $f : P \rightarrow Q$, where P, Q are Tychonov spaces, then $\beta f : \beta P \rightarrow \beta Q$ denotes its Čech extension. In the special case of discrete P, Q , where $x \in \beta P$ is an ultrafilter on P , $(\beta f)(x) \in \beta Q$ is the induced measure, $\{B \subseteq Q : f^{-1}(B) \in x\}$. Now, showing that appropriate $\Pi_n^{n+1} : \mathfrak{I}_{n+1} \rightarrow \mathfrak{I}_n$ can be chosen requires a digression:

Definition 5. *An ultrafilter x on κ is regular iff there are $E_\alpha \in x$ for $\alpha < \kappa$ such that $\bigcap_n E_{\alpha_n} = \emptyset$ whenever the α_n (for $n \in \omega$) are distinct.*

Clearly, such x are countably incomplete. Moreover,

Lemma 6. *If x is a countably incomplete good ultrafilter on κ , then x is regular.*

This is Exercise 6.1.3 of [3]; a proof is contained in the proof of Lemma 2.1 of Keisler [6]. The proof of universality of regular ultrapowers ([3, Theorem 4.3.12]) is easily modified to produce:

Lemma 7. *Suppose that $\kappa \geq 2^\lambda$ and y is any ultrafilter on λ . Let x be a regular ultrafilter on κ . Then there is an $f : \kappa \rightarrow \lambda$ such that $(\beta f)(x) = y$.*

PROOF: Since $\kappa \geq 2^\lambda$, we may list the elements of y (possibly with repetitions) as $\{B_\alpha : \alpha < \kappa\}$. Let the $E_\alpha \subseteq \kappa$ be as in Definition 5. Choose $g : \kappa \rightarrow \lambda$ such that $g(\xi)$ is some element of $\bigcap\{B_\alpha : \xi \in E_\alpha\}$ (observe that this is a finite intersection). Then $(\beta g)(x) = y$ because each $g^{-1}(B_\alpha) \supseteq E_\alpha \in x$. This g may fail to be onto, but we may now fix a set $A \in x$ with $|\kappa \setminus A| = \kappa$, and choose $f : \kappa \rightarrow \lambda$ such that $f \upharpoonright A = g \upharpoonright A$, so that $(\beta f)(x) = (\beta g)(x) = y$. □

PROOF OF THEOREM 1: We obtain the situation of Lemma 3. Fix $X_n = \beta\mathfrak{I}_n$, and fix $p_n \in \beta\mathfrak{I}_n$ to be good and non-principal (and hence countably incomplete). Applying Lemmas 6 and 7, fix $\Pi_n^{n+1} : \mathfrak{I}_{n+1} \rightarrow \mathfrak{I}_n$ so that setting $\pi_n^{n+1} = \beta(\Pi_n^{n+1})$ yields $p_n = \pi_n^{n+1}(p_{n+1})$. Then (A) follows by Lemma 4, and (B) is clear, since there is a base for the space X_n of size $2^{\mathfrak{I}_n} = \mathfrak{I}_{n+1}$. Finally, (C) holds because $(\pi_0^n)^{-1}\{p_0\} \subseteq (\mathfrak{I}_n)^*$, which is nowhere dense in $\beta\mathfrak{I}_n$. □

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