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Non-associative geometry and discrete structure of spacetime

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Abstract. A new mathematical theory, *non-associative geometry*, providing a unified algebraic description of continuous and discrete spacetime, is introduced.

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1. Introduction

The recent development of geometry has shown the importance of non-associative algebraic structures such as quasigroups, loops and odules. For instance, it is possible to say that non-associativity is the algebraic equivalent to the differential geometric concept of curvature. The corresponding construction may be described as follows. In a neighborhood of an arbitrary point on a manifold with an affine connection one can introduce the geodesic local loop which is uniquely defined by means of the parallel translation of geodesics along geodesics ([1], [2], [3]). The family of local loops constructed in this way uniquely defines a space with affine connection, but not every family of geodesic loops on a manifold defines an affine connection. It is necessary to add some algebraic identities connecting loops in different points. Later, the additional algebraic structures (so called *geoodular structures*) were introduced and the equivalence of the categories of geoodular structures and of affine connections was shown by Sabinin [S3,S4]. The main algebraic structures arising in this approach are related to non-associative algebra and the theory of quasigroups and loops.

In our paper the new mathematical theory, *non-associative geometry*, which may help us to understand the discrete structure of spacetime, is introduced. The key point is to give an algebraic (non-local) description of a manifold, which may be used in continuous and discrete cases. Non-associative geometry provides the unified algebraic description of continuous and discrete spacetime and admits all basic attributes of spacetime geometry including generalized Einstein's equations.

2. What is the non-associative geometry?

Here we survey, in brief, algebraic foundations of *non-associative geometry* due to L.V. Sabinin (see on the matter [6], [7], [8], [9]).

Definition 2.1. Let $\langle Q, \cdot \rangle$ be a groupoid with binary operation $(a,b) \mapsto a \cdot b$ and Q be a smooth manifold. Then $\langle Q, \cdot \rangle$ is called a *quasigroup* if the equations $a \cdot x = b$, $y \cdot a = b$ have unique solutions: $x = a \backslash b$, $y = b/a$. A *loop* is a quasigroup with a two-sided identity, $a \cdot e = e \cdot a = a$, for any $a \in Q$. A loop $\langle Q, \cdot, e \rangle$ with a smooth functions $\phi(a,b) := a \cdot b$ is called a *smooth loop*.

Let $\langle Q, \cdot, e \rangle$ be a smooth local loop with a neutral element e. We define

$$L_a b = R_b a = a \cdot b, \quad l_{(a,b)} = L^{-1}_{a \cdot b} \circ L_a \circ L_b, \tag{1}$$

where L_a is a *left translation*, R_b is a *right translation*, $l_{(a,b)}$ is an *associator*.

Definition 2.2. Let $\langle M, \cdot, e \rangle$ be a partial *groupoid* with binary operation $(x,y) \mapsto x \cdot y$ and neutral element e, $x \cdot e = e \cdot x = x$. If M is a smooth manifold (at least C^1-smooth), and if the operation of multiplication (at least C^1-smooth) is defined in some neighborhood U_e, then $\langle M, \cdot, e \rangle$ is called a *partial loop on* M.

Remark 2.1. The operation of multiplication is locally left and right invertible. This means that if $x \cdot y = L_x y = R_y x$ then there exist L_x^{-1} and R_x^{-1} in some neighborhood of the neutral element e:

$$L_a(L_a^{-1}x) = x, \quad R_a(R_a^{-1}x) = x.$$

The vector fields A_j defined on U_e by

$$A_j(x) = \big((L_x)_{*,e}\big)^i_j \frac{\partial}{\partial x^i} = L^i_j(x) \frac{\partial}{\partial x^i} \tag{2}$$

are called the *left basic fundamental fields*. Similarly, the *right basic fundamental fields* B_j are defined by

$$B_j(x) = \big((R_x)_{*,e}\big)^i_j \frac{\partial}{\partial x^i} = R^i_j(x) \frac{\partial}{\partial x^i} \,. \tag{3}$$

The solution of the equation

$$\frac{df^i(t)}{dt} = L^i_j(f(t))X^j, \quad f(0) = e, \tag{4}$$

is of the form $f(t) = \mathrm{Exp}tX$ defining the *exponential map*

$$\mathrm{Exp} : X \in T_e(M) \longrightarrow \mathrm{Exp}X \in M.$$

The unary operation

$$tx = \mathrm{Exp}(t\mathrm{Exp}^{-1}x), \tag{5}$$

based on the exponential map, is called the *left canonical unary operation* for $\langle M, \cdot, e\rangle$. A smooth loop $\langle M, \cdot, e\rangle$ equipped with its canonical left unary operations is called the *left canonical preodule* $\langle M, \cdot, (t)_{t\in\mathbb{R}}, e\rangle$. If one more operation is introduced,

$$x + y = \mathrm{Exp}(\mathrm{Exp}^{-1}x + \mathrm{Exp}^{-1}y), \tag{6}$$

then we obtain the *canonical left prediodule* of a loop, $\langle M, \cdot, +, (t)_{t\in\mathbb{R}}, e\rangle$. A canonical left preodule (prediodule) is called the *left odule (diodule)* if the *monoassociativity* property

$$tx \cdot ux = (t+u)x \tag{7}$$

is satisfied. In the smooth case, for an odule, the left and the right canonical operations as well as the exponential maps coincide.

Definition 2.3. Let M be a smooth manifold and

$$L : (x, y, z) \in M \mapsto L(x, y, z) \in M$$

a smooth partial ternary operation, such that $x \dot{a} y = L(x, a, z)$ defines in some neighborhood of the point a a loop with neutral a, then the pair $\langle M, L\rangle$ is called a *loopuscular structure* (*manifold*).

A smooth manifold M with a smooth partial ternary operation L and smooth binary operations $\omega_t : (a, b) \in M \times M \mapsto \omega_t(a, b) = t_a b \in M$, $(t \in \mathbb{R})$, such that $x \dot{a} y = L(x, a, y)$ and $t_a z = \omega_t(a, z)$ determine in some neighborhood of an arbitrary point a an odule with neutral element a, is called a *left odular structure* (*manifold*) $\langle M, L, (\omega_t)_{t\in\mathbb{R}}\rangle$. Let $\langle M, L, (\omega_t)_{t\in\mathbb{R}}\rangle$ and $\langle M, N, (\omega_t)_{t\in\mathbb{R}}\rangle$ be odular structures. Then $\langle M, L, N, (\omega_t)_{t\in\mathbb{R}}\rangle$ is called a *diodular structure* (*manifold*). If $x \underset{a}{+} y = N(x, a, y)$ and $t_a x = \omega_t(a, x)$ define a vector space, then such a diodular structure is called a *linear diodular structure*.

A diodular structure is said to be *geodiodular* if

(8) (the first geoodular identity) $L^{t_a x}_{u_a x} \circ L^a_{t_a x} = L^a_{u_a x} \quad (L^a_x y = L(x, a, y))$,

(9) (the second geoodular identity) $L^a_x \circ t_a = t_x \circ L^a_x$,

(10) (the third geoodular identity) $L^a_x N(y, a, z) = N(L^a_x y, x, L^a_x z)$

are true.

Definition 2.4. Let M be a C^k-smooth $(k \geq 3)$ affinely connected manifold. The following operations are given on M (locally):

$$L^a_x y = x \underset{a}{\cdot} y = \mathrm{Exp}_x \tau^a_x \mathrm{Exp}^{-1}_a y, \tag{11}$$

$$\omega_t(a, z) = t_a z = \mathrm{Exp}_a t \mathrm{Exp}^{-1}_a z, \tag{12}$$

$$N(x, a, y) = x \underset{a}{+} y = \mathrm{Exp}_a(\mathrm{Exp}^{-1}_a x + \mathrm{Exp}^{-1}_a y), \tag{13}$$

Exp_x being the exponential map at the point x and τ^a_x the parallel translation along the (unique) geodesic going from a to x. The above construction equips M with the linear geodiodular structure which is called a *natural linear geodiodular structure of an affinely connected manifold* (M, ∇).

Remark 2.2. Any C^k-smooth ($k \geq 3$) affinely connected manifold can be considered as a geodiodular structure.

Definition 2.5. Let $\langle M, L\rangle$ be a loopuscular structure of a smooth manifold M. Then the formula

$$\nabla_{X_a} Y = \left\{ \frac{d}{dt} \left([(L^a_{g(t)})_{*,a}]^{-1} Y_{g(t)} \right) \right\}_{t=0},$$
$$g(0) = a, \quad \dot g(0) = X_a,$$

Y being a vector field in the neighborhood of a point a, defines the *tangent affine connection*.

In coordinates the components of this affine connection are

$$\Gamma^i_{jk}(a) = -\left[\frac{\partial^2 (x \dot{a} y)^i}{\partial x^j \partial y^k} \right]_{x=y=a}.$$

The equivalence of the categories of geoodular (geodiodular) structures and of affine connections has been shown in [4], [5].

Definition 2.6. Let $\langle M, L\rangle$ be a loopuscular structure, then

(14) $$h^a_{(b,c)} = (L^a_c)^{-1} \circ L^b_c \circ L^a_b$$

is called the *elementary holonomy*.

Comment 2.1. The elementary holonomy is, in fact, the parallel translation along a geodesic triangle path. Consequently, it is some integral curvature. Indeed, in the smooth case, differentiating $(h^a_{(x,y)})^i$ by x^j, y^k at $a \in M$, we get the curvature tensor at $a \in M$ precisely up to numerical factor,

(15) $$R^i{}_{jkl}(a) = 2 \left[\frac{\partial^3 (h^a_{(x,y)} z)^i}{\partial x^l \partial y^k \partial z^j} \right]_{x=y=z=a}$$

Comment 2.2. For a diodular structure one can consider an elementary holonomy $h_{(a,b)} = h^e_{(a,b)}$ together with the diodule $\langle M, \underset{e}{\cdot}, \underset{e}{+}, (t_e)_{t\in\mathbb{R}}, e\rangle$ to be a so-called *holonomial diodule*, and restore this diodular structure in a unique way:

(16) $$L(x,a,y) = L^e_x h_{(a,x)} (L^e_a)^{-1} y,$$

(17) $$N(x,a,y) = L^e_a((L^e_a)^{-1} x \underset{e}{+} (L^e_a)^{-1} y),$$

(18) $$\omega_t(a,y) = t_a y = L^e_a t_e (L^e_a)^{-1} y.$$

In this case the holonomial identities

(19) $$h_{(a,b)} t_e x = t_e h_{(a,b)} x, \; h_{(a,b)}(x \underset{e}{+} y) = h_{(a,b)} x \underset{e}{+} h_{(a,b)} y \quad \text{(linearity)},$$

(20) $$h_{(a,a\cdot u_e b)} tb = l_{(a,u_e b)} tb \quad \text{(joint identity)},$$

(21) $$h_{(c\cdot t_e a, c\cdot u_e a)} h_{(c,c\cdot t_e a)} x = h_{(c,c\cdot u_e a)} x \quad (h\text{-identity}),$$

(22) $$h_{(e,q)} x = x \quad (e\text{-identity})$$

are true.

Comment 2.3. Using the definition of elementary holonomy (14) we may easily verify that elementary holonomy satisfies the odular Bianchi identities:

$$h^a_{(z,x)} \circ h^a_{(y,z)} \circ h^a_{(x,y)} = (L^a_x)^{-1} \circ h^x_{(y,z)} \circ L^a_x. \tag{23}$$

These identities can be considered as non-local form of the usual Bianchi identities and in this sense they are equivalent. Perhaps, this can be considered to be the first non-local algebraic expression of the Bianchi identities. In the linear approximation (23) generates the Bianchi identities in conventional form. This may be easily seen in the normal coordinates related to the point a.

Non-associative geometry is based on the constructions described above. In the table below we compare the basic concepts of the classical differential geometry and of the non-associative geometry.

Differential Geometry vs Non-associative Geometry

Differential Geometry	Non-associative Geometry
Tangent space $T_a(M)$	Osculating space $\langle M, \underset{a}{+}, a, (t_a)_{t\in\mathbb{R}}\rangle$
Tangent bundle structure	Osculating structure $\langle M, N, (\omega_t)_{t\in\mathbb{R}}\rangle$
Cotangent space	Co-osculating space
Parallel displacement	Left translations $L^a_x y$
Curvature $R(X,Y)Z$	Elementary holonomy $h^a_{(x,y)}z$
Bianchi identities	Odular Bianchi identities

2.1 Example: the non-associative geometry of two-dimensional sphere S^2_R

The well known two-sphere S^2_R of radius R admits a natural loop structure ([11], [12], [13]) which may be described as follows. Let $\mathbb{C}$ be a complex plane and $\zeta, \eta \in \mathbb{C}$. The non-associative multiplication $\star$ is defined by

$$\zeta \star \eta = L_\zeta \eta = \frac{\zeta + \eta}{1 - \overline{\zeta}\eta/R^2}, \quad \zeta, \eta \in \mathbb{C} \tag{24}$$

where bar denotes complex conjugation and the neutral element e coincides with the origin of the coordinate system. This loop is isomorphic to the local two-parameter loop associated with two-sphere S^2_R. The isomorphism between points of the sphere and points of the complex plane $\mathbb{C}$ is established by the stereographic projection from the south pole of the unit sphere, $\zeta = R\tan(\theta/2)e^{\mathrm{i}\varphi}$.

Remark 2.3. The entire sphere may be covered by two local (partial) loops, one of them with the neutral element at the north pole (see the above) and another with the neutral element at the south pole.

The associator is found to be

$$l_{(\zeta,\eta)}\xi = \frac{1-\zeta\overline{\eta}/R^2}{1-\eta\overline{\zeta}/R^2}\xi. \tag{25}$$

The sphere is a symmetric space and its elementary holonomy is determined by the associator: $h_{(\zeta,\eta)} = l_{(\zeta,L_\zeta^{-1}\eta)}$ ([6], [9]). The computation gives

$$h_{(\zeta,\eta)}\xi = \frac{1+\overline{\zeta}\eta/R^2}{1+\zeta\overline{\eta}/R^2}\xi. \tag{26}$$

The left invariant diodular metric on S_R^2 is given by

$$g^0(L_\zeta^{-1}\xi, L_\zeta^{-1}\eta) = g^\zeta(\xi,\eta), \tag{27}$$

where $g^0(\zeta,\eta)$ is the diodular metric tensor at the neutral element induced by the natural metric on the tangent space at the neutral element (north pole of S_R^2), and $g^\zeta(\xi,\eta)$ is the diodular metric at the point ζ. Actually (27) is an algebraic analogue of compatibility of the connection with the metric structure of S_R^2. We define the left invariant diodular metric on the two-sphere as follows:

$$g^\zeta(\xi,\eta) = 2\left(\frac{(\xi-\zeta)(\bar{\eta}-\bar{\zeta})}{(1+\bar{\zeta}\xi/R^2)(1+\bar{\eta}\zeta/R^2)} + \frac{(\bar{\xi}-\bar{\zeta})(\eta-\zeta)}{(1+\zeta\bar{\xi}/R^2)(1+\bar{\zeta}\eta/R^2)}\right).$$

In particular,

$$g^\zeta(\xi,\xi) = \frac{4|\xi-\zeta|^2}{|1+\bar{\zeta}\xi/R^2|^2}. \tag{28}$$

Let $\xi = \zeta + d\zeta$. Then (28) leads to

$$g(d\zeta,d\zeta) = \frac{4d\zeta d\overline{\zeta}}{(1+|\zeta|^2/R^2)^2},$$

the well known expression for the element of arclength on S_R^2.

Comment 2.4. Note that the same result may be obtained in another way. Let us introduce the basis of left fundamental vectors and the dual basis of one-forms:

$$\Gamma_1 = (1+|\zeta|^2)\partial_\zeta, \quad \Gamma_2 = (1+|\zeta|^2)\partial_{\bar{\zeta}}, \tag{29}$$

$$\theta^1 = \frac{d\zeta}{1+|\zeta|^2}, \quad \theta^2 = \frac{d\overline{\zeta}}{1+|\zeta|^2}. \tag{30}$$

Then the metric based on the left fundamental basis forms is given by

$$ds^2 = 4\theta^1\theta^2 = \frac{4d\zeta d\bar{\zeta}}{(1+|\zeta|^2/R^2)^2}\,. \tag{31}$$

Computation of the curvature tensor gives

$$R^{\zeta}{}_{\zeta\zeta\bar{\zeta}} = \frac{2}{R^2\left(1+|\zeta|^2/R^2\right)^2}\,, \tag{32}$$

and using (26) we find

$$R^{\zeta}{}_{\zeta\bar{\zeta}\zeta}(0) = 2\left[\frac{\partial^3(h_{(\zeta,\eta)}\xi)}{\partial\zeta\partial\bar{\eta}\partial\xi}\right]_{\zeta=\eta=\xi=0},$$

which is consistent with (15).

2.1.1 The non-associative discrete geometry of S^2_R

We start with the natural geodesic triangulation of the sphere which is specified as follows. The simplex, triangulating S^2_R, is a geodesic triangle. The geodesic lattice will be assumed to consist of a central vertex at the north pole and of the geodesic triangles attached to this vertex. To each surface vertex $\mathbf{p} = (j,k)$ we assign the polar coordinates (θ_j, φ_k), assuming $\theta_j = \pi j/n$, $\varphi_k = 2\pi k/n$ $(j,k = 0,1,2,\dots n-1)$. With such a choice we have the triangulation defined by n^2 points allocated on the surface of the sphere:

$$\zeta_{\mathbf{p}} = R\tan\left(\frac{\pi j}{2n}\right)e^{\frac{2\pi i k}{n}}. \tag{33}$$

The non-associative operation (24) now takes the form

$$\zeta_{\mathbf{pq}} = \frac{\zeta_{\mathbf{p}}+\zeta_{\mathbf{q}}}{1-\bar{\zeta}_{\mathbf{p}}\zeta_{\mathbf{q}}/R^2}\,. \tag{34}$$

Writing $\zeta_{\mathbf{pq}}$ as

$$\zeta_{\mathbf{pq}} = R\tan\left(\frac{\theta_{\mathbf{pq}}}{2}\right)e^{i\varphi_{\mathbf{pq}}}, \tag{35}$$

one obtains from (34) the following formulae which define the left translations in "spherical coordinates":

$$\theta_{\mathbf{pq}} = 2\tan^{-1}\left(\frac{1}{R}\left|\frac{\zeta_{\mathbf{p}}+\zeta_{\mathbf{q}}}{1-\bar{\zeta}_{\mathbf{p}}\zeta_{\mathbf{q}}/R^2}\right|\right), \tag{36}$$

$$\varphi_{\mathbf{pq}} = \arg(\zeta_{\mathbf{p}}+\zeta_{\mathbf{q}}) - \frac{i}{2}\ln l(\zeta_{\mathbf{p}},\zeta_{\mathbf{q}}), \tag{37}$$

$$\zeta_{\mathbf{p}} = R\tan\left(\frac{\pi j}{2n}\right)e^{\frac{2\pi i k}{n}}, \qquad \zeta_{\mathbf{q}} = R\tan\left(\frac{\pi l}{2n}\right)e^{\frac{2\pi i m}{n}}, \tag{38}$$

where

$$l(\zeta_{\mathbf{p}}, \zeta_{\mathbf{q}}) = \frac{1 - \zeta_{\mathbf{p}}\overline{\zeta}_{\mathbf{q}}/R^2}{1 - \overline{\zeta}_{\mathbf{p}}\zeta_{\mathbf{q}}/R^2}.$$

For the diodular metric we have

$$g^{\zeta_{\mathbf{p}}}(\zeta_{\mathbf{q}}, \zeta_{\mathbf{q}}) = \frac{4|\zeta_{\mathbf{p}} - \zeta_{\mathbf{q}}|^2}{|1 + \overline{\zeta}_{\mathbf{p}}\zeta_{\mathbf{q}}/R^2|^2}, \tag{39}$$

and the elementary holonomy is

$$h_{(\zeta_{\mathbf{p}},\zeta_{\mathbf{q}})}\zeta_{\mathbf{m}} = \frac{1 + \overline{\zeta}_{\mathbf{p}}\zeta_{\mathbf{q}}/R^2}{1 + \zeta_{\mathbf{p}}\overline{\zeta}_{\mathbf{q}}/R^2}\zeta_{\mathbf{m}}. \tag{40}$$

Comment 2.5. To a certain extent the information concerning the geometry of the sphere is hidden in the structure of the finite loop. The spherical symmetry is determined by the relation between the associator and elementary holonomy, $h_{(\zeta_{\mathbf{p}},\zeta_{\mathbf{q}})} = l_{(\zeta_{\mathbf{p}}, L^{-1}_{\zeta_{\mathbf{p}}}\zeta_{\mathbf{q}})}$. The smooth sphere could be regarded as the result of a "limit process" of triangulating as $n \longrightarrow \infty$. Indeed, with growing n, the number of points increases and the triangulations become finer.

In order to obtain the correct 'continuous' limit let us consider $\mathbf{q} = \mathbf{p} + \boldsymbol{\delta}$, $|\boldsymbol{\delta}| \ll n$. Let $\boldsymbol{\delta} = (l, m)$, then

$$\zeta_{\mathbf{q}} = \zeta_{\mathbf{p}} + R\left(\frac{\zeta_{\mathbf{p}}}{\overline{\zeta}_{\mathbf{p}}}\right)^{\frac{1}{2}}\left(\left(1 + \frac{|\zeta_{\mathbf{p}}|^2}{R^2}\right)\frac{\pi l}{2n} + i\frac{|\zeta_{\mathbf{p}}|}{R}\frac{2\pi m}{n}\right) + O\left(\left(\frac{|\boldsymbol{\delta}|}{n}\right)^2\right),$$

and the diodular metric takes the form

$$g^{\zeta_{\mathbf{p}}}(\zeta_{\mathbf{q}}, \zeta_{\mathbf{q}}) = R^2\left(\left(\frac{\pi l}{n}\right)^2 + \frac{4|\zeta_{\mathbf{p}}|^2}{(1 + |\zeta_{\mathbf{p}}|^2/R^2)^2}\left(\frac{2\pi m}{n}\right)^2\right) + O\left(\left(\frac{|\boldsymbol{\delta}|}{n}\right)^2\right). \tag{41}$$

Simplifying (41), we obtain

$$g^{\zeta_{\mathbf{p}}}(\zeta_{\mathbf{q}}, \zeta_{\mathbf{q}}) = R^2\left((\Delta\theta_{\mathbf{q}})^2 + \sin^2\theta_{\mathbf{p}}(\Delta\varphi_{\mathbf{q}})^2\right) + O\left(\left(\frac{|\boldsymbol{\delta}|}{n}\right)^2\right), \tag{42}$$

where $\Delta\theta_{\mathbf{q}} = \pi l/n$ and $\Delta\varphi_{\mathbf{q}} = 2\pi m/n$. The 'differential geometry' appears as the result of a limit process as $n \longrightarrow \infty$:

$$\begin{aligned} &\Delta\theta_{\mathbf{q}} \longrightarrow d\theta, \quad \Delta\varphi_{\mathbf{q}} \longrightarrow d\varphi, \\ &g^{\zeta_{\mathbf{p}}}(\zeta_{\mathbf{q}}, \zeta_{\mathbf{q}}) \longrightarrow ds^2 = R^2\left((d\theta)^2 + \sin^2\theta(d\varphi)^2\right). \end{aligned} \tag{43}$$

A similar consideration of the elementary holonomy gives

$$h_{(\zeta_{\mathbf{p}},\zeta_{\mathbf{q}})}\zeta_{\mathbf{m}} = \zeta_{\mathbf{m}}\left(1 + i\frac{\Delta(\zeta_{\mathbf{p}},\zeta_{\mathbf{q}})}{R^2} + O\left(\left(\frac{|\boldsymbol{\delta}|}{n}\right)^2\right)\right), \tag{44}$$

where

$$\Delta(\zeta_{\mathbf{p}},\zeta_{\mathbf{q}}) = \frac{2|\zeta_{\mathbf{p}}|^2\Delta\varphi_{\mathbf{q}}}{1+|\zeta_{\mathbf{p}}|^2/R^2}$$

is the area of the geodesic triangle $(e, \mathbf{p}, \mathbf{q})$. Approaching the limit as $n \longrightarrow \infty$, one restores the conventional scalar curvature $1/R^2$.

3. The algebraic generalization of vacuum Einstein's equations

Einstein's equations in a vacuum

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{45}$$

can be rewritten in the tetrad basis as follows ([10]):

$$*R_{abcd}\theta^b \wedge \theta^c \wedge \theta^d = 0, \tag{46}$$

where

$$*R_{abcd} = \frac{1}{2}\epsilon_{abmn}R^{mn}{}_{cd}$$

is the dual to the Riemann tensor (latin indices a, b, c, d are used for the tetrad basis, and are running over 0,1,2,3). Equations (45) (or (46)) mean:

$$\begin{aligned}
\{00\} &\implies & R^1{}_{212} + R^2{}_{323} + R^3{}_{131} &= 0,\\
\{11\} &\implies & R^0{}_{202} + R^2{}_{323} + R^3{}_{030} &= 0,\\
\{22\} &\implies & R^0{}_{101} + R^1{}_{313} + R^3{}_{030} &= 0,\\
\{33\} &\implies & R^0{}_{101} + R^1{}_{212} + R^2{}_{020} &= 0,\\
\{01\} &\implies & R^0{}_{221} + R^0{}_{331} &= 0,\\
\{02\} &\implies & R^0{}_{112} + R^0{}_{332} &= 0,\\
\{03\} &\implies & R^0{}_{113} + R^0{}_{223} &= 0,\\
\{12\} &\implies & R^1{}_{002} + R^1{}_{332} &= 0,\\
\{13\} &\implies & R^1{}_{003} + R^1{}_{223} &= 0,\\
\{23\} &\implies & R^2{}_{003} + R^2{}_{113} &= 0,
\end{aligned}$$

and may be written in the following form:

$$*R(X,Y)Z + *R(Y,Z)X + *R(Z,X)Y = 0, \tag{47}$$

where $X, Y, Z \in T(M)$.

Let us consider the following algebraic equation:

$$*h^e_{(x,y)}z \underset{e}{+} *\,h^e_{(y,z)}x \underset{e}{+} *\,h^e_{(z,x)}y = e \quad (\forall\, e, x, y, z), \tag{48}$$

where

$$\big(*h^e_{(x,y)}\big)^a_b = \frac{1}{2} g^{ac} \epsilon_{cbmn} g^{nl} \big(h^e_{(x,y)}\big)^m_l$$

is the dual elementary holonomy. Employing the normal coordinates with the origin at the point e and the relation (15) between the curvature and elementary holonomy, we find that in the first approximation (48) restores the Einstein equations (47).

We propose the algebraic system (48) together with (19)–(22) as a non-local generalization of the vacuum Einstein's equations (diodular Einstein's equations). They should be considered as the equations for constructing the diodule at the point e (which uniquely defines the corresponding diodular space).

Comment 3.1. The relation between odular Einstein's equations considered in the neighborhoods of the points e and a is established by means of the odular Bianchi identities (23):

$$*h^a_{(x,y)} = *\big(L^e_a \circ h^e_{(y,a)} \circ h^e_{(x,y)} \circ h^e_{(a,x)} \circ (L^e_a)^{-1}\big). \tag{49}$$

In the metric gravitational theory the odular Einstein's equations should be considered together with

$$g^e(x, y) = g^a(L^e_a x, L^e_a y) \tag{50}$$

which relates connections and metrics (here $g^a(p, q)$ is a metric tensor at $a \in M$).

In the normal coordinates with the origin at the point e we get the complete system of diodular Einstein's equations in the form:

$$\big(*h^e_{(x,y)}\big)^a_b z^b + \big(*h^e_{(y,z)}\big)^a_b x^b + \big(*h^e_{(z,x)}\big)^a_b y^b = 0, \tag{51}$$

$$\big(h^e_{(x_b,x_c)}\big)^d_f = \big((L^e_{x_c})^{-1}\big)^d_g \big(L^{x_b}_{x_c}\big)^g_h \big(L^e_{x_b}\big)^h_f. \tag{52}$$

4. Concluding remarks

In our paper we proposed a new *non-associative* approach to the classical and discrete structure of manifolds which gives the unified description of continuous and discrete spacetime. This means that at the Planck scales the standard concept of spacetime might be replaced by the diodular discrete structure which at large spacetime scales 'looks like' a differentiable manifold.

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