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# Sufficient conditions for convexity in manifolds without focal points

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*Abstract.* In this paper, local, global, strongly local and strongly global supportings of subsets in a complete simply connected smooth Riemannian manifold without focal points are defined. Sufficient conditions for convexity of subsets in the same sort of manifolds have been derived in terms of the above mentioned types of supportings.

*Keywords:* supporting of subsets, convex subsets (hypersurfaces), conjugate (focal) points, horospheres

*Classification:* 53C42

## 1. Introduction.

Convexity of subsets in Euclidean space $E^n$ has been a very interesting fruitful area of research for a long time [4], [7], [8]. A comprehensive survey of the study of sufficient conditions for convexity of subsets of $E^n$ is given in [4]. In Section 3 of [4], the subject of local supporting of subsets in $E^n$ is considered and the following results are established.

(i) An open connected set $G \subset E^n$ is convex if, for each boundary point $x \in \partial G$, there exists a local supporting hyperplane $H(x)$ passing through $x$.

(ii) A closed connected set $F \subset E^n$ possessing interior points is convex if there exists a $\varrho > 0$ such that for each $x \in \partial F$, there is a hyperplane passing through $x$ which leaves the set $F \cap U_\varrho(x)$ in a closed half-space, where $U_\varrho(x)$ is a $\varrho$-neighborhood of the point $x$ in $E^n$.

(iii) A closed connected set $F \subset E^n$ possessing interior points is convex if there is a $\varrho > 0$, such that for each $x \in \partial F$, there exists a cylinder $Z$ whose base is an $(n-1)$-dimensional ball with center $x$ and radius $\varrho$, where $\operatorname{Int}(Z) \cap F = \emptyset$. The height of the cylinder may depend on $x$.

The result (ii) above is generalized to subsets of linear topological spaces in [4].

In [4], the authors expected a more general result which seems to be an extension of the results (i)–(iii) above in a general Riemannian manifold $M$ in the condition that one could find supporting hypersurfaces in $M$ possessing behavior similar to that of hyperplanes in $E^n$.

The main goal of this paper is to show some realization of the above expected viewpoint in a complete simply connected $C^\infty$ Riemannian manifold $\widetilde{W}$ without

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focal points. In a brief word we define and study convexity of subsets of $\widetilde{W}$ in terms of local and global supportings of the same subsets. The most candidate hypersurfaces to be used in defining supportings are the horospheres of $\widetilde{W}$ as we shall see below. Actually, horospheres in $\widetilde{W}$ behave nicely (see Section 2 below).

## 2. Preliminaries.

From now on, let us take $W$ (resp. $\widetilde{W}$) to denote $C^\infty$ complete simply connected Riemannian manifold without conjugate (resp. focal) points. $M$ will denote a general $C^\infty$ Riemannian manifold. For a subset $A \subset M$, $\overline{A}$ will denote the closure of $A$ while $\partial A$ its boundary. For basic properties of $W$ and $\widetilde{W}$ we refer the reader to [3], [5], [6].

Concerning conjugate and focal points we just quote the following principal facts which we shall frequently use throughout the paper (see [2], [6]).

(a) A manifold with non-positive sectional curvatures is free from focal points.

(b) Every manifold without focal points has no conjugate points but the converse is not generally true.

(c) For each pair of points $p, q \in W$, there exists a unique geodesic segment from $p$ to $q$ and is denoted by $[pq]$. When $p$ is deleted from the geodesic segment we write $(pq]$.

Let $d(p,q)$ denote the distance between the two points $p, q \in W$. For each element $v$ of the unit sphere bundle $SW$ of $W$ and for each real number $s > 0$, let us define the real-valued functions $b_{vs} : W \to \mathbb{R}$ by $b_{vs}(q) = s - d(\gamma_v(s), q)$, where $\gamma_v$ is the maximal geodesic of $W$ with initial velocity $\gamma_v'(0) = v$. The functions $b_{vs}$ are increasing with $s$ and absolutely bounded by $d(\gamma_v(0), q)$. The Busemann function of $v$ is defined by $b_v = \lim_{s\to\infty} b_{vs}$. Each $b_v$ is $C^1$ function defined on the whole of $W$. In particular, if $W$ is $E^n$, each $b_v$ represents the usual height function in the direction of $v$. Call $H_v = b_v^{-1}(0)$ the horosphere and $\overline{D}_v = b_v^{-1}[0,\infty)$ the closed horodisc of $v$ [5].

From the above argument we may look at the horospheres (resp. horodiscs) in $W$ as geodesic spheres (resp. balls) of infinite radius.

The nice behavior — mentioned before — of horospheres in a manifold $\widetilde{W}$ without focal points may be understood if one takes into account that [3]:

(1) Each Busemann function in $\widetilde{W}$ is $C^2$ and has gradient vector field of unit length.

(2) The level hypersurfaces (horospheres) of each Busemann function in $\widetilde{W}$ form an equidistant family whose orthogonal trajectories are geodesics.

(3) If $u$ is a unit vector at $p \in W$, then $u = \operatorname{grad} b_u(p)$. Moreover, if $v = \operatorname{grad} b_u(q)$ for some $q \in W$ then $b_u$ and $b_v$ differ only by a constant. Hence, the horospheres determined by $b_u$ are the same as those determined by $b_v$.

## 3. On convexity.

A subset $B \subset M$ is convex if for each pair of points $p, q \in B$, there is a unique minimal geodesic segment $[pq]$ from $p$ to $q$ and this segment is in $B$ [2]. A subset

$K \subset M$ is a convex body if it is a convex subset of $M$ with a non-empty interior. The boundary $\partial F$ of the convex open subset $F \subset M$ is a convex hypersurface of $M$. A closed subset $A \subset M$ is called strictly convex if it is convex and the boundary $\partial A$ contains no geodesic segments.

The following results on convexity in $\widetilde{W}$ are necessary for Section 4 below which represents the main part of this work. For the proofs see [1].

**Lemma 3.1.** *Let $A \subset \widetilde{W}$ be an open subset. Then $A$ is convex if and only if $\overline{A}$ is convex.*

**Lemma 3.2.** *In $\widetilde{W}$, each geodesic ball $B(x, \lambda)$ of center $x$ and finite radius $\lambda > 0$ is a strictly convex body.*

**Lemma 3.3.** *In $\widetilde{W}$, each horodisc is a convex body.*

Notice that although a horodisc in $\widetilde{W}$ is a limit of a sequence of geodesic balls, horodisc is convex not necessarily strictly convex. Half-space in $E^n$ is a good example for this claim. Horodiscs in hyperbolic space $H^n$ are strictly convex subsets.

**Lemma 3.4.** *For a closed convex subset $B \subset W$ with smooth boundary hypersurface $\partial B$, each tangent geodesic $\gamma$ to $\partial B$ has the property*

$$\gamma \cap \operatorname{Int}(B) = \emptyset.$$

**Corollary 3.5.** *Let $\gamma$ be a maximal geodesic in $\widetilde{W}$ tangent to the horosphere $H_v$. Then $\gamma$ lies wholly in the closed subset $\widetilde{W} - (D_v \cup D_{-v})$.*

## 4. Main results.

In this section we state and prove our main results. We start by giving the definitions of types of supportings.

**Definition 4.1.** A subset $A \subset \widetilde{W}$ is globally supported by a closed horodisc $\overline{D}_v$ for $v \in S\widetilde{W}$ if

(i) $A$ is a proper subset of $\overline{D}_v$;
(ii) $\overline{A} \cap H_v \neq \emptyset$.

If in addition $\overline{A} \cap H_v$ is a single point set, then $A$ is strongly globally supported by $\overline{D}_v$.

**Definition 4.2.** A subset $A \subset \widetilde{W}$ is locally supported at the point $p \in \partial A$ by the closed horodisc $\overline{D}_v$ if $p \in H_v$ and there exists a neighborhood $U(p)$ in $\widetilde{W}$ such that $A \cap U(p)$ is globally supported by $\overline{D}_v$. If $A \cap U(p)$ is strongly globally supported at $p$ by $\overline{D}_v$, then $A$ is strongly locally supported at $p$ by $\overline{D}_v$.

From the above definitions, it is clear that if a subset $A \subset \widetilde{W}$ is globally supported by $\overline{D}_v$, then no point of $H_v$ is an interior point of $A$. Besides, if $A \subset \widetilde{W}$ is locally supported at $p \in \partial A$ by $\overline{D}_v$, then each point of $H_v$ sufficiently close to $p$ cannot be an interior point of $A$. Moreover, each global supporting horodisc for a certain subset $A \subset \widetilde{W}$ is local supporting of the same subset but the converse is not necessarily true.

**Theorem 4.3.** *Let $A$ be an open connected subset of $\widetilde{W}$ with smooth boundary hypersurface $\partial A$. Assume that $A$ is locally (resp. strongly locally) supported at each boundary point. Then $A$ is convex (resp. strictly convex).*

Proof: Firstly, we show that if $A$ is locally supported at each boundary point, then $A$ is convex.

Let us fix the following notation. At the boundary point $x \in \partial A$, $n(x)$ is the unit normal of $\partial A$ at $x$ in the interior direction of $A$.

Assume, on the contrary, that $A$ is locally supported by $\overline{D}_{n(x)}$ at each boundary point $x \in \partial A$ while $A$ is non-convex. Consequently, there exist two interior points $p, q \in A$ with a connecting geodesic segment $[pq]$ not contained wholly in $A$. We have now two possibilities to be considered separately (i) $[pq] \subset \overline{A}$ and (ii) $[pq] \not\subset \overline{A}$.

(i) $[pq] \subset \overline{A}$

Let us now move from the interior point $p$ along $[pq]$ towards $q$. Let $x$ be the first point at which $[pq]$ touches $\partial A$. It is easy to see that all the points of the geodesic subsegment $[px]$ joining $p$ and $x$ are interior points of $A$ except $x$. Let us consider the local supporting closed horodisc $\overline{D}_{n(x)}$ at $x$.

Clearly, the geodesic segment $[xp]$ is tangential to $\partial A$ at $x$. Since $A$ is locally supported by $\overline{D}_{n(x)}$ at $x$, then $x \in H_{n(x)}$ and $[xp]$ is also tangential to $H_{n(x)}$ at $x$. For a point $r \in (px)$ sufficiently close to $x$, we have $[rx) \subset D_{n(x)}$, i.e. the maximal geodesic $\gamma$ through $p$ and $x$ satisfies $\gamma \cap D_{n(x)} \neq \emptyset$ contradicting Lemma 3.4 and Corollary 3.5 (see Fig. 1).

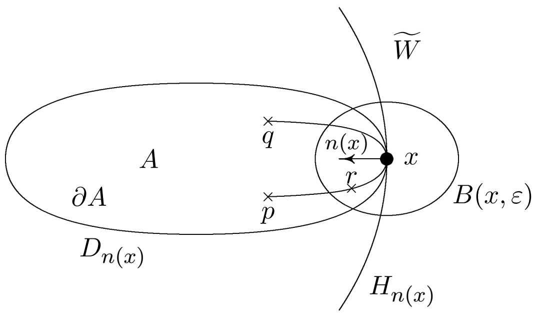


Figure 1.

(ii) $[pq] \not\subset \overline{A}$

Since $A$ is connected, then there exists a curve $\tau$ joining $p$ and $q$ such that $\tau \subset A$. Let us consider all geodesic segments joining $p$ to all the points of $\tau$. For points sufficiently close to $p$ these segments are in $A$. If we move from $p$ towards $q$ along $\tau$, we find a geodesic segment $[py]$ joining $p$ to a point $y \in A$ and this segment touches $\partial A$ at some $x$ such that all the points of the subsegment $[px]$ are interior points of

$A$ except $x \in \partial A$ (see Fig. 2).

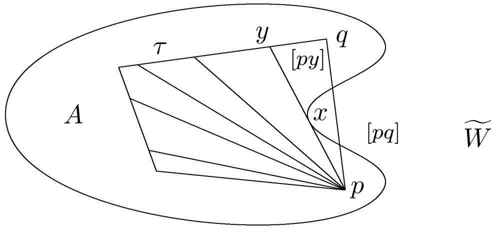


Figure 2.

We arrive again to a situation exactly as that of Fig. 1. Repeating the same argument of the case (i) we finally have that $A$ is convex.

To complete the proof assume that $A$ is strongly locally supported at each boundary point. By the above argument we have that $A$ is convex. Assume, on the contrary, that $A$ is not strictly convex. Consequently, there exists a pair of points $p, q \in \partial A$ such that $[pq] \subset \partial A$. Let us consider $y \in [pq]$ to be the middle point of $[pq]$. Since $A$ is strongly locally supported at $y$, then there exists a sufficiently small neighborhood $U(y)$ in $\widetilde{W}$ about $y$ such that $\overline{A \cap U(y)} \subset \overline{D}_{n(y)}$ and $(\overline{A \cap U(y)}) \cap H_{n(y)} = \{y\}$. Consequently, $[pq] \cap U(y) \subset \overline{D}_{n(y)}$ and $([pq] \cap U(y)) \cap H_{n(y)} = \{y\}$, which means that the maximal geodesic $\gamma$ through $p$ and $q$ satisfies (i) $\gamma$ is tangent to $H_{n(y)}$ at $y$, (ii) $\gamma \cap D_{n(y)} \neq \emptyset$ contradicting Corollary 3.4 and the proof of Theorem 4.3 is now complete (see Fig. 3). □

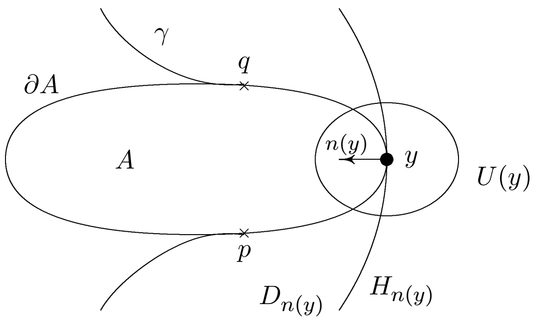


Figure 3.

Notice that we have neglected completely discussing local supporting in the outer direction $-n(x)$ at $x \in \partial A$ as the closed horodisc $\overline{D}_{-n(x)}$ cannot support $A$ locally at $x$.

We can easily construct examples in the hyperbolic space $H^n$ to show that the converse of Theorem 4.3 is not generally true.

**Theorem 4.4.** *Let $A$ be an open bounded subset of $\widetilde{W}$ with smooth boundary* *hypersurface $\partial A$. Assume that $A$ is globally* (*resp. strongly globally*) *supported at each boundary point. Then $A$ is convex* (*resp. strictly convex*).

PROOF: Firstly, we show that $A$ is connected.

Assume, on the contrary, that $A$ is disconnected. Also assume without loss of generality that $A$ is the union of two open disjoint subsets $A_1$ and $A_2$ with smooth boundary hypersurfaces $\partial A_1$ and $\partial A_2$, respectively. Notice that $\overline{A}_1 \cap \overline{A}_2 = \emptyset$ otherwise $\partial A = \partial A_1 \cup \partial A_2$ will not be a hypersurface of $\widetilde{W}$. Since both $A_1$ and $A_2$ are bounded subsets of $\widetilde{W}$, then $\overline{A}_1$ and $\overline{A}_2$ are compact subsets of $\widetilde{W}$. Let us assume that the Hausdorff distance [7] between $\overline{A}_1$ and $\overline{A}_2$ is $\lambda > 0$ and $p \in \partial A_1$ and $q \in \partial A_2$ is a closest pair of points, i.e. $d(p,q) = \lambda$. Consider the maximal geodesic $\gamma$ through the points $p, q$ parametrized by arc-length for which $p = \gamma(0)$ and $q = \gamma(\lambda)$. Clearly $\gamma$ intersects $\partial A_1$ and $\partial A_2$ at $p$ and $q$ orthogonally, respectively (see [2, p. 216]). Moreover, there exist $p' \in A_1$ and $q' \in A_2$ such that $p', q' \in \gamma$, $p' = \gamma(\mu_1)$ and $q' = \gamma(\mu_2)$ where $\mu_1 < 0$ and $\mu_2 > \lambda$ (see Fig. 4).

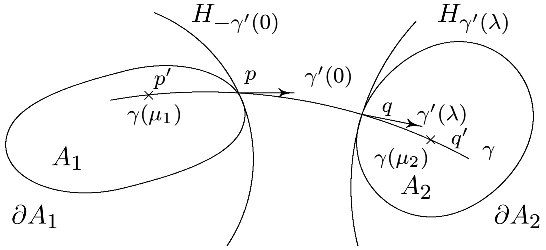


Figure 4.

The subset $A$ cannot be globally supported at either $p$ or $q$ since

$$\begin{aligned} b_{-\gamma'(0)}(p') &> 0 &\quad \text{and} \quad & b_{-\gamma'(0)}(q') < -\lambda < 0, \\ b_{\gamma'(\lambda)}(p') &< -\lambda < 0 &\quad \text{and} \quad & b_{\gamma'(\lambda)}(q') > 0, \end{aligned}$$

contradicting the assumption of the theorem. Hence $A$ is connected.

Since each global (resp. strongly global) supporting is local (resp. strongly local) we conclude by using Theorem 4.3 that $A$ is a convex (resp. strictly convex) subset of $\widetilde{W}$ and the proof of Theorem 4.4 is now complete. □

Theorem 4.4 can be proved independently of Theorem 4.3 in the following way:

(1) Prove that $A$ is connected as mentioned above.

(2) Prove that $\overline{A}$ is the intersection of convex subsets of $\widetilde{W}$, namely the supporting closed horodiscs of $\overline{A}$. Taking into account that the intersection of convex subsets is convex we obtain that $\overline{A}$ is itself convex and consequently $A$ is convex by Lemma 3.1.

It is also noteworthy that the converse of Theorem 4.4 is not generally true. Examples can also be constructed in $H^n$ to show the validity of this claim.

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