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Relatives of K-loops: Theory and examples

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Abstract. A *K-loop* or *Bruck loop* is a Bol loop with the automorphic inverse property. An overview of the most important theorems on K-loops and some of their relatives, especially Kikkawa loops, is given. First, left power alternative loops are discussed, then Kikkawa loops are considered. In particular, their nuclei are determined. Then the attention is paid to general K-loops and some special classes of K-loops such as 2-divisible ones.

To construct examples, the method of *derivation* is introduced. This has been used in the past to construct quasifields from fields. Many known methods to constructing loops can be seen as special cases of derivations. The examples given show the independence of various axioms.

Keywords: K-loop, Bol loop, Kikkawa loop, left power alternative loop, 2-divisible loop, derivation

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Introduction

K-loops came about as the additive loops of neardomains. The latter have been introduced by Karzel [14] to study sharply 2-transitive groups. Kerby and Wefelscheid have extracted from this the notion of a K-loop.[1] Indeed, from [36, V §1] it is seen that the additive loop of a neardomain is a Kikkawa loop with the additional identity $\delta_{a,b} = \delta_{a,ba}$. Kist [19, (1.8), p. 13] was probably the first who realized that K-loops in the above sense are Bol loops, and are thus Bruck loops (without 2-divisibility, though). Later Kreuzer [22] showed that the two notions are equivalent. Only then it was clear that the older notion of a Bruck loop is the same as the notion of a K-loop. We aim to give a coherent presentation of the most important theorems on K-loops and some relatives, in particular Kikkawa loops.

After some remarks on left inverse property loops, we discuss left power alternative loops in §2. Kikkawa loops are introduced in §3. They were called *symmetric loops* by Kikkawa [17], because of the role they play in the study of symmetric spaces. It turns out that with Kikkawa's topological hypotheses they are actually K-loops. We will show that a few well-known results for K-loops already hold for Kikkawa loops. In §4 we determine the nuclei of Kikkawa loops. §5 collects some basic properties of Bol loops. Up to this point we deal

[1] The "K" is used in honor of Karzel.

with (logical) ancestors of K-loops. §6 is devoted to K-loops in general, and some special classes of K-loops. We give some characterizations under hypotheses such as 2-divisibility.

The first published source, where the name K-loop has been used is [33]. UNGAR showed in this paper that the relativistic velocity addition forms a K-loop, providing an interesting example, which gave a strong impetus to the subject.

Derivation, as introduced in §7, is a method which modifies the multiplication in a group. It turned out to be useful for constructing various examples and counterexamples. This method is well-known as a tool for constructing quasifields from fields, see [15] and [16]. It has also been used, probably unconsciously, for the construction of loops. Finally, in §8, we put derivations to use. In particular, we give examples showing the independence of various axioms.

1. Preliminaries

A set L with a binary operation and an identity 1 will be called a *groupoid*. For $a \in L$ the *left translation* is given by $\lambda_a : x \mapsto ax$. If an element $a \in L$ has a single (left and right) inverse, then we say a has a *unique* inverse. This will be denoted by a^{-1}. If all elements of L have unique inverses, we will have use for the map $\iota : x \mapsto x^{-1}$, which is clearly an involution in this case.

For $a \in L$, $n \in \mathbf{N}$, we put recursively

$$a^0 := 1,\ a^n := a(a^{n-1}) \ \text{ and } \ a^{-n} := (a^{-1})^n \ \text{ if } a^{-1} \text{ exists.}$$

This clearly implies $\lambda_a^n(1) = a^n$.

Since groupoids are in general non-associative, we will have to use lots of parentheses. To save a few, we shall adopt the well-known *dot-convention*:

$$a \cdot bc = a(bc),\ ab \cdot c = (ab)c,\ a^k b = (a^k)b,\ ba^k = b(a^k) \ \text{ for } a, b, c \in L,\ k \in \mathbf{Z}.$$

An element a from a groupoid L is called

left regular if λ_a is injective;
left alternative if $a \cdot ab = a^2 b \ \ \forall\, b \in L$;
left power alternative if a and λ_a have a unique inverse and $\lambda_a^k = \lambda_{a^k},\ \ \forall\, k \in \mathbf{Z}$.

It is said to have the

left inverse property if there exists $a' \in L$ with $a' \cdot ab = b \ \ \forall\, b \in L$.

If every element of a groupoid L has one of the above properties, then the corresponding phrase will also be used for L. Moreover, we say that L is a *left (right) loop* if there is a unique solution $x \in L$ of the equation $ax = b$ ($xa = b$) for all $a, b \in L$; and that L is a *loop* if it is a left and a right loop (recall that $1 \in L$).

In a left loop L, all the left translations λ_a are bijective. Therefore, it makes sense to define for all $a, b \in L$ the *precession map*

$$\delta_{a,b} := \lambda_{ab}^{-1} \lambda_a \lambda_b.$$

Obviously, these mappings are characterized by the property

$$a \cdot bx = ab \cdot \delta_{a,b}(x) \quad \text{for all } x \in L.$$

Furthermore, let $\mathcal{D} = \mathcal{D}(L)$ be the *left inner mapping group*, which is generated by all precession maps $\delta_{a,b}$, $a, b \in L$.

The following lemmas gives an important characterization of the left inverse property. The material is taken from [5, VII.1] and [23, (2.6)]. Proofs are straightforward.

(1.1). *Let L be a groupoid. The following are equivalent:*

(I) *L satisfies the left inverse property;*
(II) *$\forall\, a \in L$ there exists a (unique) inverse $a^{-1} \in L$ and $\lambda_{a^{-1}} = \lambda_a^{-1}$;*
(III) *$\forall\, a \in L : \lambda_a^{-1} \in \lambda(L)$;*
(IV) *L is a left loop, and $\forall\, a \in L$ there exists $a' \in L$ with $a'a = 1$ and $\delta_{a',a} = \mathbf{1}$;*
(V) *L is a left loop, and $\forall\, a, a' \in L$ with $aa' = 1$ we have $\delta_{a,a'} = \mathbf{1}$;*
(VI) *L is a left loop with unique inverses, and $\forall\, a, b \in L$ we have $(ab)^{-1} = \delta_{a,b}(b^{-1}a^{-1})$.*

(1.2). *Let L be a left loop and $a \in L$. Then a satisfies the left inverse property if and only if there exists $a' \in L$ with $\lambda_{a'} = \lambda_a^{-1}$. If this is the case, then a has a unique inverse $a' = a^{-1}$, and for all $b \in L$ we have $\delta_{a^{-1},ab}\delta_{a,b} = \mathbf{1}$.* □

2. Left power alternative loops

The notion of left power alternativity and the basic content of the following lemma (but with stronger hypothesis) can be found in [17, Proposition 1.11]. The other are straightforward.

(2.1). *Let L be a left loop, and $a \in L$ with unique inverse a^{-1}. The following are equivalent:*

(I) *a is left power alternative;*
(II) *$\forall\, k, \ell \in \mathbf{Z}, x \in L : a^k \cdot a^\ell x = a^{k+\ell}x$;*
(III) *$\forall\, k, \ell \in \mathbf{Z} : \delta_{a^k,a^\ell} = \mathbf{1}$;*
(IV) *$\forall\, k \in \mathbf{Z} : \delta_{a,a^k} = \mathbf{1}$;*
(V) *$\forall\, k \in \mathbf{Z} : \delta_{a^k,a} = \mathbf{1}$, and $\delta_{a^k,a^{-1}} = \mathbf{1}$.*

In particular, a is contained in a cyclic subgroup of L, a is alternative and satisfies the left inverse property.
Moreover, if L is finite, then the order[2] $|a|$ of a divides $|L|$, the order of L. □

As indicated in the footnote to the previous theorem, we can use the notion of the *order* of an element in a left power alternative loop in a reasonable manner. Likewise, it makes sense to speak of the *exponent* of such a loop.

[2]Since a is contained in a subgroup, this notion can be taken from group theory.

(2.2). *Let L be a left loop with left inverse property. If $\delta_{a,a^k} = \mathbf{1}$ for all $a \in L$, $k \in \mathbf{N}(!)$, then L is left power alternative.*

PROOF: Let $a \in L$. In view of (2.1) (IV), all we need to show is that $\delta_{a,a^{-k}} = \mathbf{1}$ for all $k \in \mathbf{N}$. Using the left inverse property and then (1.1) we can compute

$$aa^{-k} = (a^{-1})^{-1} \cdot a^{-1}(a^{-1})^{k-1} = (a^{-1})^{k-1}$$

$$\text{thus } \delta_{a,a^{-k}} = \left(\delta_{a^{-1},(a^{-1})^{k-1}}\right)^{-1} = \mathbf{1}. \qquad \square$$

3. Kikkawa loops

Let L be a left loop. L is said to be a left A_ℓ-loop if the left inner mappings are all automorphisms of L. If L is a loop, then we speak of an A_ℓ-loop. L satisfies the *automorphic inverse property* if all $a, b \in L$ have unique inverses, and $(ab)^{-1} = a^{-1}b^{-1}$, i.e., $\iota \in \mathrm{Aut}\, L$.

A permutation α of a left loop is an automorphism if and only if $\alpha\lambda_a\alpha^{-1} = \lambda_{\alpha(a)}$ for all $a \in L$. Therefore, $\alpha\delta_{a,b}\alpha^{-1} = \delta_{\alpha(a),\alpha(b)}$ for all $\alpha \in \mathrm{Aut}\, L$, $a, b \in L$. This will be used freely in the proof of the following lemma.

A left A_ℓ-loop with the left and automorphic inverse property will be called a *left Kikkawa loop*, and a *Kikkawa loop* if it is a loop.

(3.1). *Let L be a left loop with left inverse property.*

(1) *If $\lambda_{ab}^2 = \lambda_a\lambda_b^2\lambda_a$ for all $a, b \in L$, then L satisfies the automorphic inverse property.*

(2) *If L satisfies the automorphic inverse property, then for all $a, b \in L$ the following are equivalent:*

(I) *the map $\iota : x \mapsto x^{-1}$ commutes with $\delta_{a,b}$, i.e., $\iota\delta_{a,b} = \delta_{a,b}\iota$;*

(II) $\delta_{a,b} = \delta_{a^{-1},b^{-1}}$;

(III) $\lambda_{ab}^2 = \lambda_a\lambda_b^2\lambda_a$.

Furthermore, these conditions imply $ab = \delta_{a,b}(ba)$.

(3) *If L is a left A_ℓ-loop, then for all $a, b \in L$ we have*

$$\delta_{a,b} = \delta_{b^{-1},a^{-1}}^{-1} = \delta_{b,b^{-1}a^{-1}}\,.$$

(4) *If L is a left Kikkawa loop, then the identities in (2) are satisfied for all $a, b \in L$, and*

$$\delta_{a,b}^{-1} = \delta_{b,a} \quad \textit{for all } a, b \in L.$$

PROOF: (1) [17, Proposition 1.13], (3) [17, Lemma 1.8], and (4) is easy.

(2) (I) $\Longleftrightarrow$ (II): ι is an automorphism of L, therefore

$$\iota\delta_{a,b}\iota = \delta_{\iota(a),\iota(b)} = \delta_{a^{-1},b^{-1}}.$$

This is equal to $\delta_{a,b}$ if and only if ι commutes with $\delta_{a,b}$, since ι is an involution.
(II) $\Longleftrightarrow$ (III): Using the left and automorphic inverse property, we obtain

$$\begin{aligned}\delta_{a,b} = \delta_{a^{-1},b^{-1}} &\iff \lambda_{ab}^{-1}\lambda_a\lambda_b = \lambda_{a^{-1}b^{-1}}^{-1}\lambda_{a^{-1}}\lambda_{b^{-1}} = \lambda_{ab}\lambda_a^{-1}\lambda_b^{-1}\\ &\iff \lambda_{ab}^2 = \lambda_a\lambda_b^2\lambda_a\,.\end{aligned}$$

By (1.1) and the automorphic inverse property, we have

$$\iota(ab) = \delta_{a,b}\big(\iota(b)\iota(a)\big) = \delta_{a,b}\iota(ba) = \iota\delta_{a,b}(ba), \quad \text{hence} \;\; ab = \delta_{a,b}(ba). \qquad \square$$

Remarks. 1. In the proof of "(I) $\Longleftrightarrow$ (II)", the left inverse property is not really needed, it suffices to have unique inverses.

2. Later we will construct examples of left Kikkawa loop, which are not loops.

The following lemma is due to Kreuzer [20, (1.4)]. His proof works for our more general statement.

(3.2). *Let L be a loop which satisfies the identity $(ab)^2 = a \cdot b^2 a$ for all $a, b \in L$. Then the map $\kappa : L \to L;\;\; x \mapsto x^2$ is injective if and only if L contains no elements of order 2, i.e., $a^2 = 1 \Longrightarrow a = 1$ for all $a \in L$.*

In particular, the general hypothesis is fulfilled if L is a left alternative Kikkawa loop. $\square$

Remark. 1. A group satisfies the identity $(ab)^2 = a \cdot b^2 a$ if and only if it is commutative. Of course, for commutative groups the conclusion of the lemma is well-known. It is even true for periodic groups, i.e., groups such that every element is of finite order.

2. In a Kikkawa loop the identity $(ab)^2 = a \cdot b^2 a$ is equivalent with the left alternative property.

4. The nuclei of Kikkawa loops

In every groupoid L one can define the *left, middle, right nucleus* of L, respectively,

$$\begin{aligned}\mathcal{N}_\ell(L) &:= \{a \in L; \forall\, x, y \in L : a \cdot xy = ax \cdot y\},\\ \mathcal{N}_m(L) &:= \{a \in L; \forall\, x, y \in L : x \cdot ay = xa \cdot y\},\\ \mathcal{N}_r(L) &:= \{a \in L; \forall\, x, y \in L : x \cdot ya = xy \cdot a\}.\end{aligned}$$

These are obviously semigroups with 1 (i.e., they are associative groupoids). For convenience, we shall drop the arguments if they are clear from the context, writing briefly $\mathcal{N}_\ell$, $\mathcal{N}_m$, $\mathcal{N}_r$. If L is a left loop, notice that

$$a \in \mathcal{N}_r \iff \delta_{x,y}(a) = a \;\; \text{for all } x, y \in L.$$

This will be useful occasionally.

We shall now give a proof of the fact that $\mathcal{N}_\ell = \mathcal{N}_m$ for left inverse property loops. This seems to have been proved first by Artzy [2, Corollary 2], but we were unable to find a direct algebraic proof in the literature.

(4.1). *Let L be a left regular groupoid.*

(1) *If $a \in \mathcal{N}_\ell$ has a right inverse $a' \in L$, then $a' \in \mathcal{N}_\ell$.*

(2) *If L satisfies the left inverse property, then $\mathcal{N}_\ell = \mathcal{N}_m$ is a group.*

Proof: (1) For $x, y \in L$ we have $a(a' \cdot xy) = aa' \cdot xy = xy = (a \cdot a'x)y = a(a'x \cdot y)$. Canceling a on the leftmost and rightmost expression completes the proof.

(2) Let $b, c \in L$. If $a \in \mathcal{N}_m$, then $ba \cdot a^{-1}b^{-1} = b(a \cdot a^{-1}b^{-1}) = bb^{-1} = 1$. Therefore

$$(ba)^{-1} = a^{-1}b^{-1}.$$

Let $b_1 \in L$ with $(b_1 a)^{-1} = b$, then $a^{-1}b_1^{-1} = b$. Therefore, we can compute

$$\begin{aligned} a \cdot bc = a \cdot (b_1 a)^{-1}c &= b_1^{-1}\Big(b_1\big(a \cdot (b_1 a)^{-1}c\big)\Big) \\ &= b_1^{-1}\big(b_1 a \cdot (b_1 a)^{-1}c\big) = b_1^{-1}c = (a \cdot a^{-1}b_1^{-1})c = ab \cdot c, \end{aligned}$$

so $a \in \mathcal{N}_\ell$.

For the converse, let $a \in \mathcal{N}_\ell$. By (1.1), the hypothesis of (1) is applicable. Thus $a^{-1} \in \mathcal{N}_\ell$, and $\mathcal{N}_\ell$ is a group. In a similar way as above we obtain $(ba)^{-1} = a^{-1}b^{-1}$. Now

$$ba \cdot c = ba(a^{-1} \cdot ac) = ba\big(a^{-1} \cdot b^{-1}(b \cdot ac)\big) = ba\big(a^{-1}b^{-1} \cdot (b \cdot ac)\big) = b \cdot ac,$$

hence $a \in \mathcal{N}_m$. □

Remarks. 1. Using isotopy theory one can give a very short proof for this.

2. It is not hard to see that $\mathcal{N}_\ell$ and $\mathcal{N}_m$ are groups for every left loop. They are not necessarily equal, though.

The *center* of a groupoid is defined by

$$\mathcal{Z}(L) := \{a \in \mathcal{N}_\ell \cap \mathcal{N}_m \cap \mathcal{N}_r;\ \forall\, x \in L : ax = xa\}.$$

For simplicity, the definition has been kept symmetric. In fact, it suffices to take elements from the intersection of any two of the nuclei to define the center. We make this explicit in one case.

(4.2). *Let L be a groupoid, then $\mathcal{Z}(L) = \{a \in \mathcal{N}_\ell \cap \mathcal{N}_m; \forall\, x \in L : ax = xa\}$.*

Proof: Let $a \in \mathcal{N}_\ell \cap \mathcal{N}_m$ with $ax = xa$ for all $x \in L$, then we can compute

$$xy \cdot a = a \cdot xy = ax \cdot y = xa \cdot y = x \cdot ay = x \cdot ya, \quad \text{for all } x, y \in L,$$

thus $a \in \mathcal{N}_r$. □

It should be emphasized that instead of $\mathcal{N}_\ell \cap \mathcal{N}_m$ also $\mathcal{N}_\ell \cap \mathcal{N}_r$ as well as $\mathcal{N}_m \cap \mathcal{N}_r$ qualify, with very similar proofs.

Let $\mathcal{M}_\ell$ denote the left multiplication group, generated by all the λ_a.

(4.3). *Let a be an element of a left loop L.*
(1) *If $a \in \mathcal{Z}(L)$, then λ_a centralizes $\mathcal{M}_\ell$.*
(2) *If λ_a centralizes $\mathcal{D}(L)$, then $a \in \mathcal{N}_r$.*
(3) *If L is A_ℓ, then λ_a centralizes $\mathcal{D}(L)$ if and only if $a \in \mathcal{N}_r$.*

PROOF: (1) $\lambda_a\lambda_x = \lambda_{ax} = \lambda_{xa} = \lambda_x\lambda_a$ for $x \in L$.
(2) For all $x, y \in L$ we have $\lambda_a\delta_{x,y} = \delta_{x,y}\lambda_a$. Apply this to 1 to see $\delta_{x,y}(a) = a$, hence $a \in \mathcal{N}_r$.
(3) Let $a \in \mathcal{N}_r$. For $\delta \in \mathcal{D}(L)$ we have $\delta(ax) = \delta(a)\delta(x) = a\delta(x)$. Therefore $\delta\lambda_a = \lambda_a\delta$. □

Remarks. 1. It can be shown by example that if λ_a centralizes $\mathcal{M}_\ell$, then $ax = xa$ for all $x \in L$, but a is not necessarily in $\mathcal{Z}(L)$. Indeed, there exists a loop L of order 16 with trivial center such that $\lambda(L) \cap \mathcal{Z}(\mathcal{M}_\ell) \neq \{\mathbf{1}\}$. A (computer generated) example can be obtained from the author.

2. This phenomenon does not occur in the full multiplication group $\mathcal{M}$. Then λ_a (as well as ρ_a) centralizes $\mathcal{M}$ if and only if $a \in \mathcal{Z}(L)$, see [1, Theorem 11]. This has the consequence that isotopic loops have isomorphic centers, [1, Theorem 12].

3. From (1) and (2) it is easy to see that L is a commutative group if and only if $\mathcal{M}_\ell(L)$ is a commutative group (see also [18, Lemma 2]).

The following applies in particular to K-loops, as we shall see later.

(4.4) Theorem. *Let L be a Kikkawa loop, then we have*
(1) $\mathcal{Z}(L) = \mathcal{N}_\ell = \mathcal{N}_m \subseteq \mathcal{N}_r$.
(2) *If $\mathcal{D}(L)$ acts fixed point free on $L \setminus \{1\}$, then $\mathcal{Z}(L) = \mathcal{N}_\ell = \mathcal{N}_m = \mathcal{N}_r = \{1\}$, or $L = \mathcal{Z}(L)$ is an abelian group.*

PROOF: (1) For $a \in \mathcal{N}_\ell$ we clearly have $\delta_{a,x} = \mathbf{1}$. Now,

$$ax = \delta_{a,x}(xa) = xa \quad \text{by (3.1.2) and (3.1.4).}$$

Since also $a \in \mathcal{N}_m$ by (4.1), the result comes from (4.2).

(2) follows directly from (1), since either $\mathcal{D}(L) = \{\mathbf{1}\}$ or the fixed set of $\mathcal{D}(L)$, which equals $\mathcal{N}_r$, is trivial. □

5. Bol loops

A groupoid is called *Bol* if $a(b \cdot ac) = (a \cdot ba)c$ for all $a, b, c \in L$. Plugging in $b = 1$ shows the left alternative law. ROBINSON [27] (see also [26, IV.6.5, p. 114]) showed that Bol loops are left power alternative.

We begin with an almost trivial observation.

(5.1). *For a groupoid L the following are equivalent:*
(I) *L is Bol;*
(II) $\lambda_a\lambda_b\lambda_a = \lambda_{a\cdot ba}$ *for all $a, b \in L$;*
(III) $\lambda_a\lambda(L)\lambda_a \subseteq \lambda(L)$ *for all $a \in L$.* □

The following lemma will be useful, when we construct examples later.

(5.2). *Let L be a Bol groupoid.*

(1) *If every $a \in L$ has a right inverse a^ρ, then a^ρ is also a left inverse of a.*

(2) *If L is left regular and every element a has a left inverse a^{-1}, then L is a Bol loop. In particular, L satisfies the left inverse property, and for all $a, b \in L$, we have*

$$ax = b \iff x = a^{-1}b \quad \text{and} \quad ya = b \iff y = a^{-1}(ab \cdot a^{-1}).$$

(3) *If L is a Bol loop and U a non-empty subset of L subject to the condition $\forall\, a, b \in U : ab \in U, a^{-1} \in U$, then U is a Bol subloop of L.*

Proof: Combine (1.1), the proofs of [31, Theorems 1, 2], and some straightforward calculations.

□

Remarks. 1. The theorem implies in particular that a left loop with Bol is a Bol loop.

2. Variations of this result are [23, (2.13)], and under even stronger hypothesis [11, Lemma 2]. In [8] (see also [28]) it is shown that a Bol quasigroup always has a right identity.

3. The solution y from (2) in the theorem occurs in [3, VI.6.8, p. 106] and [11, Lemma 2].

4. We were unable to decide whether the hypothesis "left regular" in (2) of the theorem is dispensable. It can be replaced by the condition, that all λ_a are surjective: For $x \in L$ choose $c \in L$ with $ac = x$. Then $a \cdot a'x = a(a' \cdot ac) = (a \cdot a'a)c = x$, and $\lambda_a \lambda_{a'} = \mathbf{1}$, hence λ_a is also injective.

5. It can be shown that in every Bol groupoid L we have $\lambda_{a^k} = \lambda_a^k$ for all $a \in L$, $k \in \mathbf{N}$.

6. K-loops

A Bol loop which satisfies the automorphic inverse property is called a *K-loop* or sometimes a *Bruck-loop*.[3]

We begin with the theorem of Kreuzer [22, 3.4], which shows that the formerly used definition of a K-loop (second part in the following theorem) is equivalent with ours.

(6.1) Theorem. *A groupoid L is a K-loop if and only if L is a Kikkawa loop, and $\delta_{a,b} = \delta_{a,ba}$ for all $a, b \in L$.*

Proof: A Bol loop has the left inverse property. If the automorphic inverse property holds, then L is an A_ℓ-loop by [22, 3.3], therefore L is a Kikkawa loop.

[3]The literature is not consistent in the definition of Bruck loop. Many authors require unique 2-divisibility.

Now it is easy to see that the Bol identity is equivalent to $\delta_{a,ba} = \delta_{b,a}^{-1}$, hence (3.1.4) implies

$$\delta_{a,ba} = \delta_{b,a}^{-1} = \delta_{a,b}.$$

The converse is even easier using (3.1.4) again. □

Remarks. 1. The crucial step in the proof is the fact that a K-loop is A_ℓ. This fact has been proved in four papers recently. FUNK and NAGY in [10, 5.1] use a geometric argument in the corresponding net. KREUZER [22] gives a direct proof, avoiding the use of pseudoautomorphisms. GOODAIRE and ROBINSON show that $\delta_{a,b}$ is a pseudoautomorphism with companion $ab \cdot a^{-1}b^{-1}$ in [12, 3.12]. This can also be derived from some remarks at the end of [30]. SABININ and SBITNEVA's proof is reproduced in [29]. All this immediately implies the A_ℓ-property for K-loops. GOODAIRE and ROBINSON's result in turn generalizes [5, VII Lemma 2.2, p. 113] for Moufang loops.

2. In various papers (e.g., [34], [35]) UNGAR uses axioms which define a Kikkawa loop together with the identity $\delta_{ab,b} = \delta_{a,b}$. He calls such structures *gyrogroups*.[4] In [29] it is shown that this last identity is equivalent with the Bol property. (A particularly simple proof results as an application of the last statement in (1.1).) Therefore gyrogroups are K-loops, as well.

The "(I) $\Longrightarrow$ (III)" of the following lemma is [11, Lemma 1]. GLAUBERMAN [11] attributes the converse to ROBINSON.

(6.2). *Let L be a Bol loop.*

(1) *The following are equivalent:*

(I) *L is a K-loop;*
(II) *$\lambda_{ab}^2 = \lambda_a \lambda_b^2 \lambda_a$ for all $a, b \in L$;*
(III) *$(ab)^2 = a \cdot b^2 a$ for all $a, b \in L$.*

(2) *If L is a K-loop, then the map $\kappa : x \mapsto x^2$ is injective if and only if L contains no elements of order 2.*

PROOF: (1) (6.1) and (3.1) show that (I) and (II) are equivalent.

(II) $\Longrightarrow$ (III): Applying both sides to 1 and using the fact that Bol loops are left alternative gives the claimed identity.

(III) $\Longrightarrow$ (II): We use the left alternative property and (5.1) to compute

$$\lambda_{ab}^2 = \lambda_{(ab)^2} = \lambda_{a \cdot b^2 a} = \lambda_a \lambda_{b^2} \lambda_a = \lambda_a \lambda_b^2 \lambda_a.$$

(2) is a direct consequence of (3.2) and (III). □

If the squaring map behaves well, we get a much stronger result.

[4] In older papers UNGAR uses the phrase "weakly associative group".

(6.3) Theorem. *Let L be a left regular groupoid with right inverses.*

(1) *For the following conditions, we have (I) $\Longrightarrow$ (II) $\Longrightarrow$ (III) $\Longrightarrow$ (IV).*

(I) *L is a K-loop;*

(II) *L is a left alternative Kikkawa loop;*

(III) *L is left alternative, satisfies the left inverse property, the automorphic inverse property, and A_ℓ;*

(IV) $\forall\, a, b \in L : \lambda_{ab}^2 = \lambda_a \lambda_{b^2} \lambda_a$.

(2) *If the map $L \to L; x \mapsto x^2$ is surjective, then the preceding conditions are equivalent.*

Proof: (1) "(I) $\Longrightarrow$ (II)" is direct from (6.1), "(II) $\Longrightarrow$ (III)" is trivial.

(III) $\Longrightarrow$ (IV): By (1.1), L is a left loop, hence (3.1.4) is applicable, and yields together with the left alternative property

$$\lambda_{ab}^2 = \lambda_a \lambda_b^2 \lambda_a = \lambda_a \lambda_{b^2} \lambda_a\,.$$

(2) (IV) $\Longrightarrow$ (I): Putting $a = 1$, we obtain $\lambda_b^2 = \lambda_{b^2}$ for all $b \in L$, i.e., L is left alternative. Let now $a, b \in L$. By assumption there exists $d \in L$ with $d^2 = b$. So we can compute

$$\lambda_a \lambda_b \lambda_a = \lambda_a \lambda_{d^2} \lambda_a = \lambda_{ad}^2 = \lambda_{(ad)^2} \in \lambda(L).$$

In view of (5.1) and (5.2), we concluded that L is a Bol loop and satisfies the left inverse property. Since L is left alternative, $\lambda_{ab}^2 = \lambda_a \lambda_{b^2} \lambda_a = \lambda_a \lambda_b^2 \lambda_a$. Now we can invoke (3.1.1) or (6.2.1) to conclude that L satisfies the automorphic inverse property. □

Remarks. 1. Part (2) of this theorem and its proof have been compiled from [25, XII.3.29, 3.34, 3.35]. The topological hypotheses used there imply 2-divisibility. This is all that is really needed.

2. (IV) is equivalent to either of the following

$$\forall\, a, b \in L : \lambda_{(ab)^2} = \lambda_a \lambda_{b^2} \lambda_a\,; \qquad \forall\, a, b \in L : \lambda_{(ab)^2} = \lambda_a \lambda_b^2 \lambda_a\,.$$

Indeed, putting $b = 1$, or $a = 1$, respectively, it is seen that both conditions imply that L is left alternative. Then the claimed equivalence is obvious.

3. The identity $\lambda_{ab}^2 = \lambda_a \lambda_b^2 \lambda_a$ for all $a, b \in L$ from (6.2.1) is true in every Kikkawa loop (see (3.1.4)). By the example in (8.2.3) a Kikkawa loop need not be left alternative. Therefore this identity is not equivalent to the identities of the previous remark.

4. Kreuzer [21, (3.5)] shows by examples that the implication "(II) $\Longrightarrow$ (I)" is not true in general. Indeed, later we present his construction of a left power alternative Kikkawa loop, which is not a Bol loop.

5. Likewise, we will construct a left alternative left Kikkawa loop, which is not a loop, showing that the implication "(III) $\Longrightarrow$ (II)" fails in general.

6. If L is a Lie loop, then the hypothesis in (II) that L be left alternative is redundant, see [17, Lemma 6.2]. Notice that from (3.1.4) one easily gets that $\delta_{a,a}^2 = \mathbf{1}$ for all $a \in L$. If L is connected, then it can also be seen that the square map is surjective. Hence KIKKAWA's connected, symmetric Lie loops are in fact K-loops (see also [25, XII.3.34]).

A left power alternative left loop L will be called (*uniquely*) *n-divisible*, $n \in \mathbf{N}$, if for every $a \in L$ there exists (exactly one) $b \in L$ with $b^n = a$. Notice that by (2.1) a, b are contained in an abelian (in fact cyclic) subgroup of L, so this notion coincides with the standard definition [9, §20]. In particular, we can write $b = a^{\frac{1}{n}}$ if L is uniquely n-divisible. In this case it also makes sense to write $a^{\frac{k}{n}}$ for every $k \in \mathbf{Z}$. Abusing language, we shall say that "$a^{\frac{k}{n}}$ is well-defined". Finally, note that a loop is n-divisible if and only if the map $x \mapsto x^n$ is surjective, it is uniquely n-divisible if and only if the map $x \mapsto x^n$ is bijective.

(6.4). *Let L be a left power alternative left A_ℓ-loop such that for a fixed rational number q the map $L \to L$; $x \mapsto x^q$ is well-defined. Then $a(a^{-1}b)^q = b(b^{-1}a)^{1-q}$ for all $a, b \in L$.*

PROOF: Using (1.1) and (2.1), we can compute

$$b^{-1} \cdot a(a^{-1}b)^q = b^{-1}a \cdot \delta_{b^{-1},a}(a^{-1}b)^q = b^{-1}a \cdot (b^{-1}a)^{-q} = (b^{-1}a)^{1-q}.$$

Multiplying by b on both sides gives the result. □

(6.5) Theorem. *Let L be a uniquely 2-divisible left power alternative left A_ℓ-loop, and let ϵ be an involutory, fixed point free automorphism of L. Then $\epsilon(x) = x^{-1}$ for all $x \in L$, and L is a K-loop.*

PROOF: By (6.4) we find

$$\epsilon\left(x\bigl(x^{-1}\epsilon(x)\bigr)^{\frac{1}{2}}\right) = \epsilon(x)\bigl(\epsilon(x)^{-1}x\bigr)^{\frac{1}{2}} = x\bigl(x^{-1}\epsilon(x)\bigr)^{1-\frac{1}{2}} = x\bigl(x^{-1}\epsilon(x)\bigr)^{\frac{1}{2}}.$$

Therefore the assumption implies $x\bigl(x^{-1}\epsilon(x)\bigr)^{\frac{1}{2}} = 1$. Using the left inverse property we get $x^{-1}\epsilon(x) = x^{-2}$, and then $\epsilon(x) = x^{-1}$. This shows that L has the automorphic inverse property. Since 2-divisible simply means that the square map $x \mapsto x^2$ is surjective, we can conclude from (6.3) that L is a K-loop. □

Remarks. 1. The proof of (6.4) only uses that the maps $x \mapsto x^q$ and $\delta_{b^{-1},a}$ commute. A_ℓ is not really needed.

2. The theorem and its proof, including the preceding lemma, are due to KIST [19, (1.2.e), (1.4.b)]. We have only modified the context to obtain the presented generalization.

(6.4) has another simple consequence.

(6.6). *Let L be a uniquely 2-divisible left power alternative A_ℓ-loop, then $\delta_{a,b} \neq \iota$ for all $a, b \in L$.*

Proof: Assume the contrary, then

$$ab^{\frac{1}{2}} = ab^{1-\frac{1}{2}} = ab \cdot \iota\left(b^{-\frac{1}{2}}\right) = ab \cdot b^{\frac{1}{2}}.$$

Canceling $b^{\frac{1}{2}}$ and then a implies $b = 1$, a contradiction. □

7. Derivations

The method of derivation it is due to Dickson. It has been used in the past to construct nearfields and later quasifields from fields and skewfields.[5] Karzel [13] axiomatized this method for groups replacing the skewfield. For an even more general setting see [36, II.1 p. 66]. Here we give a generalization which applies to constructing loops.

Let G be a group. A map $\phi : G \to \operatorname{Aut} G;\ a \mapsto \phi_a$ with $\phi_1 = \mathbf{1}$ is called a *weak derivation*. It is called a *derivation* if furthermore for all $a, b \in G$ there exists a unique $x \in G$ such that

$$x\phi_x(a) = b.$$

We have

(7.1). *Let G be a group with a weak derivation ϕ. If we let $a \circ b := a\phi_a(b)$, then*

(1) *$G^\phi := (G, \circ)$ is a left loop. The identity elements of G and G^ϕ coincide. For all $a \in G$ the right inverse of a is given by $a' := \phi_a^{-1}(a^{-1})$, i.e., $a \circ a' = 1$.*

(2) *If ϕ is a derivation, then G^ϕ is a loop.*

Proof: (1) The unique solution of the equation $a \circ x = b$ is given by $x = \phi_a^{-1}(a^{-1}b)$.

(2) is obvious. □

G^ϕ is called the *derived (left) loop*.

Remarks. 1. It is not necessary to assume the image of ϕ to be in $\operatorname{Aut} G$. The symmetric group of G would do. Derivations in the present sense have also been called *automorphic*.

2. For weaker hypotheses in (7.1.2) if ϕ maps into a finite group, see [15, (2.6)].

3. See [16] for a description of nuclei and center of derived loops.

To distinguish powers in G and G^ϕ, we denote powers in G^ϕ by $a^{\underline{k}}$, $k \in \mathbf{Z}$, e.g., $a^{\underline{3}} = a \circ (a \circ a)$. We list a bunch of straightforward properties.

[5] For the most general approach and some historic remarks see [32].

(7.2). *Let G be a group and ϕ a weak derivation. We have*

(1) $\delta_{a,b} = \phi_{a\circ b}^{-1}\phi_a\phi_b$ *for $a, b \in G$. Therefore, $\mathcal{D}(G^\phi) \subseteq \operatorname{Aut} G$.*

(2) $\sigma \in \operatorname{Aut} G$ *is an automorphism of G^ϕ if and only if $\sigma\phi_a\alpha^{-1} = \phi_{\sigma(a)}$ for all $a \in G$.*

(3) G^ϕ *is left alternative if and only if $\phi_{a\underline{2}} = \phi_a^2$ for all $a \in G$.*

(4) G^ϕ *satisfies the left inverse property if and only if $\phi_{\phi_a^{-1}(a^{-1})} = \phi_a^{-1}$ for all $a \in G$. In this case $a^{\underline{-1}} = \phi_a^{-1}(a^{-1})$.*

(5) G^ϕ *is left power alternative if and only if $\phi_{a\underline{k}} = \phi_a^k$ for all $a \in G$ and for all $k \in \mathbf{Z}$.*

(6) G^ϕ *is a Bol loop if and only if $\phi_{a\circ(b\circ a)} = \phi_a\phi_b\phi_a$ for all $a, b \in G$. In this case, ϕ is a derivation.*

(7) G^ϕ *is a group if and only if $\phi_{a\circ b} = \phi_a\phi_b$ for all $a, b \in G$. In this case, ϕ is a derivation.*

PROOF: For $a, b, x \in G$, we compute

$$a\phi_a(b)\phi_a\phi_b(x) = a\phi_a\big(b\phi_b(x)\big) = a \circ (b \circ x) = (a \circ b) \circ \delta_{a,b}(x) = a\phi_a(b)\phi_{a\circ b}\delta_{a,b}(x).$$

Rearranging terms gives the result.

(2) For $a, b \in G$ we have $\sigma(a \circ b) = \sigma(a)\sigma\phi_a(b)$ and $\sigma(a) \circ \sigma(b) = \sigma(a)\phi_{\sigma(a)}\sigma(b)$. These are equal if and only if $\sigma\phi_a(b) = \phi_{\sigma(a)}\sigma(b)$. Hence the result.

(3) G^ϕ is left alternative if and only if $\delta_{a,a} = \mathbf{1}$ for all $a \in G$. Now (1) shows the assertion.

(4) For $a \in G$ put $a' := \phi_a^{-1}(a^{-1})$. Then G^ϕ satisfies the left inverse property if and only if $\delta_{a,a'} = \mathbf{1}$ for all $a \in G$, by (1.1). Now (7.1.1) and (1) show the result.

(5) Using (2.1) and (1), the result can be obtained easily.

(6) Let G^ϕ be a Bol loop, then by (1) we can compute

$$\mathbf{1} = \delta_{b,a}\delta_{a,b\circ a} = \phi_{b\circ a}^{-1}\phi_b\phi_a\phi_{a\circ(b\circ a)}^{-1}\phi_a\phi_{b\circ a} \Longrightarrow \phi_b\phi_a\phi_{a\circ(b\circ a)}^{-1}\phi_a = \mathbf{1}.$$

Rearranging terms gives the assertion.

Conversely, for $a, b, c \in G$ we find

$$a \circ \big(b \circ (a \circ c)\big) = a\phi_a(b)\phi_a\phi_b(a)\phi_a\phi_b\phi_a(c) = a \circ (b \circ a)\phi_{a\circ(b\circ a)}(c) = \big(a \circ (b \circ a)\big) \circ c.$$

(5.2) shows that G^ϕ is a Bol loop, and thus ϕ is a derivation.

(7) follows directly from (1). $\square$

Before we proceed with the theory, we give an example, which has been referred to in previous sections.

(7.3). *Let $(G,+)$ be an abelian group, and put $\phi_0 = \mathbf{1}$ and $\phi_a = -\mathbf{1}$ for all $a \in G \setminus \{0\}$, i.e., $\phi_a(x) = -x$. Then*

(1) *G^ϕ is a left alternative left Kikkawa loop of exponent 2.*

(2) *If there exists an element $a \in G$ with $2a \neq 0$, i.e., if G is not of exponent 2, then G^ϕ is not a loop. In this case $\mathcal{D}(G^\phi) = \{\pm\mathbf{1}\}$.*

Proof: (1) By (7.1.1) G^ϕ is a left loop. Clearly, $a \circ a = 0$ for all $a \in G$, hence (7.2.3) shows that G^ϕ is left alternative and of exponent 2. Since $a^{\underline{-1}} = a$ for all $a \in G$, the left and automorphic inverse properties are trivial. (7.2.1) and (7.2.2) show A_ℓ, because $-\mathbf{1}$ centralizes every automorphism of G, and $\phi_{-a} = \phi_a$ for all $a \in G$.

(2) The equation $x \circ a = a$ has two solutions, namely 0 and $2a$. Moreover, by (7.2.1), $\delta_{2a,a} = \phi_a^{-1}\phi_{2a}\phi_a = -\mathbf{1}$, because $2a \circ a = a$. $\square$

For an epimorphism $\eta : G \to \overline{G}$, $\overline{G}$ a group, let $A_\eta := \{\alpha \in \mathrm{Aut}\, G; \eta\alpha = \eta\}$. Before we show how to use this to construct derivations, we record

(7.4). *Let G be a group and A a subset of $\mathrm{Aut}\, G$. Let N be the normal subgroup in G generated by the set $\{g^{-1}\alpha(g); g \in G, \alpha \in A\}$, and V an arbitrary normal subgroup in G. Then the following are equivalent:*

- (I) *$N \subseteq V$;*
- (II) *V is A-invariant and the action of A induced on G/V is trivial;*
- (III) *for the canonical epimorphism $\eta : G \to G/V$ we have $A \subseteq A_\eta$.*

Proof: (I) $\Longrightarrow$ (II): If $\alpha \in A$, then for $g \in V$ we have $\alpha(g) \in gV = V$. Hence V is A-invariant. Furthermore, if $g \in G$, then $\alpha(g) \in gN \subseteq gV$. Thus $\alpha(gV) = gV$.

(II) $\Longrightarrow$ (III): For all $g \in G$, $\alpha \in A$ we have $\alpha(g)V = \alpha(gV) = gV$, thus $g^{-1}\alpha(g) \in V$. This implies $\eta\big(g^{-1}\alpha(g)\big) = 1$, and $\eta\alpha(g) = \eta(g)$. Hence $\alpha \in A_\eta$.

(III) $\Longrightarrow$ (I): For all $g \in G$, $\alpha \in A$:

$$\eta\big(g^{-1}\alpha(g)\big) = \eta(g^{-1})\eta\alpha(g) = \eta(g^{-1})\eta(g) = 1.$$

Therefore $g^{-1}\alpha(g) \in V$, and $N \subseteq V$. $\square$

The following construction gives many derivations.

(7.5) Theorem. *Let G, $\overline{G}$ be groups, and let $\eta : G \to \overline{G}$ be an epimorphism. For every map $\psi : \overline{G} \to A_\eta$ with $\psi_1 = \mathbf{1}$, we have*

(1) *$\phi := \psi\eta$ is a derivation, and for all $a \in G$ there exists a map $\mu_a : G \to \ker\eta$ such that $\phi_a(x) = x\mu_a(x)$.*

(2) *For all $a, b \in G$ and all $\alpha, \beta \in A_\eta$ we have $\phi_{\alpha(a)\beta(b)} = \phi_{ab} = \phi_{a\circ b}$. Moreover, for all $k \in \mathbf{N} : \phi_{a^{\underline{k}}} = \phi_{a^k}$.*

(3) *G^ϕ satisfies the left inverse property if and only if $\psi_{u^{-1}} = \psi_u^{-1}$ for all $u \in \overline{G}$ if and only if $\phi_{a^{-1}} = \phi_a^{-1}$ for all $a \in G$. In this case, $\phi_{\underline{a^{-1}}} = \phi_{a^{-1}}$.*

(4) *G^ϕ is a Bol loop if and only if $\psi(uvu) = \psi_u\psi_v\psi_u$ for all $u, v \in \overline{G}$.*

(5) *G^ϕ is a group if and only if ψ is a homomorphism.*

PROOF: (1) The condition $\psi_1 = \mathbf{1}$ makes sure that $\phi_1 = \mathbf{1}$, hence G^ϕ is a left loop. For $a, b \in G$, consider the equation $x\phi_x(a) = b$. Applying η to both sides, gives $\eta(x) = \eta(ba^{-1})$, and then $x = b\phi_{ba^{-1}}(a^{-1})$. This is indeed a solution, since

$$\begin{aligned}\phi_x = \phi\big(b\phi_{ba^{-1}}(a^{-1})\big) &= \psi\Big(\eta(b)\eta\big(\phi_{ba^{-1}}(a^{-1})\big)\Big) \\ &= \psi\big(\eta(b)\eta(a^{-1})\big) = \psi\eta(ba^{-1}) = \phi_{ba^{-1}}\,.\end{aligned}$$

Therefore $x = b\phi_{ba^{-1}}(a^{-1})$ is the unique solution of the equation in question.

For the last assertion, let $a, x \in L$. By (7.4), $x^{-1}\phi_a(x) \in \ker\eta$. This gives the result.

(2) $\phi_{\alpha(a)\beta(b)} = \psi\big(\eta\alpha(a)\eta\beta(b)\big) = \psi\big(\eta(a)\eta(b)\big) = \phi_{ab}$. Since $a \circ b = a\phi_a(b)$, the second equation now follows. This also implies the last statement.

(3) comes directly from (7.2.4).

(4) Assume $\psi(aba) = \psi_a\psi_b\psi_a$ for all $a, b \in \overline{G}$. Using (2) we can compute

$$\phi_{a\circ(b\circ a)} = \phi_{aba} = \psi\big(\eta(a)\eta(b)\eta(a)\big) = \psi\eta(a)\psi\eta(b)\psi\eta(a) = \phi_a\phi_b\phi_a.$$

By (7.2.6) G^ϕ is a Bol loop. The converse is a consequence of a similar calculation and again (7.2.6).

(5) From (7.2.7) and (2) the result can be deduced easily. □

Derivations constructed as in the theorem will be called *η-derivations* with *factorization* $\phi = \psi\eta$. Note that the factorization is not unique. The map $\mu : G \to (\ker\eta)^G$ is called the *obstruction* of ϕ corresponding to η. It is unique given η, and it factors through η, more precisely: There exists a map $\nu : \overline{G} \to (\ker\eta)^G$ such that $\mu = \nu\eta$. This will be called the *factorization* of μ. If G is abelian, then the μ_a are homomorphisms. This gives a way to construct η-derivations on abelian groups.

(7.6). *Let G, $\overline{G}$ be abelian groups, and let $\eta : G \to \overline{G}$ be an epimorphism with $U := \ker\eta$. Let $\nu : \overline{G} \to \mathrm{Hom}(G, U)$ be a map such that for all $a \in G$, $v \in \overline{G}$,*

$$\nu_1(a) = 1, \quad \text{and} \quad U \subseteq \ker\nu_v.$$

Put $\mu := \nu\eta$, and $\phi_a := \mathbf{1} + \mu_a$ for all $a \in G$ (i.e., $\phi_a(x) = x\mu_a(x)$ for all $x \in G$). Then ϕ is an η-derivation with corresponding obstruction μ. Moreover, we have

(1) *Let $a \in G$. If $\mu_a(a^{-1}) = 1$, then $a^{\underline{-1}} = a^{-1}$.*

(2) *If $a^{\underline{-1}} = a^{-1}$, for all $a \in G$, then G^ϕ satisfies the automorphic inverse property if and only if $\mu_a = \mu_{a^{-1}}$ for all $a \in G$.*

Proof: Let $a \in G$. Clearly, ϕ_a is an endomorphism of G. We first show that ϕ_a is bijective: For $b \in G$ put $x := b\mu_a(b^{-1})$. We compute

$$\begin{aligned}\phi_a(x) = x\mu_a(x) &= b\mu_a(b^{-1})\mu_a\big(b\mu_a(b^{-1})\big) = b\mu_a(b^{-1})\nu_a\big(\eta(b)\eta\mu_a(b^{-1})\big)\\ &= b\mu_a(b^{-1})\nu_a\big(\eta(b)\big) = b\mu_a(b^{-1})\mu_a(b) = b.\end{aligned}$$

Therefore ϕ_a is surjective.

Let $x \in G$ be in the kernel of ϕ_a, i.e., $1 = \phi_a(x) = x\mu_a(x)$. This implies

$$x = \mu_a(x)^{-1} \in U, \quad \text{and so} \quad \mu_a(x) = 1.$$

Thus ϕ_a has trivial kernel and is injective.

Now $\eta\phi_a = \eta(\mathbf{1} + \mu_a) = \eta + \eta\mu_a = \eta$. Therefore $\phi_a \in A_\eta$, and (7.5.1) shows the result.

(1) follows directly from (7.2.4).

(2) $a^{-1} \circ b^{-1} = a^{-1}\phi_{a^{-1}}(b^{-1}) = a^{-1}\phi_a(b^{-1}) = (a \circ b)^{-1}$. □

Remarks. 1. The group $\overline{G}$ does not play an essential role in the construction of η-derivations. It can always be replaced by $G/\ker\eta$.

2. A special case of η-derivations is the construction of André quasifields, see [24, §12] or [15, §3].

3. η-derivations have been used in [6] and [7] to construct Bol quasifields from fields.

Let G be an abelian group with a subgroup U and a map $\mu : G \mapsto \mathrm{Hom}(G, U)$. The pair (U, μ) will be called a *derivation sprout* on G if μ factors through the canonical epimorphism $\eta : G \to G/U$, i.e., there exists a map $\nu : G/U \to \mathrm{Hom}(G, U)$ such that $\mu = \nu\eta$. Moreover, we require that for all $a \in G$

$$\mu_1(a) = 1 \quad \text{and} \quad U \subseteq \ker\mu_a.$$

Notice that this is exactly what we looked at in the preceding theorem. Hence by $\phi_a := \mathbf{1} + \mu_a$ for all $a \in G$, we obtain a derivation, the *derivation corresponding to* (U, μ).

We remark that ϕ determines μ by (7.5.1), while U is not unique in general, i.e., there might be distinct subgroups U, U' of G such that (U, μ), (U', μ) are both derivation sprouts on G. The corresponding derivations are of course the same.

(7.7). *Let ϕ be an η-derivation with factorization $\psi\eta$ on a group G, and let $\sigma \in \mathrm{Aut}\, G$. We have*

(1) *If $\sigma \in A_\eta$, then $\sigma \in \mathrm{Aut}\, G^\phi \iff \forall\, a \in G : \sigma\phi_a\sigma^{-1} = \phi_a$, i.e., σ centralizes $\phi(G)$.*

(2) *If A_η is abelian, then $A_\eta \subseteq \mathrm{Aut}\, G^\phi$, and G^ϕ is an A_ℓ-loop.*

Proof: (1) The condition $\eta\sigma = \eta$ implies $\phi_{\sigma(a)} = \phi_a$. Now (7.2.2) shows the assertion.

(2) comes directly from (1) and (7.2.2). □

8. Examples

As a major application, we give a generalization of Kreuzer's construction [21, (3.5)] of a left power alternative Kikkawa loop, which is not Bol. By (6.3) such a loop cannot be 2-divisible. In fact we can give some more examples to show independence of various axioms.

Let G, H be (additively written) abelian groups with the following properties. Assume G has a subgroup T of index 2, and T contains an element t of order 2. Let α be the endomorphism of G with kernel T and image $\{0, t\}$. These properties determine α uniquely. More specific

$$\alpha : \begin{cases} G \to G \\ x \mapsto \begin{cases} 0 & \text{if } x \in T; \\ t & \text{if } x \in G \setminus T. \end{cases} \end{cases}$$

Groups G with these properties are easy to be found, in fact every finite abelian group with order divisible by 4 does the job. For a subset M of H with $0 \notin M$ put

$$\mu_{(a,b)} := \begin{cases} \alpha \times \mathbf{0} & \text{if } (a,b) \in T \times M; \\ \mathbf{0} \times \mathbf{0} & \text{if } (a,b) \in G \times H \setminus T \times M, \end{cases}$$

where $\mathbf{0}$ denotes the zero map. Notice that $\mu_{(a,b)}$ can be viewed as an element of $\operatorname{Hom}\bigl(G \times H, T \times \{0\}\bigr)$.

(8.1). *With notation as above, $(T \times \{0\}, \mu)$ is a derivation sprout on $G \times H$. Let ϕ be the corresponding derivation, then $L := (G \times H)^{\phi}$ is an A_{ℓ}-loop.*

(1) *L is a Kikkawa loop if and only if $M = -M$.*

(2) *L is left alternative if and only if $2b \notin M$ for all $b \in H$.*

(3) *L is a left power alternative Kikkawa loop if and only if $M = -M$ and for all $b \in H,\ n \in \mathbf{N}$,*

$$nb \in M \iff n \text{ is odd and } b \in M.$$

(4) *L is a Bol loop if and only if $2H + M = M$. These conditions imply that L is a K-loop.*

(5) *L is a group if and only if L is commutative if and only if $M = \emptyset$.*

Proof: The construction guarantees that $(T \times \{0\}, \mu)$ is a derivation sprout.

The obvious facts that $2\alpha = \alpha^2 = \mathbf{0}$, clearly imply $|\mathbf{1} + \alpha \times \mathbf{0}| = 2$. From this we derive three useful properties of ϕ:

(i) $\quad \phi_x = \phi_x^{-1}, \quad \delta_{x,y} = \phi_{x+y}\phi_x\phi_y$ and $\phi_x\phi_y = \phi_y\phi_x$ for all $x, y \in L$.

Indeed, the first statement is obvious. The second comes from (7.2.1), (7.5.2) and the first. For the third statement we compute $\phi_x\phi_y\phi_x = (\mathbf{1}+\mu_x)(\mathbf{1}+\mu_y)(\mathbf{1}+\mu_x) = \mathbf{1} + \mu_y = \phi_y$, since $\mu_x\mu_y = \mathbf{0} = \mu_y\mu_x$ in all cases. By (7.7.1) L is an A_{ℓ}-loop.

(1) Assume $M = -M$. For $(a,b) \in L$ we have

$$\mu_{(a,b)}(-a,-b) = (0,0), \quad \text{since} \quad \alpha(-a) = 0 \quad \text{if} \quad a \in T.$$

From (7.6.1) we get $(a,b)^{\underline{-1}} = -(a,b)$. The hypotheses ensure us that

$$-(T \times M) = T \times M, \quad \text{so} \quad \mu_{-(a,b)} = \mu_{(a,b)}.$$

Thus (7.6.2) implies that L satisfies the automorphic inverse property. Moreover,

$$\phi_{(a,b)}^{-1} = \phi_{(a,b)} = \mathbf{1} + \mu_{(a,b)} = \mathbf{1} + \mu_{-(a,b)} = \phi_{-(a,b)}.$$

Thus by (7.5.3) L satisfies the left inverse property and is therefore a Kikkawa loop.

Conversely, if there exists $b \in M$ with $-b \notin M$, then

$$\phi_{(0,b)} = \mathbf{1} + \alpha \times \mathbf{0}, \quad \text{while} \quad \phi_{(0,-b)} = \mathbf{1}.$$

Therefore (7.5.3) shows that L does not satisfy the left inverse property, and so is not a Kikkawa loop.

(2) For all $x = (a,b) \in L$ we find

$$\delta_{x,x} = \phi_{2x}\phi_x\phi_x = \phi_{(2a,2b)} = \mathbf{1} \iff 2b \notin M,$$

because $2a \in T$. Since $\delta_{x,x} = \mathbf{1}$ for all $x \in L$ is equivalent with the left alternative property, we are done.

(3) Let L be a left power alternative Kikkawa loop. From (1) we get $M = -M$.

If $b \in H$, $n \in \mathbf{N}$, constitutes a counterexample to the displayed condition, assume that n is minimal. By (2), $n > 1$. Put $x := (0,b)$ and observe that for all $k \in \mathbf{N}$

$$\phi_{kx} \neq \mathbf{1} \iff \phi_{kx} = \mathbf{1} + \alpha \times \mathbf{0} \iff kb \in M.$$

A previous remark and (2.1) give

$$\phi_{(k+1)x}\phi_x\phi_{kx} = \delta_{x,x^{\underline{k}}} = \mathbf{1} \quad \text{for all} \quad k \in \mathbf{N}.$$

Assume first $nb \in M$, then n is even, or $b \notin M$. If $b \notin M$, then

$$\mathbf{1} = \delta_{x,x^{\underline{n-1}}} = \phi_{nx}\phi_x\phi_{(n-1)x} = \phi_{nx}\phi_{(n-1)x} \Longrightarrow (n-1)b \in M.$$

Since n was minimal, we must have $b \in M$, a contradiction. Therefore $b \in M$. If n were even, then $n-1$ would be odd and so $(n-1)b \in M$. But this implies $\delta_{x,x^{\underline{n-1}}} \neq \mathbf{1}$, a contradiction as well.

Therefore we see that $nb \notin M$. This implies $b \in M$ and n odd, because we are looking at a counterexample. Since $n-1$ is even and fulfills the displayed condition, we find $(n-1)x \notin T \times M$, and

$$\delta_{x,x^{\underline{n-1}}} = \phi_{nx}\phi_x\phi_{(n-1)x} = \phi_x \neq \mathbf{1},$$

the final contradiction. We conclude that there cannot exist a counterexample, thereby proving one direction.

For the converse, we see from (1) that L is a Kikkawa loop. Thus it remains to show that L is left power alternative. By (2.2), it suffices to prove for all $x = (a,b) \in L$, and $n \in \mathbf{N}$

$$\phi_{(n+1)x}\phi_x\phi_{nx} = \delta_{x,x^{\underline{n}}} \stackrel{!}{=} \mathbf{1}.$$

Assume first that $\phi_{nx} \neq \mathbf{1}$, then $nx = (na, nb) \in T \times M$, and the assumptions imply that n is odd, and $b \in M$. Since T is of index 2, we also must have $a \in T$. Therefore $\phi_x = \mathbf{1} + \alpha \times \mathbf{0} = \phi_{nx}$. Finally, $n+1$ is even, so $(n+1)b \notin M$ and $\phi_{(n+1)x} = \mathbf{1}$. Consequently, $\delta_{x,x^{\underline{n}}} = \mathbf{1}$ in this case.

Now for the case $\phi_{nx} = \mathbf{1}$: If $\phi_{(n+1)x} \neq \mathbf{1}$, then we can conclude as in the first case that $n+1$ is odd, and $\phi_x = \mathbf{1} + \alpha \times \mathbf{0} = \phi_{(n+1)x}$, which entails the assertion.

So we are left with $\phi_{(n+1)x} = \phi_{nx} = \mathbf{1}$. If $a \notin T$, then $\phi_x = \mathbf{1}$ by definition. If $a \in T$, then also $na, (n+1)a \in T$. Therefore $nb, (n+1)b \notin M$. One of n and $n+1$ is odd, and the assumptions imply $b \notin M$. Therefore $\phi_x = \mathbf{1}$, as well.

Summing up, we have seen that $\delta_{x,x^{\underline{n}}} = \mathbf{1}$ for every $x \in L$.

(4) From (7.2.6), (7.5.2), and (i) we see that L is Bol if and only if

$$\phi_{2x+y} = \phi_{x+y+x} = \phi_x\phi_y\phi_x = \phi_y \quad \text{for all} \quad x, y \in L.$$

Let $x = (a', b')$, $y = (a, b) \in G \times H$. Since $2a' \in T$, we have $2a' + a \in T \iff a \in T$. Therefore,

$$L \text{ is Bol if and only if } \; 2b' + b \in M \iff b \in M \quad \text{for all} \quad b, b' \in H.$$

This implies $2H + M \subseteq M$. Since trivially $M \subseteq 2H + M$, we obtain $2H + M = M$ if L is Bol.

For the converse, assume $2H + M = M$ and let $b, b' \in H$. We get

$$b \in M \Longrightarrow 2b' + b \in M \quad \text{and} \quad 2b' + b \in M \Longrightarrow b = 2(-b') + 2b' + b \in M.$$

As we have seen, this implies Bol.

If $b \in M$, then $-b = 2(-b) + b \in M$, and L is a Kikkawa loop, by (1). Hence the last assertion.

(5) If $M = \emptyset$, then $\phi_x = \mathbf{1}$ for all $x \in L$, and $L = (G \times H, +)$ is a commutative group.

From (7.5.5) one easily sees that if L is a group, then ϕ is a homomorphism. If $M \neq \emptyset$, then $\ker\phi = G \times H \setminus T \times M$ would be a proper subgroup of $G \times H$. This is not the case, since for $a \in G \setminus T$ we have $2a \in T$.

Assume $b \in M \neq \emptyset$, take $a \in T$, and $c \in G \setminus T$. Then

$$\mu_{(a,b)}(c,0) = \big(\alpha(c),0\big) = (t,0) \neq (0,0) = \mu_{(c,0)}(a,b).$$

This implies $(a,b) \circ (c,0) \neq (c,0) \circ (a,b)$, hence L is not commutative. In other words, if L is commutative, then $M = \emptyset$. □

There are examples for most of the possible combinations of axioms in the preceding theorem. All of the examples can be modified in many ways. We leave it to the reader to construct his/her favorite. The phrase "leads to" means using the construction of the preceding theorem with this H and M gives a loop with the specified properties. The proofs are done easily by checking the corresponding conditions in (8.1).

(8.2). *We continue to use notation introduced just before the preceding theorem.*

(1) $H := \mathbf{Z}_3,\ M := \{2\}$, *leads to an A_ℓ-loop which is not a Kikkawa loop, and is not left alternative.*

(2) $H := \mathbf{Z}_8,\ M := \{1\}$, *leads to a left alternative A_ℓ-loop which is not a Kikkawa loop.*

(3) $H := \mathbf{Z}_3,\ M := \{1,2\}$, *leads to a Kikkawa loop which is not left alternative.*

(4) $H := \mathbf{Z},\ M := \{1,-1\}$, *leads to a left alternative Kikkawa loop which is not left power alternative.*

(5) $H := \mathbf{Z}_4 \times \mathbf{Z}_4,\ M := \{(1,2),(3,2)\}$, *leads to a left power alternative Kikkawa loop which is not Bol.*

(6) H *any abelian group,* $M := H \setminus 2H$, *leads to a K-loop, which is a group if and only if $H = 2H$. Examples with $H \neq 2H$ are any finite abelian group of even order, $H := \mathbf{Z}$, etc.* □

Remarks. 1. Example (5) is due to Kreuzer [21, (3.5)]. He makes a slip in setting up his conditions. More specific, his (ii) is too weak. Indeed, this condition is fulfilled by our example (4), which is not left power alternative. However, for all of Kreuzer's examples, namely (5), $H = \mathbf{R}^*$, $M = \{-1\}$, and $H = \mathbf{Q}$, $M = \{2^{2k+1}; k \in \mathbf{Z}\}$, the condition in (8.1.3) does hold. Therefore they qualify for (5).

2. $H := \mathbf{Z}_8,\ M := \{1,-1\}$, gives a finite example for (4).

In the construction of (7.6), if U has a complement in G, then we get a particularly simple setup.

(8.3). *Let G be an abelian group with a subgroup U which has a complement V, i.e., $G = U \oplus V$. Every map $\nu : V \to \mathrm{Hom}(V,U)$ with $\nu_0 := \mathbf{0}$ can be extended to a map $\mu : G \to \mathrm{Hom}(G,U)$ if we put for every $a \in G$*

$$\mu_a(u+v) := \nu_w(v), \quad \text{if } a \in w+U,\ u \in U,\ v \in V.$$

(1) *(U,μ) is a derivation sprout on G. Let ϕ be the corresponding derivation.*

(2) *If G is an elementary abelian 2-group, then G^ϕ is a Bol loop. G^ϕ is of exponent 2 if and only if $\nu_v(v)=0$ for all $v\in V$.*

Proof: μ is well-defined, since $G=U\oplus V$.

(1) By construction μ factors through the canonical epimorphism $G\to G/U$. Clearly, $\mu_0=\mathbf{0}$, and $U\subseteq\ker\mu_a$ for all $a\in G$.

(2) For all $a,b\in G$ we have

$$\phi_a\phi_b=(\mathbf{1}+\mu_a)(\mathbf{1}+\mu_b)=\mathbf{1}+\mu_a+\mu_b+\mu_a\mu_b=\mathbf{1}+\mu_a+\mu_b,$$

since $\mu_a\mu_b(G)\subseteq\mu_a(U)=0$. This implies $\phi_a\phi_b=\phi_b\phi_a$ and $\phi_a^2=\mathbf{1}$. Using this and (7.5.2) we can compute

$$\phi_{a\circ(b\circ a)}=\phi_{a+b+a}=\phi_b=\phi_a\phi_b\phi_a,$$

hence G^ϕ is Bol by (7.2.6). For the last assertion let $a=u+v$ with $u\in U,\ v\in V$. We find

$$a\circ a=a+\phi_a(a)=a+a+\mu_a(a)=\mu_a(a)=\nu_v(v).$$

So $a\circ a=0$ if and only if $\nu_v(v)=0$. $\square$

Remarks. 1. Part (1) has a converse: Namely, if (U,μ) is a derivation sprout, then a corresponding map ν can be found, which gives μ as in the theorem. The details will be left to the reader.

2. The map ν is very closely related to the map of the same name in (7.6).

Collected from the literature, here are some more

Examples. 1. [4] $G=\mathbf{Z}_2^3=\mathbf{Z}_2\oplus\mathbf{Z}_2^2$, and $\nu_w:\mathbf{Z}_2^2\to\mathbf{Z}_2,\ (x,y)\mapsto w_1w_2y$, where $w=(w_1,w_2)$.

2. [27, Example 2] as above, but $\nu_w(x,y)=w_1(w_2+1)y$.

3. [20] $G=\mathbf{Z}_2^4=\mathbf{Z}_2\oplus\mathbf{Z}_2^3$, and $\nu_w:\mathbf{Z}_2^3\to\mathbf{Z}_2,(x,y,z)\mapsto w_1w_3y+w_1w_2z$, where $w=(w_1,w_2,w_3)$.

The last two examples are of exponent 2, the first one is not.

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