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## On a class of commutative groupoids determined by their associativity triples

ALEŠ DRÁPAL

*Abstract.* Let  $G = G(\cdot)$  be a commutative groupoid such that  $\{(a, b, c) \in G^3; a \cdot bc \neq ab \cdot c\} = \{(a, b, c) \in G^3; a = b \neq c \text{ or } a \neq b = c\}$ . Then  $G$  is determined uniquely up to isomorphism and if it is finite, then  $\text{card}(G) = 2^i$  for an integer  $i \geq 0$ .

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For a groupoid  $G = G(\cdot)$  denote by  $\mathbf{Ns}(G)$  the set of its non-associative triples, i.e.  $\mathbf{Ns}(G) = \{(a, b, c) \in G^3; a \cdot bc \neq ab \cdot c\}$ . If  $\mathcal{V}$  is a variety of groupoids and  $S$  a non-empty set, then it can be a non-trivial problem to determine all such  $N \subseteq S^3$  that  $N = \mathbf{Ns}(G)$  for a groupoid  $G = S(\cdot) \in \mathcal{V}$ . For example, it is known [1], [2] that  $\mathbf{Ns}(G) \neq \{(a, a, a); a \in G\}$  for any non-empty groupoid  $G$ .

In the present short note we investigate the case when  $\mathcal{V}$  is the variety of the commutative groupoids and  $\mathbf{Ns}(G) = \{(a, b, c) \in G^3; a = b \neq c \text{ or } a \neq b = c\}$ . We shall show that all such non-trivial groupoids can be obtained by a slight modification of a 2-elementary Abelian group and that these groupoids are determined up to isomorphism by  $\text{card}(G)$ . Moreover, whenever  $G$  is finite and non-trivial, then  $\text{card}(G) = 2^i$  for an integer  $i \geq 1$ .

Note that  $a \cdot ba = ab \cdot a$  for any  $a, b \in G$  whenever  $G$  is a commutative groupoid. The set  $\{(a, b, c) \in G^3; a = b \neq c \text{ or } a \neq b = c\}$  thus covers all  $(a, b, c) \in G^3$  such that  $\text{card}\{a, b, c\} \leq 2$  and  $a \cdot bc \neq ab \cdot c$  can occur.

**Theorem 1.** *For an Abelian group  $G(+)$  and each  $0 \neq e \in G$  define on the set  $G$  a commutative groupoid  $G_e$  by  $0 \cdot 0 = e$ ,  $a \cdot b = a + b$  and  $a \cdot 0 = 0 \cdot a = 0$  for any  $a, b \in G \setminus \{0\}$ . If  $G(+)$  is 2-elementary, then  $\mathbf{Ns}(G_e) = \{(a, b, c) \in G^3; a = b \neq c \text{ or } a \neq b = c\}$ . Conversely, if  $G(\cdot)$  is a commutative groupoid where  $a \cdot bc \neq ab \cdot c$  if and only if  $a = b \neq c$  or  $a \neq b = c$ , and  $\text{card}(G) > 1$ , then there exist a 2-elementary Abelian group  $G(+)$  and an element  $0 \neq e \in G$  such that  $G(\cdot) = G_e$ . Moreover,  $G_e$  is isomorphic to  $G_f$  for any choice of  $e, f \in G$ ,  $e \neq 0 \neq f$ .*

PROOF: Only the converse part of the theorem requires a proof. Let us hence assume that  $G(\cdot)$  is a commutative groupoid,  $\text{card}(G) > 1$  and  $\mathbf{Ns}(G(\cdot)) = \{(a, b, c) \in G^3; a = b \neq c \text{ or } a \neq b = c\}$ . As  $G$  is commutative, we have

$$(1) \quad a \cdot ba = ab \cdot a \text{ for any } a, b \in G.$$

Let  $a = bc$ , where  $a, b, c \in G$  are pair-wise distinct. If  $c \neq ab$ , then  $aa \cdot b = (bc \cdot a) \cdot b = (b \cdot ca) \cdot b = b(ca \cdot b) = b(c \cdot ab) = bc \cdot ab = a \cdot ab$ . Hence  $c = ab$  and we have

(2) If  $a = bc$ ,  $b \neq a \neq c$  and  $b \neq c$ , then  $b = ac$  and  $c = ab$ .

Further, we shall prove

(3) If  $a = bc$ ,  $b \neq a \neq c$  and  $b \neq c$ , then  $a^2 = b^2 = c^2$  and  $a^2 \notin \{a, b, c\}$ .

To see this, observe that  $c^2 = ab \cdot c = a \cdot bc = a^2$  by (2) and that  $a^2 = a$  implies  $a \cdot bb = a \cdot aa = a = cb = ab \cdot b$ .

If  $a \in G$  is such that  $a, a^2, a^3$  are pair-wise distinct, we obtain from (3)  $a^2 \notin \{a^3, a^2, a\}$ , a contradiction. Therefore it holds

(4)  $a = a^2$  or  $a^2 = a^3$  or  $a = a^3$  for any  $a \in G$ .

Let  $a, b, c \in G$  be again pair-wise distinct and  $a = bc$ . Then  $a \neq a^2$  by (3), and  $a^3 = a$  implies  $a \cdot bb = a \cdot aa = a^3 = a = c \cdot b = ab \cdot b$ . Hence we have

(5) If  $a = bc$ ,  $b \neq a \neq c$  and  $b \neq c$ , then  $a \cdot a^2 = b \cdot a^2 = c \cdot a^2 = a^2 = b^2 = c^2$ .

We shall now order the set  $G$  by  $a < b$  iff  $ab = b$  and  $a \neq b$ . From  $a < b$  and  $b < a$  it follows  $b = ab = ba = a$  and from  $a < b < c$  we obtain  $ac = a \cdot bc = ab \cdot c = bc = c$ . Therefore  $<$  really is a (sharp) ordering of  $G$ .

Let again  $a, b, c \in G$  be pair-wise distinct and with  $a = bc$ . If  $e < a$ , then  $ec = e \cdot ab = ea \cdot b = ab = c$ . Consequently, we have

(6) Let  $a = bc$ ,  $b \neq a \neq c$  and  $b \neq c$ . If  $e < a$ , then  $e < b$  and  $e < c$ .

Conversely, suppose that  $a < e$ . Then  $b \neq e \neq c$ ,  $eb = ea \cdot b = e \cdot ab = ec$  and  $eb \cdot c = e \cdot bc = ea = e$ . From  $eb = c$  it follows  $ec = c$ ,  $e < c$  and by (2) and (6)  $e < a$ . Therefore  $eb \neq c$ . If  $eb, c, e$  are pair-wise distinct, then  $a < e$  implies by (6) that  $a < c$ , a contradiction. It follows  $eb = e$  and we obtain

(7) Let  $a = bc$ ,  $b \neq a \neq c$  and  $b \neq c$ . If  $a < e$ , then  $b < e$  and  $c < e$ .

For  $a, b \in G$  put  $(a, b) \in r$  iff  $a \neq b$  and  $a \neq ab \neq b$ , and denote by  $\sim$  the least equivalence containing the relation  $r$ . From (6) and (7) we get by induction immediately

(8) Let  $a, b, e \in G$  and let  $a \sim b$ . Then  $a < e$  iff  $b < e$ , and  $e < a$  iff  $e < b$ .

Denote by  $\mathcal{E}$  the set of equivalence classes of  $\sim$ . By the definitions of  $\sim$  and  $<$  we have either  $a \sim b$ , or  $a < b$ , or  $b < a$  for any  $a, b \in G$ . Hence it follows from (8) that  $<$  induces a linear ordering of  $\mathcal{E}$ . Suppose that  $(\mathcal{E}, <)$  has no maximum element. Then for  $a \in G$  we can choose  $b \in G$  with  $a < b$ ,  $a^2 < b$ . Then  $b \cdot aa = b \cdot a^2 = b = ba = ba \cdot a$ , a contradiction.

Let  $U \in \mathcal{E}$  be the maximum element of  $(\mathcal{E}, <)$  and suppose that  $a, b \in U$ ,  $a \neq b$ . Then  $a \neq ab \neq b$ , and we obtain  $a^2 > a$  by (3) and (5). This is a contradiction, and hence  $U$  contains exactly one element, say  $u$ .

For  $u \neq a \in G$  we have  $ua = u$ , and thus  $u = ua \cdot a \neq u \cdot a^2$  provides  $a^2 = u$ . Therefore  $a < b < u$  would imply  $a \cdot bb = au = u = bb = ab \cdot b$ , which contradicts our hypothesis. It follows that the equivalence  $\sim$  has exactly two classes and by (7) we have  $ab \notin \{a, b, u\}$  for any  $a, b \in G$ ,  $a \neq b$ ,  $a \neq u \neq b$ . Moreover,  $u^2 = aa \cdot u \neq a \cdot au = u$ .

Put now  $u = 0$  and define  $G(+)$  by  $a+0 = 0+a = a$  for any  $a \in G$  and  $a+b = ab$  for  $a, b \in G$ ,  $a \neq 0 \neq b$ . Clearly,  $a + (b + c) = (a + b) + c$  whenever  $0 \in \{a, b, c\}$ , and by (2) also when  $a = b$  or  $b = c$ . Similarly,  $a + (b + ab) = a + a = ab + ab = ab + (a + b)$  for  $a, b \in G$ ,  $a \neq b$ ,  $a \neq 0 \neq b$ . Finally,  $a + (b + c) = a + bc = a \cdot bc = ab \cdot c = ab + c = (a + b) + c$  when  $a, b, c \in G$  are pair-wise distinct and  $c \neq ab$ . It follows that  $G(+)$  is a 2-elementary Abelian group and we see that  $G(\cdot) = G_{u2}$ .  $\square$

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