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Abstract. It is shown that divisible effect algebras are in one-to-one correspondence with unit intervals in partially ordered rational vector spaces.

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1. Introduction

Effect algebras as partial algebraic structures with a partially defined operation $\oplus$ and constants 0 and 1 have been introduced as an abstraction of the Hilbert-space effects, i.e., self-adjoint operators between 0 and I on a Hilbert space ([10]). Hilbert-space effects play an important role in the foundations of quantum mechanics and measurement theory ([17], [5]). An equivalent algebraic structure, so-called difference posets (or D-posets) with the partial operation $\ominus$ have been introduced in [16].

From the structural point of view, effect algebras are a generalization of boolean algebras, MV-algebras, orthomodular lattices, orthomodular posets, orthoalgebras. For relations among these structures and some other related structures see, e.g., [8]. Besides of quantum mechanics, they found applications in mathematical logic ([7], [6]), fuzzy probability theory ([4]), K-theory of C*- algebras ([18], [19]).

A very important subclass of effect algebras are so-called interval effect algebras, which are representable as intervals of the positive cone in a partially ordered abelian group. Examples of this kind are, e.g., Hilbert-space effects, MV-algebras, effect algebras with ordering set of states ([2]), effect algebras with the Riesz decomposition property ([20]), convex effect algebras ([14], [15]). We note that a general characterization of interval effect algebras among effect algebras is an open problem.

In the present paper, we prove that divisible effect algebras are interval effect algebras. In the proof, "word method" is used, following [1], [21], [20]. This method is compared with the method used for convex effect algebras in [14]. Completions in order-unit norm and their relations to state spaces are studied. We note that the special case of divisible MV-algebras has been treated in [9].

2. Effect algebras and interval effect algebras

Let $(E;\oplus,0,1)$ be an effect algebra, i.e., $\oplus$ is a partially defined binary operation and $0,1$ are constants $(0\neq 1)$, such that the following axioms are satisfied for any a,b,c in E:

(E1) $a\oplus b=b\oplus a$ (commutativity)
(E2) $(a\oplus b)\oplus c=a\oplus(b\oplus c)$ (associativity)
(E3) For every $a\in E$ there is a unique $b\in E$ such that $a\oplus b=1$ (supplementation)
(E4) $a\oplus 1$ is defined iff $a=0$ (zero-one law).

We denote the element b from (E3) by a', and call it the **orthosupplement** of a. The equalities in (E1) and (E2) are to be understood in such a way that if one side exists, then the other exists and they are equal. Owing to (E2), we need not write brackets in expressions like $a\oplus b\oplus c$ or $a_1\oplus a_2\oplus\cdots\oplus a_n$, the latter being defined recurrently.

Define, for $n\in\mathbb{N}$, na as follows: $1a=a$, $na=(n-1)a\oplus a$ if $(n-1)a$ and $(n-1)a\oplus a$ are defined. If there is a greatest $n\in\mathbb{N}$ such that na is defined, then this n is called the **isotropic index** of a, denoted by $\iota(a)$. If na is defined for all $n\in\mathbb{N}$, we put $\iota(a)=\infty$. Define also $0a:=0$ for every $a\in E$.

Let us write $a\perp b$ iff $a\oplus b$ is defined. Then $\perp$ is a binary relation on E, the domain of $\oplus$. Let us also define the binary relation $\leq$ by $a\leq b$ iff there is $c\in E$ such that $a\oplus c=b$. It turns out that the element c is unique, and we shall write $c=b\ominus a$ iff $a\oplus c=b$. Then $\leq$ is a partial order on E. Moreover, $a\perp b$ iff $a\leq b'$. From this it follows that $a\perp b$ and $a_1\leq a$, $b_1\leq b$ imply $a_1\perp b_1$.

In every effect algebra E, there hold the following **cancellation property**: $a\oplus c\leq b\oplus c$ implies $a\leq b$, and **positivity property**: $a\oplus b=0$ implies $a=0=b$.

Let E,F be effect algebras. A mapping $h:E\to F$ is a **morphism** if $a\perp b$ implies $h(a)\perp h(b)$ and $h(a\oplus b)=h(a)\oplus h(b)$, and $h(1)=1$. Then clearly, $h(a')=h(a)'$, $h(b)\ominus h(a)=h(b\ominus a)$ whenever $a\leq b$, and $a\leq b$ implies $h(a)\leq h(b)$. A morphism is a **monomorphism** if $a\perp b$ if and only if $h(a)\perp h(b)$. A surjective monomorphism is an isomorphism.

Let (G,G^+) be a partially ordered abelian group (additively written) with positive cone, and choose an element $a\in G^+$. Consider the interval $[0,a]\subseteq G^+$. Define a partial operation $\oplus$ on $[0,a]$ as follows: $x\perp y$ if $x+y\leq a$, and then $x\oplus y=x+y$. It is easy to check that with this operation $\oplus$ and with a as a unit element, $[0,a]$ is an effect algebra. A very important class of effect algebras, so called interval effect algebras, arise this way.

An effect algebra E is an **interval effect algebra** if there is a partially ordered abelian group (G,G^+), an element $a\in G^+$, and an isomorphism $h:E\to[0,a]$.

Example 1. The interval $[0,1]$ of the real line $\mathbb{R}$ is an interval effect algebra. More generally, let $[0,1]^X$ be the set of all functions from a set X to the unit interval $[0,1]$. As an interval of $\mathbb{R}^X$, it is an interval effect algebra. Notice that the above examples are also examples of MV-algebras.

Example 2. Let H be a Hilbert space. Consider the group of all bounded self-adjoint operators $\mathcal{B}_s(H)$ on H. The interval $\mathcal{E}(H) := [\theta, I]$, where θ is the zero and I is the identity operator, is an interval effect algebra. Elements of $\mathcal{E}(H)$ are called *Hilbert space effects.*

3. Divisible effect algebras

We say that an effect algebra E is **divisible** if for each $a \in E$ and each $n \in \mathbb{N}$ there is a unique $x \in E$ such that $a = nx$. We shall write $x = (1/n)a$.

Observe that if $m \leq n$, then $a = n(1/n)a = m(1/n)a \oplus (n-m)(1/n)a$, hence $m(1/n)a$ exists. We shall write $(m/n)a = m((1/n)a)$ whenever $m/n \leq 1$. In the next lemma, we collect some simple properties of divisible effect algebras.

Lemma 1. *Let $(E; \oplus, 0, 1)$ be a divisible effect algebra.*

(i) *If $m, n \geq 1$ and $a \in E$, then*

$$\frac{1}{n}(\frac{1}{m}a) = \frac{1}{mn}a.$$

(ii) *If $m, n \geq 2$ and $a \in E$, then $(1/m)a \perp (1/n)a$ and*

$$\frac{1}{m}a \oplus \frac{1}{n}a = (m+n)(\frac{1}{mn}a) = \frac{m+n}{mn}a.$$

(iii) *If $a, b \in E$ and $a \perp b$ then for any $n \in \mathbb{N}$, $(1/n)a \perp (1/n)b$ and*

$$\frac{1}{n}a \oplus \frac{1}{n}b = \frac{1}{n}(a \oplus b).$$

(iv) *If $a \leq b$, then for any $n \in \mathbb{N}$,*

$$\frac{1}{n}a \leq \frac{1}{n}b.$$

(v) *If $m \leq n$, then for any $a \in E$,*

$$\frac{1}{n}a \leq \frac{1}{m}a.$$

(vi) *If na is defined for $n \in \mathbb{N}$, $a \in E$, and $m \geq n$, then*

$$\frac{1}{m}(na) = n(\frac{1}{m}a) = \frac{n}{m}a.$$

(vii) *If na is defined for some $n \in \mathbb{N}$, then for any $m \in \mathbb{N}$, $n((1/m)a)$ is defined, and*
$$n(\frac{1}{m}a) = \frac{1}{m}(na).$$

(viii) *If for some $n \in \mathbb{N}$ and $a, b \in E$, $(1/n)a = (1/n)b$, then $a = b$.*

(ix) *If for some $m, n \in \mathbb{N}$ and $0 \neq a \in E$, $(1/m)a = (1/n)a$, then $m = n$.*

(x) *If $m, n \geq 2$, then for any $a, b \in E$, $(1/n)a \perp (1/m)b$.*

Proof: (i) By the definition of divisibility we obtain
$$mn(\frac{1}{n}(\frac{1}{m}a)) = m[n(\frac{1}{n}(\frac{1}{m}a))] = m(\frac{1}{m}a) = a,$$
which entails the desired result.

(ii) We have $(1/m)a = n((1/n)(1/m)a) = n((1/nm)a)$ by (i). Similarly $(1/n)a = m((1/nm)a)$. Since $m + n \leq nm$, $(n + m)((1/mn)a)$ exists, and
$$(n+m)\frac{1}{nm}a = n\frac{1}{nm}a \oplus m\frac{1}{nm}a = \frac{1}{m}a \oplus \frac{1}{n}a.$$

(iii) From $a = n((1/n)a)$, $b = n((1/n)b)$ we obtain
$$a \oplus b = n(\frac{1}{n}a) \oplus n(\frac{1}{n}b) = n(\frac{1}{n}a \oplus \frac{1}{n}b)$$
by commutativity and associativity. Hence $(1/n)(a \oplus b) = (1/n)a \oplus (1/n)b$.

(iv) Find $c \in E$ such that $a \oplus c = b$. Then by (iii), $(1/n)a \oplus (1/n)c = (1/n)b$, hence $(1/n)a \leq (1/n)b$.

(v) By (i), $(1/n)a = m((1/mn)a)$, $(1/m)a = n((1/nm)a)$, and $m \leq n$ implies $m((1/nm)a) \leq n((1/nm)a)$, hence $(1/n)a \leq (1/m)a$.

(vi) We have $na = n(m(1/m)a) = mn((1/m)a)$, hence $n((1/m)a) = (1/m)(na)$.

(vii) Since $(1/m)a \leq a$, $n((1/m)a)$ is defined. Moreover, $m(n(1/m)a) = nm((1/m)a) = na$, hence $n((1/m)a) = (1/m)(na)$.

(viii) By definition, $a = n((1/n)a) = n((1/n)b) = b$.

(ix) From $(1/n)a = (1/m)a$ it follows that $a = n((1/n)a) = n((1/m)a) = m((1/m)a)$. It follows by cancellation that $|m - n|((1/m)a) = 0$. Since by (viii) $(1/m)a \neq 0$, it follows by positivity that $m = n$.

(x) If $n, m \geq 2$, then $(1/n)1 \perp (1/m)1$ by (ii). Since by (iv), $(1/n)a \leq (1/n)1$ and $(1/m)b \leq (1/m)1$, we have $(1/n)a \perp (1/m)b$. □

Example 3. Observe that a necessary condition for an effect algebra to be divisible is the following: If na and nb exist and $na \leq nb$, then $a \leq b$. Let us consider an effect algebra $\{0, a, b, 1\}$ with the partial operation $\oplus$ such that $0 \oplus x = x$, $x \in \{0, a, b, 1\}$, $a \oplus a = 1 = b \oplus b$ (so-called diamond). Then obviously, $(1/2)1$ is not unique, and so the diamond is not divisible. According to [11], the diamond is an interval effect algebra; the interval $[(0,0),(2,0)]$ in the lexicographic product $\mathbb{Z} \times_{lex} \mathbb{Z}_2$, where $\mathbb{Z}$ is the additive group of integers with the standard positive cone, and $\mathbb{Z}_2$ is the additive group of integers modulo 2 ordered by the trivial cone $\{0\}$.

4. Embedding into a group

In this section, we use the "word technique" developed by Baer [1] and Wyler [21], and introduced to the context of effect algebras by Ravindran [20]. Starting with a divisible effect algebra E, we will find a partially ordered abelian group $G(E)$ with an order unit u such that E is isomorphic (as effect algebras) to the interval $[0, u]$ in $G(E)^+$.

Definition 1. Let $(E; \oplus, 0, 1)$ be an effect algebra.

(i) A sequence $A = (a_1, a_2, \ldots, a_n)$, $n \geq 1$ of finite length with entries from E is called a **word** in E. We denote by $W(E)$ the set of all words. We define an **addition** $+$ in $W(E)$ by the concatenation, that is

$$+ : W(E) \times W(E) \to W(E)$$
$$((a_1, \ldots, a_m), (b_1, \ldots, b_n)) \mapsto (a_1, \ldots, a_m, b_1, \ldots, b_n).$$

(ii) We call two words A and B **directly similar**, written $A \sim B$, if one is of the form $(a_1, \ldots, a_n)$, $n \geq 2$, and the other one is of the form $(a_1, \ldots, a_p \oplus a_{p+1}, \ldots, a_n)$, $1 \leq p \leq n-1$.

We call two words A and B **similar**, written $A \simeq B$ if there are words $A_0, A_1, \ldots, A_k$, $k \geq 0$, such that $A = A_0 \sim A_1 \sim \cdots \sim A_k = B$. In such a case we say that A and B are **connected by a chain of length** k.

We set for $a_1, \ldots, a_n \in E$, $n \geq 1$,

$$[a_1, a_2, \ldots, a_n] := \{A \in W(E) : A \simeq (a_1, \ldots, a_n)\}$$

and

$$C(E) := \{[a_1, \ldots, a_n] : a_1, \ldots, a_n \in E, n \geq 1\}.$$

The following lemma has been proved, in the context of effect algebras, by Ravindran [20]. We note that part (ii) has been proved by Baer [1] in a more general context.

Lemma 2. *Let $(E; \oplus, 0, 1)$ be an effect algebra.*

(i) *Similarity in $W(E)$ is an equivalence relation compatible with $+$. If we denote the induced operation again by $+$, $(C(E); +)$ is a semigroup with the neutral element $[0]$.*

(ii) *For any $a_1, a_2, \dots, a_n, b \in E$, $n \geq 1$, $(a_1, \dots, a_n) \simeq (b)$ if and only if $a_1 \oplus \dots \oplus a_n$ exists and equals b.*

Proof: (i) $\simeq$ is by construction an equivalence relation. From $A_1 \simeq A$, $B_1 \simeq B$ it follows $A_1 + B_1 \simeq A + B$, so $+$ is definable in $C(E)$. As $W(E)$ is associative, so is $C(E)$. It has $[0]$ as neutral element, since $[a_1, \dots, a_n] + [0] = [a_1, \dots, a_n, 0] = [a_1, \dots, a_n]$.

(ii) If for a word $(x_1, \dots, x_n)$ the sum of its entries $x_1 \oplus \dots \oplus x_n$ exists, the same is true for any word directly similar to $(x_1, \dots, x_n)$. So the "only if" part follows by induction on the length of the connecting chain. The "if" part is trivial. $\square$

Lemma 3. *Let $(E; \oplus, 0, 1)$ be a divisible effect algebra.*

(i) *For any $a, b \in E$,*

$$(a, b) \simeq (b, a).$$

(ii) *For any word $A = (a_1, \dots, a_n)$ and $k \in \mathbb{N}$, define*

$$\frac{1}{k} A := (\frac{1}{k} a_1, \dots, \frac{1}{k} a_n).$$

For any $A, B \in W(E)$ and any $k \in \mathbb{N}$,

$$A \simeq B \ \text{ iff } \ \frac{1}{k} A \simeq \frac{1}{k} B.$$

Proof: (i) Using divisibility, Lemma 1(x), and commutativity of E, we obtain

$$\begin{aligned}(a, b) &\sim (\frac{1}{2}a, \frac{1}{2}a, b) \sim (\frac{1}{2}a, \frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}b) \\ &\sim (\frac{1}{2}a, \frac{1}{2}a \oplus \frac{1}{2}b, \frac{1}{2}b) = (\frac{1}{2}a, \frac{1}{2}b \oplus \frac{1}{2}a, \frac{1}{2}b) \\ &\sim (\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}a, \frac{1}{2}b) \dots \sim (b, a).\end{aligned}$$

(ii) If $A \sim B$, then owing to $(1/k)(a \oplus b) = (1/k)a \oplus (1/k)b$ we have $(1/k)A \sim (1/k)B$. The proof then goes by induction with respect to the length of the connecting line.

Conversely, $(1/k)A \simeq (1/k)B$ implies $k((1/k)A) \simeq k((1/k)B)$ by the definition of $+$, and using (i) we arrive at $A \simeq B$. $\square$

Lemma 4. *Let $(E;\oplus,0,1)$ be a divisible effect algebra. Then $C(E)$ satisfies the following.*

(i) *Let $\bar{a},\bar{b}\in C(E)$. From $\bar{a}+\bar{b}=[0]$ it follows $\bar{a}=\bar{b}=[0]$.*
(ii) *Let $\bar{a},\bar{b}\in C(E)$. Then $\bar{a}+\bar{b}=\bar{b}+\bar{a}$.*
(iii) *Let $\bar{a},\bar{b},\bar{c}\in C(E)$. If $\bar{a}+\bar{c}=\bar{b}+\bar{c}$ then $\bar{a}=\bar{b}$.*

Consequently, $(C(E);+)$ is an abelian semigroup with $[0]$ as neutral element.

Proof: (i) Let $\bar{a}=[a_1,\dots,a_n]$, $\bar{b}=[b_1,\dots,b_m]$. Then

$$\bar{a}+\bar{b}=[a_1,\dots,a_n,b_1,\dots,b_m].$$

From $(a_1,\dots,a_n,b_1\dots,b_m)\simeq(0)$ it follows by (ii) of Lemma 2 that $a_1\oplus\cdots\oplus a_n\oplus b_1\oplus\cdots\oplus b_m=0$, and by positivity in E, $a_i=b_j=0$, $1\le i\le n$, $1\le j\le m$. Hence $\bar{a}=\bar{b}=[0]$.

(ii) Follows from Lemma 3 by induction.

(iii) It suffices to prove it for $\bar{c}=[c]$, for lengths $n>1$ the cancellation can be done subsequently.

Assume that $(a_1,\dots,a_m,c)\simeq(b_1,\dots,b_n,c)$. Without any loss of generality we may assume that $m\le n$. By Lemma 1(x),

$$\frac{1}{n+1}a_1\oplus\cdots\oplus\frac{1}{n+1}a_m\oplus\frac{1}{n+1}c\quad\text{and}\quad\frac{1}{n+1}b_1\oplus\cdots\oplus\frac{1}{n+1}b_n\oplus\frac{1}{n+1}c$$

exist, and by Lemma 3(ii),

$$\begin{aligned}\frac{1}{n+1}(a_1,\dots,a_m,c)&\simeq\frac{1}{n+1}a_1\oplus\cdots\oplus\frac{1}{n+1}a_m\oplus\frac{1}{n+1}c\\&\simeq\frac{1}{n+1}(b_1,\dots,b_n,c)\\&\simeq\frac{1}{n+1}b_1\oplus\cdots\oplus\frac{1}{n+1}b_n\oplus\frac{1}{n+1}c.\end{aligned}$$

By Lemma 2(ii),

$$\frac{1}{n+1}b_1\oplus\cdots\oplus\frac{1}{n+1}b_n\oplus\frac{1}{n+1}c=\frac{1}{n+1}a_1\oplus\cdots\oplus\frac{1}{n+1}a_m\oplus\frac{1}{n+1}c.$$

By cancellativity in E, $(1/n+1)b_1\oplus\cdots\oplus(1/n+1)b_n=(1/n+1)a_1\oplus\cdots\oplus(1/n+1)a_m$. Hence $(1/n+1)(a_1,\dots,a_m)\simeq(1/n+1)(b_1,\dots,b_n)$, and by Lemma 3(ii), $(a_1,\dots,a_m)\simeq(b_1\dots,b_n)$.

The rest of the proof follows by Lemma 2. □

Define on $C(E)\times C(E)$ a binary operation $+$ by

$$(\bar{a},\bar{b})+(\bar{c},\bar{d})=(\bar{a}+\bar{c},\bar{b}+\bar{d})$$

for $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in C(E)$.

Define on $C(E) \times C(E)$ a binary relation $\approx$ by

$$(\bar{a}, \bar{b}) \approx (\bar{c}, \bar{d}) \ \Leftrightarrow \ \bar{a} + \bar{d} = \bar{b} + \bar{c}$$

and put

$$\langle \bar{a}, \bar{b} \rangle := \{(\bar{c}, \bar{d}) : (\bar{c}, \bar{d}) \approx (\bar{a}, \bar{b})\},$$

$$G(E) := \{\langle \bar{a}, \bar{b} \rangle : \bar{a}, \bar{b} \in C(E)\},$$

$$G(E)^+ := \{\langle \bar{a}, [0] \rangle : \bar{a} \in C(E)\},$$

$$\epsilon : C(E) \to G(E),\ \bar{a} \mapsto \langle \bar{a}, [0] \rangle.$$

Clearly, $([0], [0]) \approx (\bar{a}, \bar{a})$ for arbitrary $\bar{a} \in C(E)$, and $\theta := \langle [0], [0] \rangle$ is the neutral element with respect to $+$.

Lemma 5. *Let $(E; \oplus, 0, 1)$ be a divisible effect algebra. Then the relation $\approx$ on $C(E) \times C(E)$ is an equivalence relation that is compatible with $+$. If $+$ is the derived operation and $G(E)^+$ is the positive cone, then $(G(E); +, \leq)$ is an abelian ordered group. Moreover, $G(E) = G(E)^+ - G(E)^+$. The mapping $\epsilon : C(E) \to G(E)$, $\bar{a} \mapsto \langle \bar{a}, [0] \rangle$ establishes a semigroup isomorphism between $C(E)$ and $G(E)^+$.*

Proof: The proof follows from Lemma 4 and, e.g., [12, Theorem II.4]. $\square$

Theorem 1. *Let $(E; \oplus, 0, 1)$ be a divisible effect algebra. Then $\pi : E \to G(E)$, $a \mapsto \langle [a], [0] \rangle$ determines an isomorphism between $(E; +, 0, 1)$ and the interval $[\langle [0], [0] \rangle, \langle [1], [0] \rangle]$ of $G(E)^+$, where $\langle [1], [0] \rangle$ is an order unit of $G(E)$. In particular, E is an interval effect algebra.*

Proof: First we prove that π is injective. Assume that $\langle [a], [0] \rangle = \langle [b], [0] \rangle$. By injectivity of ϵ from Lemma 5, and by Lemma 2(ii), this yields $a = b$.

By the surjectivity of ϵ, any positive element of $G(E)$ is of the form $\langle \bar{a}, [0] \rangle$ for some $\bar{a} \in C(E)$. Now $\langle \bar{a}, [0] \rangle \leq \langle [1], [0] \rangle$ means for some other positive element $\langle \bar{b}, [0] \rangle$ with $\bar{b} \in C(E)$ that $\langle \bar{a}, [0] \rangle + \langle \bar{b}, [0] \rangle = \langle 1, [0] \rangle$. But then $\bar{a} + \bar{b} = [1]$, since by Lemma 5, ϵ is a semigroup morphism. It follows by Lemma 2(ii) that $\bar{a} = [a]$, $\bar{b} = [b]$ for some $a, b \in E$, hence $\langle \bar{a}, [0] \rangle = \pi(a)$. This proves that π is surjective.

It remains to prove that π preserves $+$. Observe that for $a, b \in E$, $a \oplus b$ exists in E and equals c iff $(a, b) \sim c$ iff $[c] = [a] + [b]$ iff $\langle [c], [0] \rangle = \langle [a], [0] \rangle + \langle [b], [0] \rangle$ iff $\pi(c) = \pi(a) + \pi(b)$. Here we also used the fact that ϵ from Lemma 5 is a semigroup isomorphism.

It remains to show that $\langle [1], [0] \rangle$ is an order unit of $G(E)$. Since $G(E) = G(E)^+ - G(E)^+$ and $G(E)^+$ is isomorphic with $C(E)$, it is sufficient to show that any $\bar{a} = [a_1, \dots, a_n] \in C(E)$ lies below a multiple of $[1]$. By Lemma 3(i), we conclude that $\underbrace{[1,1,\dots,1]}_{n-\text{times}} = [a_1, a_1', \dots, a_n, a_n'] = [a_1, \dots, a_n] + \bar{b}$ for some $\bar{b} \in C(E)$, hence $\bar{a} \le \underbrace{[1] + \ldots + [1]}_{n-\text{times}}$. □

Recall that a map $f : E \to K$, where E is an effect algebra and K is any abelian group, is called a K-valued measure if $f(a \oplus b) = f(a) + f(b)$ whenever $a \oplus b$ is defined in E. The following theorem has been proved in [2].

Theorem 2. *Let E be an interval effect algebra. Then there exists a unique (up to isomorphism) partially ordered abelian group G with generating cone G^+ (in the sense that $G = G^+ - G^+$) and an element $u \in G^+$ such that the following conditions are satisfied:*

(i) *E is isomorphic to the interval effect algebra $G^+[0,u]$,*
(ii) *$G^+[0,u]$ generates G^+ (in the sense that every element in G^+ is a finite sum of elements of $G[0,u]$),*
(iii) *every K-valued measure $f : E \to K$ can be extended uniquely to a group homomorphism $f^* : G \to K$.*

The group G from the above theorem is called a **universal group** (or a **a unigroup**) for E. Note that u is an order-unit for G.

Theorem 3. *Let E be a divisible effect algebra. The group $(G(E), u)$ (where $u := \langle [1], [0] \rangle$) constructed in Theorem 1 is the universal group for E.*

PROOF: Condition (i) from Theorem 2 follows by Theorem 1. Condition (ii) follows by the construction of the group $G(E)$ and the fact that every word can be expressed as a finite $+$-sum of words of length 1. To prove (iii), let K be any abelian group, and $f : E \to K$ a K-valued measure on E. Let $\pi : E \to G(E)$, $a \mapsto \langle [a], [1] \rangle$ be the isomorphism between E and $[\theta, u]$ (where $\theta := \langle [0], [0] \rangle$) from Theorem 1. We have to prove that there is a group homomorphism $f^* : G(E) \to K$ such that $f = f^* \circ \pi$. Define $f^* : \pi(E) \to K$ by $f^* \circ \pi(a) = f(a)$. As π is an isomorphism between E and $\pi(E) = [\theta, u] \subset G(E)$, f^* is a well-defined K-valued measure on $[\theta, u]$. Clearly, $f^*(u) = f(1)$. Let g be any element of $G(E)^+$. By (ii), there are $x_1, \dots, x_n$ in E such that $g = \pi(x_1) + \cdots + \pi(x_n)$. Extend the definition of f^* by putting $f^*(g) = f(x_1) + \ldots + f(x_n)$. To prove that f^* is well-defined, assume that there are $y_1, \dots, y_m \in E$ with $g = \pi(y_1) + \cdots + \pi(y_m)$. Without loss of generality we may assume that $m \le n$. Then the elements

$$\frac{1}{n} x_1 \oplus \cdots \oplus \frac{1}{n} x_n,\ \frac{1}{n} y_1 \oplus \cdots \oplus \frac{1}{n} y_m$$

exist in E, and the equality

$$\frac{1}{n} g = \frac{1}{n}(\pi(x_1) + \cdots + \pi(x_n)) = \frac{1}{n}(\pi(y_1) + \cdots + \pi(y_m))$$

implies, because π is an isomorphism, that

$$\frac{1}{n}x_1 \oplus \cdots \oplus \frac{1}{n}x_n = \frac{1}{n}y_1 \oplus \cdots \oplus \frac{1}{n}y_m.$$

Therefore

$$\begin{aligned} f^*(g) &= f(x_1) + \cdots + f(x_n) = nf(\frac{1}{n}x_1 \oplus \cdots \oplus \frac{1}{n}x_n) \\ &= nf(\frac{1}{n}y_1 \oplus \cdots \oplus \frac{1}{n}y_m) = f(y_1) + \cdots + f(y_m). \end{aligned}$$

Additivity of f^* is clear. Now every $g \in G(E)$ is of the form $g = g^+ - g^-$ with $g^+, g^- \in G(E)^+$. So extend f^* by defining $f^*(g) = f^*(g^+) - f^*(g^-)$. If there is another representation $g = h^+ - h^-$, $h^+, h^- \in G(E)^+$, then $g^+ - g^- = h^+ - h^-$ implies $g^+ + h^- = g^- + h^+$, which yields $f^*(g^+) - f^*(g^-) = f^*(h^+) - f^*(h^-)$. This proves that $f^* : G(E) \to K$ is a well-defined homomorphism which extends f. Uniqueness is clear from the construction. $\square$

5. Embedding into a rational vector space

Recall that a torsion-free abelian group is **divisible** if for any $g \in G$ and any $n \in \mathbb{N}$ there is a unique $x \in G$ such that $nx = g$. We write $x = (1/n)g$.

Let $(E; \oplus, 0, 1)$ be a divisible effect algebra. For a word $A = (a_1, \dots, a_k)$, $k \in \mathbb{N}$, and $\alpha \in \mathbb{Q}$, $\alpha = \frac{p}{q}$, define

$$\begin{aligned} \alpha A :&= p(\frac{1}{q}a_1, \dots, \frac{1}{q}a_k) \\ &= \underbrace{(\frac{1}{q}a_1, \dots, \frac{1}{q}a_k) + \cdots + (\frac{1}{q}a_1, \dots, \frac{1}{q}a_k)}_{p-\text{times}}. \end{aligned}$$

Lemma 6. *Let $(E; \oplus, 0, 1)$ be a divisible effect algebra, $A, B \in C(E)$, $m, n, p, q \in \mathbb{N}$.*

(i) *If $p/q = m/n$, then $(p/q)A \simeq (m/n)A$.*

(ii) *If $A \simeq B$, then $\alpha A \simeq \alpha B$, $0 \le \alpha \in \mathbb{Q}$.*

Proof: (i) Let $A = (a_1, \dots, a_k)$. Then

$$\begin{aligned} \frac{p}{q}A &= p(\frac{1}{q}a_1, \dots, \frac{1}{q}a_k) \simeq pn(\frac{1}{nq}a_1, \dots, \frac{1}{nq}a_k) \\ &= mq(\frac{1}{nq}a_1, \dots, \frac{1}{nq}a_k) \simeq m(\frac{1}{n}a_1, \dots, \frac{1}{n}a_k) \\ &= \frac{m}{n}(a_1, \dots, a_k) = \frac{m}{n}A. \end{aligned}$$

(ii) If $A \sim B$, i.e., $A = (a_1, \dots, a_k)$, say, $B = (a_1, \dots, a_p \oplus a_{p+1}, \dots a_k)$, then $\alpha A \sim \alpha B$ owing to

$$\frac{1}{n}(x \oplus y) = \frac{1}{n}x \oplus \frac{1}{n}y,$$

$x, y \in E$. The statement then can be proved by induction on the minimal length of the chain connecting A and B. □

For $X \in C(E)$, $X = [a_1, \dots, a_k]$, $\alpha \in \mathbb{Q}$, $\alpha \geq 0$, define

$$\alpha X := [\alpha(a_1, \dots, a_k)].$$

By Lemma 6, αX is well defined.

Proposition 1. *For any $X, Y \in C(E)$ and nonnegative $\alpha, \beta \in \mathbb{Q}$, the following statements hold.*

(i) $\alpha(X + Y) = \alpha X + \alpha Y$.
(ii) $\alpha(\beta X) = (\alpha\beta)X$.
(iii) $(\alpha + \beta)X = \alpha X + \beta X$.
(iv) $1X = X$, $0X = [0]$.

Proof: (i) Let $\alpha = p/q$, $X = [a_1, \dots, a_k]$, $Y = [b_1, \dots, b_l]$. Then $X + Y = [a_1, \dots, a_k, b_1 \dots, b_l]$, and hence

$$\begin{aligned}\frac{p}{q}(X + Y) &= [p(\frac{1}{q}a_1, \dots, \frac{1}{q}a_k, \frac{1}{q}b_1, \dots, \frac{1}{q}b_l)]\\ &= [p(\frac{1}{q}a_1, \dots, \frac{1}{q}a_k)] + [p(\frac{1}{q}b_1 \dots, \frac{1}{q}b_l)] = \frac{p}{q}X + \frac{p}{q}Y.\end{aligned}$$

(ii) Let $\alpha = p/q$, $\beta = m/n$, $X = [a_1, \dots, a_k]$. Then

$$\begin{aligned}\alpha(\beta X) &= \frac{p}{q}[\frac{m}{n}(a_1, \dots, a_k)] = [\frac{p}{q}m(\frac{1}{n}a_1, \dots, \frac{1}{n}a_k)]\\ &= [mp(\frac{1}{qn}a_1, \dots, \frac{1}{qn}a_k)] = \frac{mp}{qn}X = (\alpha\beta)X.\end{aligned}$$

(iii) Let $\alpha = p/q$, $\beta = m/n$, $X = [a_1, \dots, a_k]$. Then by (ii),

$$\begin{aligned}\alpha X + \beta X &= [p(\frac{1}{q}a_1, \dots, \frac{1}{q}a_k)] + [m(\frac{1}{n}a_1, \dots, \frac{1}{n}a_k)]\\ &= [pn(\frac{1}{qn}a_1 \dots, \frac{1}{qn}a_k)] + [mq(\frac{1}{qn}a_1, \dots, \frac{1}{qn}a_k)]\\ &= [(pn + mq)(\frac{1}{qn}a_1, \dots, \frac{1}{qn}a_k)] = (\alpha + \beta)X.\end{aligned}$$

(iv) is clear. □

Corollary 1. *Let $(E;\oplus,0,1)$ be a divisible effect algebra. The group $G(E)$ from Theorem 1 can be endowed with a structure of an ordered vector space over the field $\mathbb{Q}$ of rational numbers.*

Proof: For $\langle\bar{a},\bar{b}\rangle \in G(E)$ and $\lambda \in \mathbb{Q}$, define

$$\lambda\langle\bar{a},\bar{b}\rangle = \begin{cases} \langle\lambda\bar{a},\lambda\bar{b}\rangle & \text{if } \lambda \geq 0 \\ \langle(-\lambda)\bar{b},(-\lambda)\bar{a}\rangle & \text{if } \lambda \leq 0. \end{cases}$$

It is straightforward to show that this operation is well defined and makes $G(E)$ a rational vector space ordered by the (rational) positive cone $G(E)^+$. □

Recall that a partially ordered abelian group G is **unperforated** if $nx \geq 0$ for some $n \in \mathbb{N}$ and $x \in G$ implies $x \geq 0$. It is clear that if $(G;u)$ bears a structure of a rational vector space, it must be divisible and unperforated.

By [3, Lemma 1.6.8], every abelian torsion-free group G can be embedded into a rational vector space H such that for all $x \in H$, there is $n \geq 0$ with $nx \in G$. The proof consists in considering the direct product $G \times \mathbb{N}$ and an equivalence relation $(a,n) \equiv (b,m)$ if $ma = nb$. Let H be the quotient with respect to this relation. Denote by $\frac{a}{n}$ the image of (a,n) in H. If we define operation $+$ on H by $\frac{a}{n} + \frac{b}{m} = \frac{ma+nb}{mn}$, the group H has all required properties. The group H is called the **divisible hull** of G. It is unique up to isomorphism.

Proposition 2. *An effect algebra E can be embedded into a divisible effect algebra F if and only if E is an interval effect algebra with an unperforated unigroup.*

Proof: Let $E = [0,a]$, where $a \in G^+$, G being an unperforated partially ordered abelian group, and let H be its divisible hull. Let us define $\frac{a}{n} \leq \frac{b}{m}$ if $ma \leq nb$ in G. Then H becomes a divisible unperforated partially ordered group. Define $h : G \to H$ by $h(x) = \frac{x}{1}$. It is clear that h is an embedding (as ordered groups), and E is embedded into $[0,h(a)]$.

Conversely, let E be embedded into an interval effect algebra F with a unigroup $(G(F);u)$. We can identify F with the interval $[0,u]$ in $G(F)^+$, so that E is a subalgebra of $[0,u]$. Let $G(E)$ be the subgroup generated by E in $G(F)$, and let $G(E)^+$ consist of all finite sums of elements of $E \subset G(F)^+$. Evidently, $0,u \in E \subseteq G(E)^+ \subseteq G(F)^+ \cap G(E)$, $G(E)^+ + G(E)^+ \subseteq G(E)^+$, $G(E)^+ \cap G(E)^+ = \{0\}$. Hence $G(E)^+$ is the generating cone in $G(E)$, and $G(E)$ is partially ordered by it. If $p \in E$, then $p' = u - p \in E$, so $p, u-p \in [0,u] \cap G(E)^+ \subseteq G(F)^+$. Suppose that $p \in [0,u] \cap G(E)^+$. Then $p, u-p \in G(E)^+ \subseteq G(F)^+$, and there exists $p_1,\dots,p_n \in E \subseteq [0,u]$ such that $p = \sum_i p_i$, and $p \leq u$ in $G(E)$. Therefore $p = \bigoplus_i p_i$ in $[0,u]$, and since E is a sub-effect algebra of $[0,u]$, it follows that $p \in E$. Hence E can be identified with $[0,u] \cap G(E)^+$, and $G(E)$ is a subgroup of $G(F)$. Since $G(F)$ is unperforated, $G(E)$ is also unperforated. □

6. Relations to convex effect algebras

Recall that an effect algebra E is **convex** [14] if for every $a \in E$ and every $\lambda \in [0,1]$ there exists an element $\lambda a \in E$ such that the following conditions hold.

(C1) If $\alpha, \beta \in [0,1]$ and $a \in E$, then $\alpha(\beta a) = (\alpha\beta)a$.

(C2) If $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$ and $a \in E$, then $\alpha a \perp \beta a$ and $(\alpha+\beta)a = \alpha a \oplus \beta a$.

(C3) If $a, b \in E$ with $a \perp b$ and $\lambda \in [0,1]$, then $\lambda a \perp \lambda b$ and $\lambda(a \oplus b) = \lambda a \oplus \lambda b$.

(C4) If $a \in E$, then $1a = a$.

Clearly, every convex effect algebra is divisible.

A map $(\lambda, a) \mapsto \lambda a$ that satisfies (C1)–(C4) is called a **convex structure** on E. It has been shown in [14, Theorem 3.1], that every convex effect algebra E is isomorphic (as convex effect algebras) to an interval $[\theta, u]$ that generates a real ordered linear space $(V; V^+)$ (which is unique up to order isomorphism), and the effect algebra order coincides with the linear space order restricted to $[\theta, u]$.

In what follows, we express $C(E)$ in another form, which is close to the method used in [14].

Define the set $\hat{K} \subseteq \mathbb{Q} \times E$ by

$$\hat{K} := \{(\alpha, a) : \alpha \geq 1, \alpha \in \mathbb{Q}, a \in E\}.$$

For $(\alpha, a), (\beta, b) \in \hat{K}$ define the relation $\sim$ on $\hat{K}$ by

$$(\alpha, a) \sim (\beta, b) \ \text{ if } \ \beta^{-1}a = \alpha^{-1}b.$$

Clearly, $\sim$ is reflexive and symmetric. To prove transitivity, suppose that $(\alpha, a) \sim (\beta, b)$ and $(\beta, b) \sim (\gamma, c)$. Then $\beta^{-1}a = \alpha^{-1}b$ and $\gamma^{-1}b = \beta^{-1}c$. Then $\beta^{-1}(\gamma^{-1}a) = (\gamma^{-1}\alpha^{-1})b = \beta^{-1}(\alpha^{-1}c)$ and by Lemma 1(viii), $\gamma^{-1}a = \alpha^{-1}c$. Thus $(\alpha, a) \sim (\gamma, c)$, so $\sim$ is an equivalence relation. Denote the equivalence class containing (α, a) by $[\alpha, a]$ and let

$$\tilde{K} := \{[\alpha, a] : (\alpha, a) \in \hat{K}\}.$$

For $\beta \geq 0$, $\beta \in \mathbb{Q}$, define

$$\beta[\alpha, a] = \begin{cases} [\beta\alpha, a] & \text{if } \beta \geq 1 \\ [\alpha, \beta a] & \text{if } \beta < 1. \end{cases}$$

We also define an operation $+$ on $\tilde{K}$ by

$$[\alpha, a] + [\beta, b] := [\alpha + \beta, \frac{\alpha}{\alpha+\beta}a \oplus \frac{\beta}{\alpha+\beta}b].$$

Using the same methods as in [14], it can be shown that $\tilde{K}$ is an abstract cone over the rationals with a zero $\tilde{\theta} = [1, 0]$.

Define a mapping $\eta : \tilde{K} \to C(E)$ by

$$\eta([\alpha, a]) = n[(\frac{1}{m}a)], \quad \text{where } \ \alpha = \frac{n}{m} \geq 1.$$

Proposition 3. (i) *The mapping η is a bijection from $\tilde{K}$ onto $C(E)$.*

(ii) *For any $0 \leq \beta \in \mathbb{Q}$, $\eta(\beta[\alpha, a]) = \beta\eta([\alpha, a])$ and $\eta([\alpha, a]+[\beta, b]) = \eta([\alpha, a])+\eta([\beta, b])$.*

Proof: (i) First we prove that η is well defined. Let $(\alpha, a) \sim (\beta, b)$, $\alpha = n/m \geq 1$, $\beta = q/p \geq 1$. Then $\beta^{-1}a = \alpha^{-1}b$, i.e., $(p/q)a = (m/n)b$. Then we have

$$n(\frac{1}{m}a) = \underbrace{(\frac{1}{m}a, \dots, \frac{1}{m}a)}_{n-\text{times}} = \underbrace{(\frac{1}{m}\frac{qm}{np}b, \dots, \frac{1}{m}\frac{qm}{np}b)}_{n-\text{times}}$$
$$= n(\frac{q}{np}b) \simeq qn(\frac{1}{np}b) \simeq q(\frac{1}{p}b).$$

Hence $\eta[(n/m), a] = \eta[(q/p), b]$.

In a similar way we prove injectivity of η. To prove surjectivity, let $\bar{a} = [a_1, \dots, a_k] \in C(E)$. Then $x := \frac{1}{k}a_1 \oplus \dots \oplus \frac{1}{k}a_k$ exists in E, and $(a_1, \dots, a_k) \simeq k(x)$. Therefore $\bar{a} = \eta[k, x]$.

(ii) The proof is routine, we leave it to the reader. □

From the proof of Theorem 3.1 in [14] and Proposition 3 the following corollary can easily be derived.

Corollary 2. *Let E be a convex effect algebra. Then the ordered rational vector space $(Q(E); Q(E)^+)$ from Corollary 1 is a rational subspace of the real linear space $(V; V^+)$ from Theorem 3.1 in [14].*

Remark. The word method can be used also to construct a real linear space for a convex effect algebra. For any word $W = (a_1, \dots, a_n)$, define the multiplication by a nonnegative real number α by

$$\alpha W = k(\frac{\alpha}{k}a_1, \dots, \frac{\alpha}{k}a_n)$$

where $k - 1 < \alpha \leq k$, $k \in \mathbb{N}$. Then we check that this multiplication yields a well-defined multiplication by nonnegative real numbers on $C(E)$, and that $C(E)$ with respect to this multiplication and the operation $+$ forms an abstract positive cone.

7. Embedding into a real linear space

Recall that a function $s : E \to [0, 1] \subset \mathbb{R}$ from an effect algebra E to the unit interval is a **state** on E if $s(a \oplus b) = s(a) + s(b)$ whenever $a \perp b$, and $s(1) = 1$. Denote by $\mathcal{S}(E)$ the set of all states (i.e. the state space) of E. In particular, a state can be considered as an $\mathbb{R}$-valued measure on E.

Let $(G; e)$ be an ordered abelian group with order unit. Recall that a **state** on $(G; e)$ is any normalized positive homomorphism s from $(G; e)$ to $(\mathbb{R}; 1)$. That is, $s(g) \geq 0$ whenever $g \in G^+$ and $s(e) = 1$. Denote by $\mathcal{S}(G; e)$ the set of all states on (G, e), i.e., the state space of $(G; e)$.

Theorem 3 implies the following result.

Theorem 4. *Let E be a divisible effect algebra and $(G(E);u)$ its unigroup. Then every state on E uniquely extends to a state on $(G(E);u)$.*

Let $(G;u)$ be an ordered abelian group with order unit. For every $x \in G$, the evaluation map $\tilde{x} : \mathcal{S}(G;u) \to \mathbb{R}$ (so that $\tilde{x}(s) = s(x)$ for all $s \in \mathcal{S}(G;u)$) is affine and continuous with respect to the pointwise operations and the product topology on $\mathbb{R}^G$, so that it defines a map:

$$\phi : G \to \mathrm{Aff}\,(\mathcal{S}(G;u)),$$

known as the **natural map** from G to the set of all affine continuous functions on $\mathcal{S}(G;u)$.

Recall that the **order-unit norm** (relative to u) is defined, for any $x \in G$, by the function

$$\|x\|_u := \inf\{\frac{k}{n} : k,n \in \mathbb{N}, -ku \le nx \le ku\}.$$

The function $\|.\|_u$ has the following norm-like properties ([13]) (we write $\|.\|$ instead $\|.\|_u$ if no confusion may happen).

(a) $\|mx\| = |m|\|x\|$.
(b) $\|x+y\| \le \|x\| + \|y\|$.
(c) If $-y \le x \le y$, then $\|x\| \le \|y\|$.

If G is nonzero, then

(d) $\|u\| = 1$.
(e) $\|x\| = \max\{|s(x)| : s \in \mathcal{S}(G;u)\} = \|\phi(x)\|$.

The rule $d(x,y) = \|x-y\|$ defines a pseudometric d on G. It is usually referred to the corresponding topology on G as the norm-topology. In general, the pseudometric d is not a metric, and hence the norm-topology on G is not Hausdorff. The **norm-completion** of G is the (Hausdorff) completion of G with respect to its order-unit norm. As the group operation in G is uniformly continuous with respect to the order-unit norm by property (b), and multiplication by rational numbers is uniformly continuous by (a), the norm-completion of a divisible abelian group G is a divisible abelian group. If $\kappa : G \to \bar{G}$ is the canonical map, then $\kappa(G^+)$ is a cone in $\bar{G}$, and so the closure of $\kappa(G^+)$ is a cone as well. We make $\bar{G}$ a pre-ordered abelian group with positive cone $\bar{G}^+$ equal to the closure of $\kappa(G^+)$ in $\bar{G}$.

Proposition 4. *Let $(G;u)$ be a nonzero divisible partially ordered abelian group with order-unit, and let $\phi : G \to \mathrm{Aff}\,(\mathcal{S}(G;u))$ be the natural map. Then the subgroup $\phi(G)$ is dense in $\mathrm{Aff}\,(\mathcal{S}(G;u)$. If, in addition, G is unperforated, then $\phi(G^+)$ is dense in $\mathrm{Aff}\,(\mathcal{S}(G;u))^+$.*

Proof: By [13, Theorem 7.9], for any (nonzero) partially ordered abelian group $(G;u)$, the set

$$H := \{\phi(x)/2^n : x \in G, n \in \mathbb{N}\}$$

is dense in $\mathrm{Aff}\,(\mathcal{S}(G;u))$. By divisibility, $H \subseteq \phi(G)$, and so $\phi(G)$ is dense in $\mathrm{Aff}\,(\mathcal{S}(G;u))$.

Similarly, if G is unperforated, then the set

$$H^+ := \{\phi(x)/2^n : x \in G^+, n \in \mathbb{N}\}$$

is dense in $\mathrm{Aff}\,(\mathcal{S}(G;u))^+$. Since $H^+ \subseteq \phi(G^+)$ if G is divisible, it follows that $\phi(G^+)$ is dense in $\mathrm{Aff}\,(\mathcal{S}(G;u))^+$. □

Theorem 5. *Let $(G;u)$ be a nonzero unperforated divisible group with order-unit. Let $\bar{G}$ be the norm-completion of G, and let $\kappa : G \to \bar{G}$ be the canonical map. Let $\phi : G \to \mathrm{Aff}\,(\mathcal{S}(G;u))$ be the natural map. Then $\bar{G}$ is isomorphic with $\mathrm{Aff}\,(\mathcal{S}(G;u))$.*

Proof: Put $A := \mathrm{Aff}\,(\mathcal{S}(G;u))$. Since $\|\phi(x)\| = \|x\|$ for all $x \in G$, ϕ is uniformly continuous. Consequently, there is a unique uniformly continuous map $f : \bar{G} \to A$ such that $f\kappa = \phi$, and we may observe that f is a group homomorphism. As $\kappa(G)$ is dense in $\bar{G}$, and $f\kappa(G) = \phi(G) \subseteq A$, we see that $f(\bar{G}) \subseteq A$. Similarly, since $\kappa(G^+)$ is dense in $\bar{G}^+$, and $f\kappa(G^+) \subseteq A^+$, we have $f(\bar{G}^+) \subseteq A^+$. Thus f is a positive homomorphism from $\bar{G}$ to A. Given any $x \in \bar{G}$ with $f(x) = 0$, choose a sequence $\{x_1, x_2 \dots\}$ in G such that $\kappa(x_i) \to x$. Then $\phi(x_i) \to 0$ in A. As $\|x_i\| = \|\phi(x_i)\|$ for all i, we obtain $\|x_i\| \to 0$, whence $x = 0$. Thus f is injective.

By Proposition 4, $\phi(G^+)$ is dense in A^+. Hence, given any $p \in A^+$, there exists a sequence $\{x_1, x_2, \dots\}$ in G^+ such that $\phi(x_i) \to p$. Since

$$\|x_i - x_j\| = \|\phi(x_i) - \phi(x_j)\|$$

for all i, j, x_i form a Cauchy sequence in G^+. Then there exists $x \in \bar{G}^+$ such that $\kappa(x_i) \to x$, and $f(x) = p$ by continuity of f. Thus $f(\bar{G}^+) = A^+$. As A is directed (and has an order unit 1), we may conclude from $f(\bar{G}^+) = A^+$ that $f(\bar{G}) = A$. Therefore f is an isomorphism of $\bar{G}$ onto A as ordered groups. □

In the situation of Theorem 5 we may identify $\bar{G}$ with $\mathrm{Aff}\,(\mathcal{S}(G;u))$ and κ with ϕ. Certainly $\bar{G}$ is partially ordered and $\kappa(u) = 1$ is an order unit in $\bar{G}$. As the order-unit norm in $\bar{G}$ (with respect to the order-unit 1) coincides with the supremum norm, $\bar{G}$ is norm-complete. Moreover, the order-unit metric in $\bar{G}$ coincides with the metric obtained from the completion process; namely, given elements x and y in $\bar{G}$, expressed as limits of sequences $\{\kappa(x_i)\}$ and $\{\kappa(y_i)\}$ with $x_i, y_i \in G$, the value $\|x - y\| = \lim_i \|x_i - y_i\|$.

Recall that a partially ordered abelian group G is **archimedean** provided that whenever $x, y \in G$ such that $nx \le y$ for all $n \in \mathbb{N}$, then $x \le 0$. It is well known that an abelian partially ordered group is archimedean if and only if it is isomorphic to a subgroup of $\mathbb{R}^X$ for some set X.

In analogy with the definition for convex effect algebras ([15]), we will say that a divisible effect algebra E is **archimedean** if whenever $a, b, c \in E$ with $a \perp b$ and $c \le a \oplus (1/n)b$ for every $n \in \mathbb{N}$, then $c \le a$.

Let S be a set of states on an effect algebra E. Then S is said to be **separating** if $s(a) = s(b)$ for all $s \in S$ implies $a = b$ $(a, b \in E)$. And S is said to be **ordering** if $s(a) \leq s(b)$ for all $s \in S$ implies $a \leq b$ $(a, b \in E)$.

The following two theorems are analogous to [15, Theorem 3.6] and [15, Theorem 3.5].

Theorem 6. *Let E be a divisible effect algebra and $G(E); u)$ its unigroup. Then the following statements are equivalent.* (a) *E possesses an ordering set of states.* (b) *E is archimedean.* (c) *$G(E)$ is archimedean.*

Proof: To show that (a) implies (b) assume that S is an ordering set of states on E. Let a, b, c be elements of E with $a \perp b$ and $c \leq a \oplus (1/n)b$ for every $n \in \mathbb{N}$. For every $s \in S$ we have $s(c) \leq s(a) + n^{-1}s(b)$ for all $n \in \mathbb{N}$. Hence $s(c) \leq s(a)$ for all $s \in S$ so $c \leq a$.

To show that (b) implies (c) assume that E is archimedean, so that $[\theta, u]$ is archimedean. Suppose that $nx \leq y$, $x, y \in G(E)$, for all $n \in \mathbb{N}$. Since u is an order-unit, there is $m \in \mathbb{N}$ such that $y \leq mu$, hence $nx \leq mu$ for all $n \in \mathbb{N}$. It follows that $n(1/m)x \leq u$ for all $n \in \mathbb{N}$. Let $(1/m)x = v - z$ with $v, z \geq 0$, so that $v \leq z + n^{-1}u$ for all $n \in \mathbb{N}$. If $v = 0$ or $z = 0$, then $x \leq 0$, and we are finished. So suppose that $v, z \neq 0$. Since u is an order unit, there are $m, k \in \mathbb{N}$ such that $z + (1/k)u \leq mu$. Then $(1/m)z \perp (1/mk)u$ and

$$\frac{1}{m}v \leq \frac{1}{m}z \oplus n^{-1}\frac{1}{mk}u$$

and since E is archimedean, it follows that $(1/m)v \leq (1/m)z$, hence $v \leq z$. It follows that $x \leq 0$.

Now assume that $(G(E); u)$ is archimedean. By [13, Theorem 7.7], the natural map $\phi : (G(E), u) \to \mathrm{Aff}\,(\mathcal{S}(G(E); 1))$ provides an isomorphism of $G(E)$ onto a subgroup of $\mathrm{Aff}\,(\mathcal{S}(G(E); u))$ as ordered groups. It follows that $a \leq b$ $(a, b \in E$, identified with $[\theta, u])$ iff $\phi(a) \leq \phi(b)$, hence $\phi(a)(s) \leq \phi(b)(s)$ for every $s \in \mathcal{S}(G(E), u)$. It follows that $a \leq b$ iff $s(a) \leq s(b)$ for all $s \in \mathcal{S}(E)$, so $\mathcal{S}(E)$ is ordering for E. □

Analogously we prove the following.

Theorem 7. *Let E be a divisible effect algebra and $(G(E); u)$ its unigroup. The following statements are equivalent:* (a) *E possesses a separating set of states.* (b) *The order-unit norm on $(G(E); u)$ is actually a norm.*

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