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Abstract. The paper surveys the known results concerning the question: "For what values of n does there exist a nonassociative Moufang loop of order n?"

Proofs of the newest results for n odd, and a complete resolution of the case n even are also presented.

Keywords: Moufang loop, order, nonassociative
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1. Introduction and preliminaries

The question above and the equivalent question, "For what integers, n, must every Moufang loop of order n be associative?" have long been of interest.

Since Artin observed that the loop of units of any alternative ring is a Moufang loop ([22]), examples of finite nonassociative Moufang loops were known right from the start. For example, the non-zero Cayley numbers form a Moufang loop under multiplication, and the subloop consisting of

$$\{\pm 1, \pm i, \pm j, \pm k, \pm e, \pm ie, \pm je, \pm ke\}$$

is a nonassociative Moufang loop of order $2^4 = 16$.

The simplest result on nonexistence may be found in [7], where it is shown that every Moufang loop of prime order must be a group. In [4], the first author extended this result to show that Moufang loops of order p^2, p^3, p prime, must be associative. Since there are nonassociative Moufang loops of order 2^4 [see above] and 3^4 ([1] or [2]), it would seem that no extension of the results above is possible. However, in [8], Leong showed that a Moufang loop of order p^4, with $p > 3$, must be a group. This is the best one can do, because Wright showed in [21] that there exists a nonassociative Moufang loop of order p^5, for any prime p.

If one allows more than one prime, the first author showed that Moufang loops of order pq, where p and q are distinct primes, must be associative ([4]). M. Purtill [16] extended the result to Moufang loops of orders pqr, and p^2q, (p, q and r distinct odd primes), although the proof of the latter result has a flaw in the case $q < p$; see [17]. Then Leong and his students produced a spate of papers, [14], [15], [9], [10], [11], culminating in [12], in which Leong and the second author showed the following:

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1.1. Any Moufang loop of order $p^\alpha q_1^{\alpha_1} \dots q_n^{\alpha_n}$, with $p < q_1 < \dots < q_n$ odd primes and with $\alpha \leq 3$, $\alpha_i \leq 2$, is a group, and the same is true with $\alpha = 4$, provided that $p > 3$.

Finally, the second author, in his doctoral dissertation [18], showed the following:

1.2. For p and q any odd primes, there exists a nonassociative Moufang loop of order pq^3 if and only if $q \equiv 1 \pmod{p}$.

Since there exist nonassociative Moufang loops of order 3^4 and of order p^5 for any prime p, and since the direct product of a nonassociative Moufang loop and a group is a nonassociative Moufang loop, the only remaining unresolved cases for n odd are the following:

$$n = p_1^{\alpha_1} \dots p_k^{\alpha_k} q^\beta r_1^{\gamma_1} \dots r_s^{\gamma_s},$$

where

$p_1 < \dots < p_k < q < r_1 < \dots < r_s$ are distinct odd primes; $k \geq 1$;
$\alpha_i \leq 4$ ($\alpha_1 \leq 3$ if $p_1 = 3$); $3 \leq \beta \leq 4$; $\gamma_i \leq 2$;
$q \not\equiv 1 \pmod{p_i}$ for all $i = 1, \dots, k$; and
$p_j \not\equiv 1 \pmod{p_i}$ for all $i < j$ with $3 \leq \alpha_j \leq 4$.

For n odd, we also have the following results which will be needed below:

1.3 ([7])**.** If L is a Moufang loop of odd order and if K is a subloop of L, and π is a set of primes which divide $|L|$, then

(a) $|K|$ divides $|L|$.
(b) If K is a minimal normal subloop of L, then it is an elementary abelian group.
(c) L contains a Hall π-subloop.

1.4 ([12])**.** If L is a nonassociative Moufang loop of odd order and if all of the proper quotient loops of L are groups, then L_a, the subloop of L generated by all associators, is a minimal normal subloop of L.

1.5 ([9])**.** If L is a Moufang loop of odd order and if every proper subloop of L is a group and if there exists a minimal normal Sylow subloop in L, then L is a group.

1.6 ([11])**.** Let L be a Moufang loop of odd order such that every proper subloop of L is associative. Suppose that K is a minimal normal subloop which contains L_a, and that Q is a Hall subloop of L such that $(|K|, |Q|) = 1$ and $Q \lhd KQ$. Then L is a group.

For n even, many cases are handled by a construction of the first author ([4]) which produces a nonassociative Moufang loop, $M(G, 2)$ of order $2m$ for any nonabelian group G of order m. Thus, if there exists a nonabelian group of order m, then there exists a nonassociative Moufang loop of order $n = 2m$. In particular, since the dihedral group D_r is not abelian, we get a nonassociative

Moufang loop of order $4r$, for each $r \geq 3$. This leaves the case $n = 2m$, for m odd and for which every group of order m is abelian.

The following result ([14]) will also be needed below:

1.7. Any Moufang loop L of order $2m$, with m odd must contain a (normal) subloop of order m.

Finally, we can characterize those odd m for which every group of order m is abelian. (We would like to thank Anthony Hughes for suggesting this lemma and for his helpful advice regarding its proof.)

Lemma 1.8. *If $m = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, with $p_1 < \cdots < p_k$ odd primes and $\alpha_i > 0$, for all i, then every group of order m is abelian if and only if the following conditions hold:*

(i) $\alpha_i \leq 2$, *for all* $i = 1, \dots, k$,
(ii) $p_j^{\alpha_j} \not\equiv 1 \pmod{p_i}$, *for any* i *and* j.

PROOF: Note that, since the direct product of a nonabelian group with any group is a nonabelian group, if there exists a nonabelian group of order s and if $s \mid m$, then there exists a nonabelian group of order m. Since there exists a nonabelian group of order p^3 for any prime p, (i) is necessary. Similarly, since $\left|Aut(C_q)\right| = q-1$, and $\left|Aut(C_q \times C_q)\right| = \left(q^2-1\right)\left(q^2-q\right)$, there exists a nonabelian group of order pq if $q \equiv 1 \pmod{p}$ and one of order pq^2 if $q^2 \equiv 1 \pmod{p}$. Thus (ii) is necessary.

To see that these conditions are sufficient, suppose that G is a group of order m, with m as above. For each $j = 1, \dots, k$, let P_j be a p_j-Sylow subgroup of G.

By condition (ii), $(m, p_j^{\alpha_j} - 1) = 1$.

Claim: $N_G(P_j) = C_G\left(P_j\right)$.

Suppose not. For $g \in N_G(P_j) - C_G\left(P_j\right)$, conjugation by g induces a non-trivial automorphism θ of P_j. Since P_j is an abelian group, θ^s is the identity mapping on P_j, whenever $g^s \in P_j$. In particular, since $|g| \mid m$, θ^m is the identity map. Hence, $|\theta| \mid m$. On the other hand, $|\theta| \mid \left|Aut(P_j)\right|$, so $|\theta| \mid \left(m, |Aut(P_j)|\right)$. If $\alpha_j = 1$, $\left|Aut(P_j)\right| = p_j - 1$. But $\left(m, p_j - 1\right) = 1$, so $|\theta| = 1$, contrary to assumption. Therefore, $\alpha_j = 2$. If P_j is cyclic, $\left|Aut(P_j)\right| = p_j(p_j-1)$; and if P_j is elementary abelian, $\left|Aut(P_j)\right| = p_j(p_j^2-1)\left(p_j-1\right)$. In either case, since $(m, p_j^{\alpha_j} - 1) = 1$ and $\left(p_j - 1\right) \mid \left(p_j^2 - 1\right)$, we also have $(m, (p_j^{\alpha_j} - 1)\left(p_j-1\right)) = 1$. Therefore $\left(m, |Aut(P_j)|\right) = p_j$ and $|\theta| = p_j$. Hence $g^{p_j} \in C_G\left(P_j\right)$. Thus, $\frac{N_G(P_j)}{C_G(P_j)}$ is a p_j-group contained in $\frac{N_G(P_j)}{P_j}$. (Recall that P_j is abelian, since $\alpha_j = 2$.) But then $p_j \mid \left|\frac{N_G(P_j)}{P_j}\right|$, and so $p_j^3 \mid \left|N_G\left(P_j\right)\right|$, contradicting the assumption that $p_j^3 \nmid |G|$. This establishes the claim.

Since $N_G(P_j) = C_G(P_j)$ for all j, by Burnside's Theorem ([20, p. 137]), each P_j has a normal p_j-complement, which we denote by N_j.

$\left|\frac{G}{N_j}\right| = |P_j| = p_j^{\alpha_j}$, where $\alpha_j \leq 2$, so $\frac{G}{N_j}$ is abelian. Let $\varphi : G \to \frac{G}{N_1} \times \frac{G}{N_2} \times \cdots \times \frac{G}{N_k}$ be defined by $g\varphi = (gN_1, gN_2, \ldots, gN_k)$. Clearly φ is a homomorphism, and $\ker(\varphi) = \{g \mid gN_j = N_j, \text{ for all } j\} = N_1 \cap N_2 \cap \cdots \cap N_k$.

Therefore, $\frac{G}{N_1 \cap N_2 \cap \cdots \cap N_k} \cong G\varphi \subseteq \frac{G}{N_1} \times \frac{G}{N_2} \times \cdots \times \frac{G}{N_k}$. But, for each j, $N_1 \cap N_2 \cap \cdots \cap N_k \subseteq N_j \subseteq G$, so $|N_1 \cap N_2 \cap \cdots \cap N_k| \mid |G| = m$, and yet, for each j, $|N_1 \cap N_2 \cap \cdots \cap N_k| \mid |N_j|$, which is p_j-free. This implies that $|N_1 \cap N_2 \cap \cdots \cap N_k| = 1$. Thus $G \cong G\varphi \subseteq \frac{G}{N_1} \times \frac{G}{N_2} \times \cdots \times \frac{G}{N_k}$, which is abelian, as required. □

2. New results

We divide this section into two parts: n odd, and $n = 2m$, m odd.

n odd.

Theorem 2.1. *If L is a Moufang loop of order $p_1p_2\ldots p_kq^3$, with $p_1, p_2, \ldots, p_k$ and q distinct odd primes, and if $q \not\equiv 1 \pmod{p_1}$ and, for each $i > 1$, $q^2 \not\equiv 1 \pmod{p_i}$, then L is a group.*

Proof: Suppose not. Let k be the smallest positive integer for which there exists a nonassociative Moufang loop of order $p_1p_2\ldots p_kq^3$, with $p_1, p_2, \ldots, p_k$ and q distinct odd primes, and with $q \not\equiv 1 \pmod{p_1}$ and $q^2 \not\equiv 1 \pmod{p_i}$ for each $i > 1$; and let L be such a loop. By 1.2, $k \geq 2$.

Let H be a proper subloop of L. By 1.3 (a), $|H| = p_{j_1}p_{j_2}\ldots p_{j_s}q^\beta$, where either $\beta < 3$, or $s < k$. If $\beta < 3$, then H is a group by 1.1; and if $s < k$, then H is a group by the minimality of k. Thus, every proper subloop of L is a group. The same applies to any proper quotient loop of L. Therefore, by 1.4 and 1.3 (b), L_a is a minimal normal subloop of L and is an elementary abelian group. By 1.5, if L is not a group, then L_a cannot be a Sylow subloop of L, and so $|L_a| \neq q^3$, and $|L_a| \neq p_i$, for any i. But, by 1.3 (a), $|L_a|$ must divide $|L|$, so, since L_a is an elementary abelian group, $|L_a| = q$ or q^2. Therefore, by 1.3 (c), L contains a subgroup X_j of order p_j. Let n_k denote the number of p_k-Sylow subgroups of L_aX_k. By the Sylow theorems, $n_k \equiv 1 \pmod{p_k}$, so $(n_k, p_k) = 1$. Also n_k divides $|L_aX_k|$. But, since $L_a \lhd L$, $|L_aX_k| = p_kq$ or p_kq^2, so, in either case, $n_k \mid q^2$. If $n_k \neq 1$, then $n_k = q$ or q^2 and so, in either case, $q^2 \equiv 1 \pmod{p_k}$, contrary to assumption. Therefore, $n_k = 1$, and so $X_k \lhd L_aX_k$. But X_k is a Hall subloop of L, and $(|L_a|, |X_k|) = 1$. Therefore, by 1.6, L is a group, contrary to assumption. The theorem now follows. □

This leaves us with the question: What happens if $q^2 \equiv 1 \pmod{p_i}$ for some i? If $q \equiv 1 \pmod{p_i}$, then, by 1.2, there exists a nonassociative Moufang loop of order p_iq^3. Thus, we may assume that, for all i, $q \not\equiv 1 \pmod{p_i}$, but that

$q \equiv -1 \pmod{p_i}$, for some i. If there is only one such i, then, by reordering if necessary, we can assume that it is p_1, and we have a group, by Theorem 2.1. Therefore, we are left with the case $k \geq 2$, $q \equiv -1 \pmod{p_1}$, and $q \equiv -1 \pmod{p_k}$ (with no assumption about the relationship between q and p_i for $1 < i < k$, other than $q \not\equiv 1 \pmod{p_i}$). The smallest such open case is $n = 3 \cdot 5 \cdot 29^3$.

$n = 2m$**,** m **odd.**

Suppose that L is a Moufang loop of order $2m$, m odd, and that L contains a (normal) abelian subgroup M of order m.

Let u be an element of $L - M$. Then $L =< u, M >$, and every element of L can be expressed in the form mu^α, where $m \in M$ and $0 \leq \alpha \leq 1$. Let T_u denote the inner mapping of L corresponding to conjugation by u. That is, for x in L, $xT_u = u^{-1}xu$. Since M is a normal subloop, T_u maps M to itself. Let θ be the restriction of T_u to M. That is, for every m in M, $m\theta = u^{-1}mu$, and $mu = u(m\theta)$. By diassociativity, $m^2\theta = u^{-1}m^2u = u^{-1}muu^{-1}mu = (m\theta)^2$. Also, since u^2 must be in M, and since M is abelian, u^2 is in the center of M. Thus, $m\theta^2 = u^{-1}(u^{-1}mu)u = u^{-2}mu^2 = m$; so θ^2 is the identity mapping and $\theta^{-1} = \theta$.

By Lemma 3.2 on page 117 of [3], T_u is a semiautomorphism of L. That is, for x, y in L, $(xyx)T_u = (xT_u)(yT_u)(xT_u)$. In particular, for m_1, m_2 in M, $(m_1m_2m_1)\theta = (m_1\theta)(m_2\theta)(m_1\theta)$. But M is abelian, so $(m_1^2m_2)\theta = (m_1\theta)^2(m_2\theta) = (m_1^2\theta)(m_2\theta)$. Since M is of odd order and since the order of an element of a finite Moufang loop must divide the order of the loop, every element of M is of odd order and hence has a square root. (That is, if $|m| = 2k+1$, then $(m^{k+1})^2 = m$.) Thus, for any m, m' in M, $(mm')\theta = [(m'')^2m']\theta = [(m'')^2\theta](m'\theta) = (m\theta)(m'\theta)$, where m'' is the square root of m. Thus θ is an automorphism of M.

For m_1 and m_2 in M, let $x = (m_1u)m_2$, let $y = m_1(m_2u)$, and let $z = (m_1u)(m_2u)$. Then, by the Moufang identities and the fact that M is an abelian group, $xu = [(m_1u)m_2]u = m_1(um_2u) = m_1[u^2(m_2\theta)] = m_1[(m_2\theta)u^2] = [m_1(m_2\theta)]u^2$, so that

$$(m_1u)m_2 = x = [m_1(m_2\theta)]u.$$

Similarly,

$$\begin{aligned} uy &= u[m_1(m_2u)] = u[m_1(u(m_2\theta))] = (um_1u)(m_2\theta) = [u^2(m_1\theta)](m_2\theta) \\ &= u^2[(m_1\theta)(m_2\theta)], \end{aligned}$$

so that

$$m_1(m_2u) = y = u[(m_1\theta)(m_2\theta)] = [(m_1\theta)(m_2\theta)]\theta u.$$

Finally, $zu = [(m_1u)(m_2u)]u = m_1(um_2u^2) = m_1[u(m_2u^2)]$, so that

$$uzu = u\{m_1[u(m_2u^2)]\} = (um_1u)(m_2u^2) = [u^2(m_1\theta)](m_2u^2) = [(m_1\theta)m_2]u^4.$$

Thus, $(z\theta)u^2 = u^2(z\theta) = uzu = [(m_1\theta)m_2]u^4$, so $z\theta = [(m_1\theta)m_2]u^2$, and

$$(m_1u)(m_2u) = z = [(m_1\theta)m_2]\theta u^2.$$

As in [5], we can summarize these results as follows: For $0 \le \alpha, \beta \le 1$,

$$(m_1u^\alpha)(m_2u^\beta) = [(m_1\theta^\beta)(m_2\theta^{\alpha+\beta})]\theta^\beta \cdot u^{\alpha+\beta}.$$

But θ is an endomorphism of M, and θ^2 is the identity, so

$$\begin{aligned}(m_1u^\alpha)(m_2u^\beta) = [(m_1\theta^\beta)(m_2\theta^{\alpha+\beta})]\theta^\beta u^{\alpha+\beta} &= [(m_1\theta^{2\beta})(m_2\theta^{\alpha+2\beta})]u^{\alpha+\beta} \\ &= [m_1(m_2\theta^\alpha)]u^{\alpha+\beta}.\end{aligned}$$

But then, for any m_1u^α, m_2u^β, m_3u^γ in L,

$$\begin{aligned}[(m_1u^\alpha)(m_2u^\beta)](m_3u^\gamma) &= \{[m_1(m_2\theta^\alpha)]u^{\alpha+\beta}\}(m_3u^\gamma) \\ &= \{[m_1(m_2\theta^\alpha)]m_3\theta^{\alpha+\beta}\}u^{\alpha+\beta+\gamma},\end{aligned}$$

and

$$\begin{aligned}(m_1u^\alpha)[(m_2u^\beta)(m_3u^\gamma)] &= (m_1u^\alpha)\{[m_2(m_3\theta^\beta)]u^{\beta+\gamma}\} \\ &= \{m_1[m_2(m_3\theta^\beta)]\theta^\alpha\}u^{\alpha+\beta+\gamma} = \{m_1[(m_2\theta^\alpha)(m_3\theta^{\alpha+\beta})]\}u^{\alpha+\beta+\gamma} \\ &= \{[m_1(m_2\theta^\alpha)](m_3\theta^{\alpha+\beta})\}u^{\alpha+\beta+\gamma}.\end{aligned}$$

Thus L is associative.

We have proved the following:

Theorem 2.2. *Every Moufang loop L of order $2m$, m odd, which contains a (normal) abelian subgroup M of order m is a group.*

We can now settle the question of for which values of $n = 2m$ must every Moufang loop of order n be a group.

Corollary 2.3. *Every Moufang loop of order $2m$ is associative if and only if every group of order m is abelian.*

PROOF: We may assume that $m \ge 6$, since there are no nonabelian groups of order less than 6, and no nonassociative Moufang loops of order less than 12 ([6]).

If there exists a nonabelian group G of order m, then the loop $M_n(G, 2)$ is a nonassociative Moufang loop of order $n = 2m$. As shown above, this takes care of all even values of m, since the dihedral group of order m is not abelian.

Now consider $n = 2m$, m odd, and suppose that every group of order m is abelian. By 1.7, any Moufang loop L of order n must contain a normal subloop M of order m. Since there exists a nonabelian group of order p^3, for any prime p, m cannot be divisible by p^3 for any prime p. But then, M must be associative, by 1.1. Furthermore, since all groups of order m are abelian, M is an abelian group. But then, by the theorem, L is a group. □

Applying Lemma 1.8, we obtain the following:

Corollary 2.4. *Every Moufang loop of order $2m$ is associative if and only if*

$m = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, *where $p_1 < \cdots < p_k$ are odd primes and where*

(i) $\alpha_i \leq 2$, *for all $i = 1, \dots, k$,*
(ii) $p_j \not\equiv 1 \pmod{p_i}$, *for any i and j,*
(iii) $p_j^2 \not\equiv 1 \pmod{p_i}$, *for any i and any j with $\alpha_j = 2$.*

3. Some questions

We might wonder whether all of the hypotheses of Theorem 2.2 are really necessary.

Clearly it is necessary that M be abelian, since the $M(G, 2)$ construction of [4] provides examples of nonassociative Moufang loops when M is not abelian.

The proof of the theorem clearly uses the fact that m is odd, but might there be a different proof which gives us the result for m even as well? We thank E.G. Goodaire for noting that the loop $M_{32}(D_4 \times C_2, 2)$ provides a counterexample. This nonassociative Moufang loop contains an abelian normal subgroup of index two which is isomorphic to $C_2 \times C_2 \times C_2 \times C_2$.

How about the fact that M is of index two? In the proof of the theorem, we do not really need u^2 to be an element of M. All that is needed is that u^2 commutes with every element of M and that it associates with every pair of elements of M. That is, what is needed is that u^2 is in the center of $< u^2, M >$. We could therefore prove the following:

Corollary 3.1. *If a Moufang loop L contains a normal abelian subgroup M of odd order m, such that L/M is cyclic, and if $u^2 \in Z(< u^2, M >)$, for uM some generator of L/M, then L is a group.*

Can we dispose with the assumption that $u^2 \in Z(< u^2, M >)$? That is, if a Moufang loop L contains a normal abelian subgroup M of odd order m, such that L/M is cyclic, must L be a group?

The answer in general is no. When $q \equiv 1 \pmod 3$, there exists a nonassociative Moufang loop L of order $3q^3$, constructed in [18], which contains a normal abelian subgroup M of order q^3, with $L/M \cong C_3$. (Note also that, in this example, $(|M|, |L/M|) = 1$, so that even this additional condition would not suffice to guarantee that L is a group.) However, if $p > 3$, the subgroup of order q^3 in the nonassociative Moufang loop of order pq^3, $q \equiv 1 \pmod p$, is not abelian, so the question is still open for $|L/M| > 3$.

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