Commentationes Mathematicae Universitatis Carolinae

Robert El Bashir; Aleš Drápal

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Commentationes Mathematicae Universitatis Carolinae, Vol. 35 (1994), No. 4, 597--606

Persistent URL: http://dml.cz/dmlcz/118701

Quasitrivial left distributive groupoids

Robert El Bashir, Aleš Drápal

Abstract. Left distributive quasitrivial groupoids are completely described and those of them which are subdirectly irreducible are found. There are also determined all left distributive algebras $A = A(*, \circ)$ such that $A(*)$ is a quasitrivial groupoid.

Keywords: left distributive groupoid, quasitrivial groupoid

Classification: 20N02

An algebra $A = A(*, \circ)$ with two binary operations $*$ and $\circ$ is said to be a *left distributive algebra* (or an LD-algebra) [LavFr], [DehAd] if

(P1) $$(a \circ b) \circ c = a \circ (b \circ c)$$

(P2) $$(a \circ b) * c = a * (b * c)$$

(P3) $$a \circ b = (a * b) \circ a$$

(P4) $$a * (b \circ c) = (a * b) \circ (a * c)$$

for any $a, b, c \in G$. The left distributive law

$$a * (b * c) = (a * b) * (a * c)$$

is a consequence of identities (P2–3). A groupoid fulfilling this law is called *left distributive* (or an LD-groupoid).

A groupoid $B = B(*)$ is said to be *quasitrivial* if

$$a * b \in \{a, b\}$$

for any $a, b \in B$.

In this paper we determine all quasitrivial LD-groupoids. We also determine all LD-algebras $A(*, \circ)$ such that $A(*)$ is quasitrivial and all subdirectly irreducible quasitrivial LD-groupoids. We show that subdirectly irreducible quasitrivial LD-groupoids form a proper class.

The groupoid $A(*)$ with $a*b = b$ for all $a, b \in A$ will be called *discrete*. Discrete groupoids are quasitrivial and left distributive. (Such groupoids are often called semigroups of left units or semigroups of right zeros.)

Let G be a group and put $a * b = aba^{-1}$ for any $a, b \in G$. Then $G(*, \cdot)$ is an LD-algebra. Suppose that A is a quasitrivial subgroupoid of $G(*)$. Then $ab = ba$

for any $a, b \in A$, and we see that $A(*)$ is discrete. We shall show that there are many quasitrivial LD-groupoids that are not discrete.

Quasitrivial groupoids that are both left and right distributive have been described in [JeKe] by Ježek and Kepka. Kepka has also studied [KepQ] quasitrivial groupoids in the general case of linear identities (i.e. identities in which each variable occurs exactly once at both sides).

Our paper is a modest contribution to the ongoing investigation of left distributive structures. While the deepest results concern free monogenerated LD-groupoids [LavFr], [DehBr], idempotent LD-groupoids have recently received also some attention [DKM]. (A groupoid is *idempotent*, if $a * a = a$ for all $a \in A$. Quasitrivial groupoids are idempotent.)

For each quasitrivial groupoid $A = A(*)$ define relation $\gamma = \gamma_A$ by

$$(a, b) \in \gamma \iff a * b = a.$$

Lemma 1. *Let $A = A(*)$ be a quasitrivial groupoid. Then $a * b = a * (a * b) = (a * b) * b$ for any $a, b \in A$.*

By a *quasiordering* we mean any reflexive and transitive relation. A quasiordering $\leq$ of a set M will be called *downward rectified*, if $a \in M$ and $b \in M$ are comparable whenever there exists $c \in M$ with $a \leq c$ and $b \leq c$ ($a \in M$ and $b \in M$ are said to be *comparable* if $a \leq b$ or $b \leq a$).

Proposition 1. *A quasitrivial groupoid $A(*)$ is left distributive iff γ_A is a downward rectified quasiordering of A.*

Proof: Suppose first that γ is a downward rectified quasiordering. For $a, b, c \in A$ put $l = a * (b * c)$ and $r = (a * b) * (a * c)$.

(i) $(a, b) \in \gamma$ and $(b, c) \in \gamma$. Then $(a, c) \in \gamma$ by transitivity of γ, and hence $l = a = r$.

(ii) $(a, b) \in \gamma$ and $(b, c) \notin \gamma$. Then $l = a * c = a * (a * c) = r$.

(iii) $(a, b) \notin \gamma$ and $(b, c) \in \gamma$. If $(a, c) \notin \gamma$, then $l = b = r$. Since γ is downward rectified, $(a, c) \in \gamma$ implies $(b, a) \in \gamma$, and we have $l = b = r$ again.

(iv) $(a, b) \notin \gamma$ and $(b, c) \notin \gamma$. In this case $l = a * c$ and $r = b * (a * c)$. If $(a, c) \notin \gamma$, then $l = c = r$. If $(a, c) \in \gamma$, then $(b, a) \in \gamma$ implies $(b, c) \in \gamma$ by transitivity of γ. Thus $(b, a) \notin \gamma$ and $l = a = r$.

On the other hand suppose that $A(*)$ is quasitrivial and left distributive. If $(a, b) \in \gamma$ and $(b, c) \in \gamma$, then $a*c = a*(a*c) = (a*b)*(a*c) = a*(b*c) = a*b = a$. The relation γ is therefore transitive. Furthermore, let $(a, c) \in \gamma$, $(b, c) \in \gamma$ and $(a, b) \notin \gamma$. Then $b * a = (a * b) * (a * c) = a * (b * c) = a * b = b$. It follows that γ is downward rectified. □

Let $A_i = A_i(*)$, $i \in I$ be pairwise disjoint left distributive groupoids. Define a groupoid $V = V(A_i; i \in I)$ on $\cup(A_i; i \in I)$ so that

$$a * b = \begin{cases} b & \text{if } a \in A_i,\ b \in A_j \text{ and } i \neq j, \\ a *_i b & \text{if } a, b \in A_i. \end{cases}$$

Lemma 2. *Let A_i, $i \in I$ be pairwise disjoint LD-groupoids. Then $V = V(A_i; i \in I)$ is also an LD-groupoid. If all A_i, $i \in I$ are idempotent (or quasitrivial), then V is idempotent (or quasitrivial), too.*

Proof: Only the left distributivity requires a proof. For $a, b, c \in V$ put $l = a * (b * c)$ and $r = (a * b) * (a * c)$. Suppose that $a \in A_i$, $b \in A_j$ and $c \in A_k$. If $i = j = k$, then $l = r$ by the hypothesis. If i, j, k are pairwise distinct or $i = j \neq k$, then $l = c = r$. If $i \neq j = k$, then $l = b *_j c = r$, and if $i = k \neq j$, then $l = a *_i c = r$. □

Let $A(*)$ be a quasitrivial groupoid and denote by ρ the least equivalence containing γ. The equivalence classes of ρ are called *components* of $A(*)$. A quasitrivial groupoid with only one component is said to be *connected.*

Corollary 1. *If $A = A(*)$ is a quasitrivial LD-groupoid and A_i, $i \in I$ are its components, then $A = V(A_i; i \in I)$.*

Lemma 3. *Let $A(\circ)$ be a semigroup and $A(*)$ a discrete LD-groupoid. Then $A(*, \circ)$ is an LD-algebra iff $A(\circ)$ is commutative.*

Proof: If $A(*, \circ)$ is an LD-algebra, then $a \circ b = (a * b) \circ a = b \circ a$ for any $a, b \in A$. If $A(\circ)$ is commutative, then the axioms of LD-algebras clearly hold. □

Lemma 4. *Let $S(\circ)$ and $T(\circ)$ be disjoint semigroups. Extend $\circ$ to $U = S \cup T$ so that $s \circ t = s = t \circ s$ for any $s \in S$, $t \in T$. Then $U(\circ)$ is a semigroup again.*

Stepping out of our main line, we note:

Proposition 2. *Let $C(*, \circ)$ and $H(*, \circ)$ be disjoint LD-algebras, and suppose that $H(*)$ is discrete. For $A = C \cup H$ define $A(*, \circ)$ so that:*

(i) *$C(*, \circ)$ and $H(*, \circ)$ are subalgebras of $A(*, \circ)$ and*

(ii) *if $c \in C$ and $h \in H$, then $c \circ h = h \circ c = c = h * c$ and $c * h = h$.*

Then $A(*, \circ)$ is an LD-algebra again.

Proof: Fix such $a, b, c \in A$ that $\{a, b, c\} \cap C \neq \emptyset \neq \{a, b, c\} \cap H$. Assume first $a \in H$, then $a \in C$ and $b \in H$, and finally $a, b \in C$ and $c \in H$. In each of these cases, (P2–4) can be verified immediately. (P1) follows from Lemma 4. □

For a quasitrivial LD-groupoid $A = A(*)$ and $a, b \in A$ write $a||_A b$ (or just $a||b$), if a and b are not comparable with respect to γ_A.

Lemma 5. *Let $A(*)$ be a quasitrivial LD-groupoid. Then $*$ is associative iff*

$$(\dagger) \qquad a||b \text{ and } (b, c) \in \gamma \quad \Longrightarrow \quad b = c$$

holds for any $a, b, c \in A$.

Proof: Let $*$ be associative and suppose that $a||b$ and $(b, c) \in \gamma$ for some $a, b, c \in A$. Then $(a, c) \notin \gamma$ because γ is downward rectified. Hence $b = b * (a * c) = (b * a) * c = c$. On the other hand let (†) be satisfied by all $a, b, c \in A$. Fix

$a, b, c \in A$ and put $l = a * (b * c)$ and $r = (a * b) * c$. Assume first $(a, b) \in \gamma$. If $(b, c) \in \gamma$, then $l = a = r$. If $(b, c) \notin \gamma$, then $l = a * c = r$. Assume now $(a, b) \notin \gamma$. If $(b, c) \in \gamma$, then $l = b = r$. If $(b, c) \notin \gamma$, then $l = a * c$ and $r = c$. Thus only the case $(a, c) \in \gamma$, $(b, c) \notin \gamma$ and $(a, b) \notin \gamma$ need to be considered. Then $(b, a) \notin \gamma$ by the transitivity of γ, and hence (†) provides $a = c$. □

Call an LD-groupoid $A(*)$ *quasilinear*, if it is quasitrivial and $(a, b) \in \gamma$ or $(b, a) \in \gamma$ for any $a, b \in A$.

Lemma 6. *Let $A(*)$ be a quasilinear LD-groupoid. Put $a \circ b = a * b$ for any $a, b \in A$. Then $A(*, \circ)$ is an LD-algebra.*

Proof: Let $a, b \in A$. If $(a, b) \in \gamma$, then $(a * b) * a = a = a * b$, and if $(a, b) \notin \gamma$, then $(b, a) \in \gamma$ and $(a * b) * a = b = a * b$ too. $A(*)$ is associative by Lemma 5 and (P1–4) follow. □

Let $H(\circ)$ be a commutative semigroup and $I \subseteq H$ its ideal. I is said to be *prime*, if $a \circ b \in I$ implies $a \in I$ or $b \in I$ for any $a, b \in H$. The set of all prime ideals will be denoted $\mathcal{P}(H(\circ))$. Note that $\emptyset$ and H belong to $\mathcal{P}(H(\circ))$.

For disjoint LD-algebras $C = C(*, \circ)$ and $H = H(*, \circ)$, $H(\circ)$ commutative, and a mapping $\theta : C \to \mathcal{P}(H(\circ))$, define on $A = C \cup H$ operations $*$ and $\circ$ so that:

(A1) $C(*, \circ)$ and $H(*, \circ)$ are subalgebras of $A(*, \circ)$.
(A2) $h \circ c = c \circ h = c = h * c$ if $h \in H$ and $c \in C$.
(A3) $c * h = c$ if $h \in H$, $c \in C$ and $h \in \theta(c)$.
(A4) $c * h = h$ if $h \in H$, $c \in C$ and $h \notin \theta(c)$.

The algebra $A(*, \circ)$ will be denoted $A(C, H, \theta)$.

Lemma 7. *Let $C = C(*, \circ)$ and $H = H(*, \circ)$ be disjoint LD-algebras. Suppose that $C(*)$ is quasilinear with $a \circ b = a * b$ for all $a, b \in C$ and that $H(*)$ discrete. Furthermore, let $\theta : C \to \mathcal{P}(H(\circ))$ be a mapping such that $\theta(b) \subseteq \theta(a)$ for any $a, b \in C$ with $(a, b) \in \gamma_C$. Then $A(C, H, \theta)$ is an LD-algebra.*

Proof: (P1) holds by Lemma 4. Fix now $a, b, c \in A = H \cup C$ such that $C \cap \{a, b, c\} \neq \emptyset \neq H \cap \{a, b, c\}$. If $a \in H$, then (P2–4) can be verified immediately. Let $a \in C$ and assume $b \in H$. Then $a \circ b = a$ and $(a * b) \circ a$ is $b \circ a = a$ or $a \circ a = a$. This proves (P3). Now $(a \circ b) * c = a * c = a * (b * c)$ and if $c \in C$, then $a * (b \circ c) = a * c = a \circ (a * c)$ by Lemma 1. Thus $a * (b \circ c) = (a * b) \circ (a * c)$ for $c \in C$, and for $c \in H$ we obtain $a * (b \circ c) = b \circ c = (a * b) \circ (a * c)$, if $b \circ c \notin \theta(a)$. If $b \circ c \in \theta(a)$, then $b \in \theta(a)$ or $c \in \theta(a)$, and hence $a * (b \circ c) = a = (a * b) \circ (a * c)$.

Assume $b \in C$ and $c \in H$. Then $a * (b \circ c) = a * b$ and $(a * b) \circ (a * c)$ equals $a * b$ or $(a * b) \circ a$. By Lemma 6 $(a * b) \circ a = a * b$, and hence (P4) is true. Put now $l = (a \circ b) * c = (a * b) * c$ and $r = a * (b * c)$. Assume first $(a, b) \in \gamma$. If $c \notin \theta(a)$, then $c \notin \theta(b) \subseteq \theta(a)$ and $l = c = r$. If $c \in \theta(a)$, then $l = a$ and r is $a * b = a$ or $a * c = a$. For $(a, b) \notin \gamma$ we distinguish the cases $c \in \theta(b)$ and $c \notin \theta(b)$. If $c \in \theta(b)$, then $l = b * c = b = a * b = r$. If $c \notin \theta(b) \supseteq \theta(a)$, then $l = c = r$. □

For a quasitrivial LD-groupoid $A(*)$ define its *core* as the set of all $a \in A$ such that there exists $b \neq a$ with $(a,b) \in \gamma_A$. If C is the core of A, then call its complement $H = A \backslash C$ *hull* of A. There is $h * a = a$ for any $h \in H$ and $a \in A$. For every $c \in C$ denote by H_c the set of all $h \in H$ with $(c,h) \in \gamma_A$.

Lemma 8. *Let $A(*)$ be a quasitrivial LD-groupoid with a core C and a hull H. If $a, b \in C$ and $(a,b) \in \gamma$, then $H_b \subseteq H_a$.*

Proof: If $h \in H_b$, then $(b,h) \in \gamma$, and thus by transitivity $(a,h) \in \gamma$ too. □

Lemma 9. *Let $A(*, \circ)$ be an LD-algebra and suppose that $A(*)$ is quasitrivial, $C \subseteq A$ is its core and $H = A \backslash C$ its hull. Then:*

(i) *$C(*)$ is quasilinear,*
(ii) *$c \circ d = c * d$ for any $c, d \in C$,*
(iii) *$H(\circ)$ is a commutative subsemigroup of $A(\circ)$,*
(iv) *$H_c \in \mathcal{P}(H(\circ))$ for any $c \in C$,*
(v) *$A(*, \circ) = A(C, H, \theta)$, if $\theta(c) = H_c$ for any $c \in C$.*

Proof: The proof is divided into a series of separate steps:

(1) If $(a,b) \in \gamma$ and $a \neq b$, then $a \circ b = a = a * b$.
This follows from $a = a * (b * b) = (a \circ b) * b$.

(2) If $(a,b) \notin \gamma$, then $a \circ b = b \circ a$.
Clearly, $a \circ b = (a * b) \circ a = b \circ a$.

(3) If $(b,c) \in \gamma$, $b \neq c$ and $a \| b$, then $a \circ b = b \circ a = b$.
We have $a * (b * c) = b = (a \circ b) * c$. There is $b \neq c$, and so $b = a \circ b$. By (2) $a \circ b = b \circ a$.

(4) $C(*)$ is quasilinear.
Suppose there are $a, b \in C$ with $a \| b$. Let $(a,c) \in \gamma$ and $(b,d) \in \gamma$. By (3) $a = a \circ b = b$, a contradiction.

(5) If $a, b \in C$, then $a \circ b = a * b$.
For $a = b$ let $h \in A$ be such that $a \neq h$ and $(a,h) \in \gamma$. By (1) $a = a \circ h = (a * h) \circ a = a \circ a$. Assume $a \neq b$. If $(a,b) \in \gamma$, use (1). If $(a,b) \notin \gamma$, then $(b,a) \in \gamma$ by (4) and $a \circ b = b \circ a = b$ by (2) and (1).

(6) If $b \in C$ and $a \in H$, then $a \circ b = b \circ a = b$.
There exists $c \in A$ with $(b,c) \in \gamma$ and $b \neq c$. If $a \| b$, use (3). If $(b,a) \in \gamma$, use (1) and (2).

(7) If $g, h \in H$, then $g \circ h = h \circ g \in H$.
By (2), $g \circ h = h \circ g$. Assume $g \circ h \in C$. Then there exists $c \in A$ with $c \neq g \circ h$ and $(g \circ h, c) \in \gamma$. Then $c = g * (h * c) = (g \circ h) * c = g \circ h$, a contradiction.

(8) $H_c \in \mathcal{P}(H(\circ))$ for any $c \in C$.
Let $h \in H_c$ and $g \in H$. Then $c*(h \circ g) = (c*h) \circ (c*g) = c \circ (c*g)$. However, $(c,g) \in \gamma$ implies $c \circ (c * g) = c$, and $(c,g) \notin \gamma$ implies $c \circ (c * g) = c$, too. H_c is therefore an ideal. Suppose now that $g \circ h \in H_c$ for $g, h \in H$ and neither $g \in H_c$ nor $h \in H_c$. Then $c * (g \circ h) = c \neq g \circ h = (c * g) \circ (c * h)$, a contradiction.

To conclude note that (i) is (4), (ii) is (5), (iii) is (7), (iv) is (8), (A1) follows from (ii) and (iii) and (A2–4) follow from (6) and the definitions of H and H_c. $\square$

If $\leq$ linearly orders a set S, then $\min_{\leq}$ is a commutative associative quasitrivial binary operation and every ideal of $S(\min_{\leq})$ is prime. Combining Lemma 3, Lemma 7, Lemma 8 and Lemma 9 we can thus state:

Proposition 3. *Let $A(*)$ be a quasitrivial LD-groupoid with a core C. A binary operation $\circ$ on A, such that $A(*, \circ)$ is an LD-algebra, can be defined iff $C(*)$ is quasilinear.*

Moreover, if $C(*)$ is quasilinear, then $\circ$ can be always chosen to be quasitrivial, too.

Proposition 4. *Let $A(*)$ be a quasitrivial LD-groupoid with a quasilinear core C and a hull H. If $\circ$ is a commutative associative binary operation on H, and $\theta : C \to \mathcal{P}(H(\circ))$ a mapping such that $\theta(b) \subseteq \theta(a)$ for $a, b \in C$ with $(a, b) \in \gamma$, and if $a \circ b$ is defined to equal $a * b$ for all $a, b \in C$, then $A(C, H, \theta)$ is an LD-algebra. Moreover, all binary operations $\circ$ on A such that $A(*, \circ)$ is an LD-algebra, can be obtained in this way.*

We turn now our attention to the congruences of quasitrivial LD-groupoids. At the beginning we formulate several easy lemmas pertaining to quasitrivial groupoids in general. Fix a quasitrivial groupoid $A = A(*)$. For $B \subseteq A$ denote ε_B the equivalence on A given by $(a, b) \in \varepsilon_B$ iff $\{a, b\} \subseteq B$ or $a = b$. Furthermore, denote (generically) by $\mathcal{E}$ the set of all $B \subseteq A$ such that ε_B is a congruence of $A(*)$, and by $\mathcal{E}_2$ the subset of $\mathcal{E}$ consisting of all $B \in \mathcal{E}$ with $\operatorname{card}(B) \geq 2$. Finally, put $E(A) = \cap(B;\ B \in \mathcal{E}_2)$.

Lemma 10. *Let $A = A(*)$ be a quasitrivial groupoid and σ an equivalence on A. Then σ is a congruence of A if and only if $(a, a') \in \sigma$, $(b, b') \in \sigma$, $(a, b) \notin \sigma$ and $(a, b) \in \gamma$ imply $(a', b') \in \gamma$ for any $a, b, a', b' \in A$.*

Lemma 11. *Let $B \subseteq A$. Then $B \in \mathcal{E}$ if and only if*

$$(a, b) \in \gamma \implies (a, b') \in \gamma \qquad \text{and} \qquad (b, a) \in \gamma \implies (b', a) \in \gamma$$

for any $b, b' \in B$ and $a \in A \backslash B$.

Lemma 12. *If σ is a congruence of $A(*)$ and B is an equivalence class of σ, then $B \in \mathcal{E}$.*

Lemma 13. *$A(*)$ is subdirectly irreducible iff $E(A)$ contains at least two elements or* $\operatorname{card}(A) \leq 1$.

Lemma 14. *If $B \in \mathcal{E}$ intersects at least two different components of $A(*)$, then it can be expressed as a union of components of $A(*)$. On the other hand, every union of components of $A(*)$ belongs to $\mathcal{E}$.*

Lemma 15. *A disconnected quasitrivial groupoid $A(*)$ is subdirectly irreducible iff it contains exactly two components, one of them subdirectly irreducible and the other one consisting of just one element. If A contains more than two elements and is disconnected and subdirectly irreducible, and if B is its non-trivial component, then $E(A) = E(B)$.*

From here on assume that $A(*)$ is a quasitrivial LD-groupoid and denote by η the kernel of the quasiordering γ; i.e. $(a, b) \in \eta$ iff $(a, b) \in \gamma$ and $(b, a) \in \gamma$. Note that γ is an ordering of A iff $\eta = \mathrm{id}_A$.

From Lemma 10, Lemma 11 and from the transitivity of γ one obtains:

Lemma 16.

(i) *η is a congruence of $A(*)$.*

(ii) *If D is an equivalence class of η and $B \subseteq D$, then $B \in \mathcal{E}$.*

(iii) *If η contains a class with at least three elements, then $E = \emptyset$.*

(iv) *If η contains at least two distinct classes D_1, D_2 with $\mathrm{card}(D_i) \geq 2$, $1 \leq i \leq 2$, then $E = \emptyset$.*

(v) *If η contains a class with at least two elements, then $A(*)$ is simple iff $\mathrm{card}(A) = 2$.*

For every $a \in A$ denote by $[\,a\,]$ the set $\{b \in A;\ (a, b) \in \gamma\}$.

Lemma 17. *$[\,a\,] \in \mathcal{E}$ for every $a \in A$.*

PROOF: Let $(a, b) \in \gamma$, $(a, b') \in \gamma$ and $(a, c) \notin \gamma$. Then $(b, c) \notin \gamma$ and from $(c, b) \in \gamma$ we deduce that c and a must be comparable with respect to γ. Thus $(c, a) \in \gamma$ and $(c, b') \in \gamma$ by transitivity. By Lemma 11 $[\,a\,]$ belongs to $\mathcal{E}$. □

A quasitrivial LD-groupoid $A(*)$ will be called *linear*, if γ_A is a linear ordering (i.e. $A(*)$ is quasilinear and $\eta = \mathrm{id}_A$).

Lemma 18. *If the core of $A(*)$ is not linear and η is id_A, then $E(A)$ is $\emptyset$.*

PROOF: By our hypothesis there can be found incomparable elements a and b in the core of $A(*)$. Both $[\,a\,]$ and $[\,b\,]$ belong to $\mathcal{E}_2$ and $[\,a\,] \cap [\,b\,] = \emptyset$. □

A subset Q of a linearly ordered set $(P, \leq)$ will be called *downward dense* (in P), if $\emptyset \neq Q \cap \{x \in P;\ a \leq x < b\}$ for any $a, b \in P$, $a < b$.

For an LD-groupoid $A(*)$ with a core C put $\overline{C} = \{B \subseteq C;\ B = \{b \in C;\ (b, e) \in \gamma\}$ for some $e \in A\}$, order $\overline{C}$ by inclusion, denote the ordering of $\overline{C}$ by $\overline{\gamma}$, and assume that $\eta = \mathrm{id}_C$. Then $c \to \{b \in C;\ (b, c) \in \gamma\}$ embeds (C, γ) into $(\overline{C}, \overline{\gamma})$. Using this embedding, identify C with a subset of $\overline{C}$. Let H be the hull of $A(*)$. We extend $\overline{\gamma}$ to $\overline{C} \cup H$ in the following way: If $\{a, b\} \subseteq H \cup \overline{C}$ intersects H, then $(a, b) \in \overline{\gamma}$ iff either $a = b$, or $a \in \overline{C}$, $b \in H$ and $(c, b) \in \gamma$ for any $c \in C$ with $(c, a) \in \overline{\gamma}$. Then $\overline{\gamma}$ is an ordering of $\overline{C} \cup H$ and $\gamma = \overline{\gamma} \cap (A \times A)$. By the definition of $\overline{C}$, for any $h \in H$ there exists $\sup_{\overline{\gamma}}\{c \in \overline{C};\ (c, h) \in \overline{\gamma}\}$ and this supremum is in $\overline{C}$. For any $a \in \overline{C}$ denote $\mathrm{card}\{h \in H;\ a = \sup_{\overline{\gamma}}\{c \in \overline{C};\ (c, h) \in \overline{\gamma}\}\}$ by $\deg(a)$. Note that $\deg(a) = 0$ implies $a \in C$ for any $a \in \overline{C}$. If $B \subseteq C$, then denote by B'

the set $\{c \in \overline{C};\ (c,b) \in \overline{\gamma}$ for some $b \in B\}$. If $s = \sup_{\overline{\gamma}} B$ exists and $s \neq \sup_{\overline{\gamma}} C$, put $\overline{B} = B' \cup \{s\}$, otherwise define $\overline{B}$ as B'.

Proposition 5. *Let $A = A(*)$ be a connected quasitrivial LD-groupoid with a core C and a hull H, and assume that $\eta = \mathrm{id}_A$. Put $S = \{h \in H;\ (a,h) \in \gamma$ for all $a \in C\}$, $M = \{c \in C;\ (a,c) \in \gamma$ for all $a \in C\}$ and $C^* = C \backslash M$. Then:*

- (i) *If C is linear, $\mathrm{card}(S) = 2$, $\deg(c) \leq 1$ for all $c \in \overline{C^*}$, and if the set $\{c \in \overline{C^*};\ \deg(c) = 1\}$ is downward dense in $\overline{C}$, then $E(A) = S$.*
- (ii) *If C is linear, $\mathrm{card}(S) \leq 1$, $\deg(c) \leq 1$ for all $c \in \overline{C^*}$, and if the set $\{c \in \overline{C^*};\ \deg(c) = 1\}$ is downward dense in $\overline{C}$, then $E(A) = S \cup M$.*
- (iii) *If C is linear, $\mathrm{card}(S) = 1$, $\deg(c) \leq 1$ for all $c \in \overline{C^*}$, and if the set $\{c \in \overline{C^*};\ \deg(c) = 1\}$ is downward dense in $\overline{C^*}$ and there exists $m \in C^*$ with $\deg(m) = 0$ and $(c,m) \in \gamma$ for all $c \in C^*$, then $E(A) = M$.*
- (iv) *$E(A) = \emptyset$ in all other cases.*

In particular, $\mathrm{card}(E(A)) \leq 2$.

Proof: Assume that $E(A) \neq \emptyset$. We shall show that then one of the cases (i)–(iii) applies and, in parallel, we shall compute $E(A)$ in these cases.

C is linear by Lemma 18. Moreover, by Lemma 11 every subset of S belongs to $\mathcal{E}$, and thus $\mathrm{card}(S) \leq 2$. As $\mathrm{card}([\,c\,]) \geq 2$ for every $c \in C$, $E(A)$ is contained in $\cap([\,c\,];\ c \in C) = S \cup M$. Put $K = S$, if $\mathrm{card}(S) = 2$, and $K = S \cup M$, if $\mathrm{card}(S) \leq 1$. We have proved $K \supseteq E(A)$.

For $a \in \overline{C^*}$ consider a set $B = \{h \in H;\ a = \sup_{\overline{\gamma}}\{x \in \overline{C};\ (x,h) \in \overline{\gamma}\}\}$. B belongs to $\mathcal{E}$ by Lemma 11, and as $B \cap K = \emptyset$, we see that $\deg(a) \leq 1$ for all $a \in \overline{C^*}$.

Suppose now that there exist $a, b \in \overline{C}$ such that $a \neq b$, $(a,b) \in \overline{\gamma}$ and $\deg(x) = 0$ for every $x \in \overline{C}$ with $(a,x) \in \overline{\gamma}$, $(x,b) \in \overline{\gamma}$ and $x \neq b$. Put $D = \{x \in \overline{C};\ (a,x) \in \overline{\gamma}$ and $(x,b) \in \overline{\gamma}\}$. Note that any $x \in D$, $x \neq b$, is in C. For every $h \in H$ there can be found $c \in \overline{C}$ such that $c = \sup_{\overline{\gamma}}\{y \in \overline{C};\ (y,h) \in \overline{\gamma}\}$. Thus by Lemma 11 $D \cap C$ belongs to $\mathcal{E}$ and for every $c, d \in D$ the set $\{x \in D;\ (d,x) \in \overline{\gamma},\ (x,c) \in \overline{\gamma}$ and $x \neq c\}$ also belongs to $\mathcal{E}$. If $b \notin C$, then $D \cap C$ has infinitely many elements and $E(A) = \emptyset$. Therefore $D \subseteq C$ can be assumed, and we see that $E(A) = \emptyset$ if $M \neq \{b\}$. Thus either there exist no $a, b \in \overline{C}$ with $a \neq b$, $(a,b) \in \overline{\gamma}$ and $\deg(x) = 0$ for any $x \in \overline{C}$ such that $(a,x) \in \overline{\gamma}$, $(x,b) \in \overline{\gamma}$ and $x \neq b$, or $M = \{b\}$ and $m = a$ is such that $(c,m) \in \gamma$ for all $c \in \overline{C^*}$ and $\deg(m) = 0$. Put $F = K$ in the former case, and $F = M \cap K$ in the latter case. We have proved that $\{c \in \overline{C^*};\ \deg(c) = 1\}$ is downward dense in $\overline{C}$ or $\overline{C^*}$, respectively. We have also proved that F contains $E(A)$, if some of the cases (i)–(iii) applies.

It remains to show $F = E(A)$. Take $k \in K$ and assume $k \notin J$ for some $J \in \mathcal{E}_2$. As $S = \emptyset$ implies $M = \emptyset$, and thus $F = \emptyset$, assume also $S \neq \emptyset$. Let $j, s \in J$ be such that $j \neq s$ and $s \in S$. As $j \in S$ provides $K \subseteq S$, we have $j \notin S$. For $j \in C$ we obtain $k \in J$ by $(j,k) \in \gamma$, $(s,k) \notin \gamma$ and by Lemma 11. Hence $J \cap C = \emptyset$. If $j \in H \backslash S$, then there can be found $c \in C$ with $(c,j) \notin \gamma$. As $(c,s) \in \gamma$, $c \in J$, again by Lemma 11. We have proved $S \cap J = \emptyset$.

Suppose now that $h, j \in J$ are such that $j \neq h$ and $h \in H \backslash S$. If $j \in C$, $s \in S$, then $(j, s) \in \gamma$, $(h, s) \notin \gamma$, and Lemma 11 provides $s \in J$. If $j \in H$, then the sets $\{a \in C;\ a \leq j\}$ and $\{a \in C;\ a \leq h\}$ are different by our degree assumption. Therefore we can assume that there exists $c \in C$ with $(c, j) \in \gamma$ and $(c, h) \notin \gamma$. From Lemma 11 we obtain $J \subseteq C$.

If $J \subseteq C$, and $a, b \in J$ are such that $(a, b) \in \gamma$ and $a \neq b$, note first that for any $c \in C$ with $(a, c) \in \gamma$, $(c, b) \in \gamma$ and $c \neq b$ we have $c \in J$ by $(b, c) \notin \gamma$ and Lemma 11. Consider now $x \in \overline{C}$ such that $(a, x) \in \gamma$, $(x, b) \in \gamma$ and $x \neq b$. If $\deg(x) = 1$, then there exists $h \in H$ with $(x, h) \in \overline{\gamma}$ and $(b, h) \notin \overline{\gamma}$. Thus $(a, h) \in \gamma$, $(b, h) \notin \gamma$, and hence from Lemma 11 we obtain $h \in J$, a contradiction with $J \subseteq C$. Therefore $\deg(x) = 0$ for any $x \in \overline{C}$ with $(a, x) \in \gamma$, $(x, b) \in \gamma$, $x \neq b$, and by the density assumption, $J = \{a, b\} = D$. □

Proposition 6. *Let $A = A(*)$ be a quasitrivial LD-groupoid with a non-trivial kernel η. $A(*)$ is subdirectly irreducible if and only if the following conditions are satisfied:*

(i) *There exists only one equivalence class of η with more than one element (denote this class by B).*

(ii) $\mathrm{card}(B) = 2$.

(iii) *The natural homomorphism $A \to A/\eta$ maps B to $E(A/\eta)$.*

If A is subdirectly irreducible, then $E(A) = B$.

Proof: Assume $E(A) \neq \emptyset$. By Lemma 16 η contains no class with three elements and at most one class with two elements. Hence there exists an equivalence class B as required by (i) and (ii). Identify A/η with $A' = (A \backslash B) \cup \{B\}$. If $C \in \mathcal{E}'_2$ and $B \notin C$, then $C \in \mathcal{E}_2$ and $E(A) = \emptyset$ by $B \in \mathcal{E}_2$. Therefore B has to be mapped inside $E(A')$.

On the other hand, let A be an LD-groupoid satisfying (i)–(iii). Then $B \subseteq E(A)$. If $C \in \mathcal{E}_2$ and $B \cap C = \emptyset$, then $C \in \mathcal{E}'_2$, a contradiction to $B \in E(A')$. Hence $B \cap C \neq \emptyset$ for every $C \in \mathcal{E}_2$. Assume now that $B = \{a, b\}$ and there exists $C \in \mathcal{E}_2$ with $a \in C$ and $b \notin C$. If $c \in C$ and $c \neq a$, then $(a, b) \in \gamma$ implies $(c, b) \in \gamma$ by Lemma 11. Similarly, $(b, c) \in \gamma$, and thus $(b, c) \in \eta$ and $b = c$. Therefore $B = E(A)$. □

From Proposition 5, Proposition 6 and Lemma 16 we obtain:

Corollary 2. *If $A = A(*)$ is a quasitrivial LD-groupoid, then* $\mathrm{card}(E(A)) \leq 2$.

Corollary 3. *A quasitrivial LD-groupoid $A(*)$ is simple iff* $\mathrm{card}(A) \leq 2$.

Proof: Every simple groupoid is subdirectly irreducible. If $A(*)$ is subdirectly irreducible and $\mathrm{card}(A) > 2$, then it contains a non-trivial congruence $\varepsilon_{E(A)}$. □

Propositions 5 and 6 together with Lemma 15 and Lemma 13 provide a complete characterization of subdirectly irreducible quasitrivial LD-groupoids.

By Proposition 5 there are subdirectly irreducible quasitrivial LD-groupoids for every cardinality κ. This contrasts with the case of both sided distributivity, in which every subdirectly irreducible quasitrivial groupoid contains at most four elements (observe that a quasitrivial LD-groupoid $A = A(*)$ is right distributive if and only if the set $B = \{b \in A$; there exists $a \in A$ with $(b,a) \in \gamma$ and $(b,a) \notin \eta\}$ is linearly ordered by γ, if $(b,a) \in \gamma$ for every $b \in B$ and $a \in A\backslash B$, and if $A\backslash B$ is either discrete, or a block of η).

By Proposition 3, for every subdirectly irreducible quasitrivial LD-groupoid $A = A(*)$ there exists a binary operation $\circ$ on A such that $A(*, \circ)$ is an LD-algebra.

The following problems seem to be open.

1. Is the variety generated by quasitrivial LD-groupoids characterized by the identities $a*(b*c) = (a*b)*(a*c)$, $a*a = a$, $(a*b)*b = a*b$ and $a*(a*b) = a*b$?
2. Which of the quasitrivial LD-groupoids are included in the variety of LD-groupoids generated by conjugation in groups (cf. [DKM])?
3. For which LD-groupoids $A(*)$ there can be defined a commutative associative operation $\circ$ on A such that $A(*, \circ)$ is an LD-algebra?

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Department of Algebra, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic

E-mail: drapal@karlin.mff.cuni.cz

(*Received* January 7, 1994)