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VECTOR FIELDS AND CONNECTIONS ON *TM*

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Summary. In this paper we describe the set $C_\Gamma^\infty TM$ of all vector fields Z on TM which determine, by first order natural procedures, connections on TM. We construct all natural differential operators of the first order from $C_\Gamma^\infty TM$ into the space of all connections on TM.

Keywords: vector field, connection, differential operator.

AMS Subject Classification: 53B05, 58A20.

1. INTRODUCTION

It is well known, see [3], [7], that every spray on a smooth manifold M determines a linear connection without torsion on TM. This fact was extended to the case of arbitrary differential equations of the second order on M, see [1; 8]. We have constructed all connections on TM (non linear in general), naturally associated in the first order with a differential equation of the second order. Now we study the problem of a geometrical construction of connections on TM by any vector field on TM. This is possible only in some cases. We characterize the set $C_\Gamma^\infty TM$ of such vector fields and construct all natural differential operators of the first order from $C_\Gamma^\infty TM$ into the space of all connections on TM. Our considerations are in the category C^∞.

Let $C^\infty Y$ denote the set of all smooth sections of a fibre manifold $\pi: Y \to M$. We recall some equivalent definitions of a connection Γ on Y that we will use.

1. Let JY be the space of all 1-jets of local sections of Y. A chart (x^i, y^α) on Y induces the chart $(x^i, y^\alpha. y_i^\alpha)$ on JY. A connection on Y is a section $\Gamma: Y \to J^1 Y$, $\bar{x}^i = x^i$, $\bar{y}^\alpha = y^\alpha$, $\bar{y}_i^\alpha = \Gamma_i^\alpha(x, y)$. The local functions Γ_i^α on TM on TM will be called the Christoffel functions of Γ.

2. A connection Γ on Y is given by a 1-form h_Γ on Y with values in TY such that $h_\Gamma(X) = 0$ for $X \in VY$ and $T\pi(Z) = T\pi\, h_\Gamma(Z)$ for any $Z \in TY$, where Tf denotes the tangent map of $f: M \to N$ and VY is the space of vertical vectors on Y. Put $v_\Gamma: = \mathrm{Id}_{TY} - h_\Gamma$. The forms h_Γ and v_Γ are said to be the horizontal and vertical forms of Γ, respectively. In terms of coordinates, $h_\Gamma = \mathrm{d}x^i \otimes \partial/\partial x^i + \Gamma_j^\alpha\, \mathrm{d}x^j \otimes \otimes \partial/\partial y^\alpha$. It is clear that h_Γ can be interpreted as an element of $C^\infty(T(TM) \otimes T^*M \to \to TM)$.

Since $JY \to Y$ is an affine bundle associated with the vector bundle $VY \otimes T^*M \to \to Y$, we conclude that if Γ is a connection on Y and $\gamma \in C^\infty(VY \otimes T^*M \to Y)$ then $\Gamma + \gamma$ is also a connection on Y.

2. $T(TM)$-VALUED FORMS ON TM AND CONNECTIONS ON TM

Let $p_M \colon TM \to M$ denote the tangent vector projection onto M. Let $p_M^* TM$ be the p_M-pull-back of TM. Then $C^\infty(TM \otimes_{TM} T^*M \to TM)$ carries the algebra structure of all vector bundle morphisms on $p_M^* TM$ over id_{TM}, i.e. if $a, b \in C^\infty(TM \otimes \otimes_{TM} T^*M \to TM)$ then $a \,.\, b$ denotes the composition of the maps a and b. We will use the identification $C^\infty(VTM \otimes T^*M \to TM) \equiv C^\infty(TM \otimes_{TM} T^*M \to TM)$, which is implied by the canonical identification $VTM \equiv TM \times_M TM \equiv p_M^* TM$.

A local chart (x^i) on M induces the charts (x^i, x_1^i) on TM and $(x^i, x_1^i, \mathrm{d}x^i, \mathrm{d}x_1^i)$ on $p_{TM} \colon TTM \to TM$. In these charts the canonical involution i_2 and the canonical morphism v on $T(TM)$ are of the form $i_2(x^i, x_1^i, \mathrm{d}x^i, \mathrm{d}x_1^i) = (x^i, \mathrm{d}x^i, x_1^i, \mathrm{d}x_1^i)$ and $v = \mathrm{d}x^i \otimes \partial/\partial x_1^i$.

Denote by $C_v^\infty(T(TM) \otimes T^*(TM) \to TM)$ the space of all $T(TM)$ – valued forms A on TM such that the restriction of the map $A \colon T(TM) \to T(TM)$ to VTM is a vector bundle morphism on VTM over id_{TM}. In coordinates, $A = a_j^i(x, x_1)\, \mathrm{d}x^j \otimes \otimes \partial/\partial x^i + (e_j^i(x, x_1)\, \mathrm{d}x^j + h_j^i(x, x_1)\, \mathrm{d}x_1^j) \otimes \partial/\partial x_1^i$ and then $v \,.\, A = a_j^i\, \mathrm{d}x^j \otimes \partial/\partial x_1^i$, $A \,.\, v = h_j^i\, \mathrm{d}x^j \otimes \partial/\partial x_1^i$. It means that v, $v \,.\, A$, $A \,.\, v$ can be considered as elements of $C^\infty(TM \otimes_{TM} T^*M \to TM)$.

Let V, W be vector spaces. Let $B \in (W \otimes W^*) \otimes (V \otimes V^*)$. Denote by B_0 the linear map $W \otimes V^* \to W \otimes V^*$ given by the tensor contraction of $B \otimes X$, $X \in W \otimes V$. We say that B is regular if B_0 is regular. A form $A \in C_v^\infty(T(TM) \otimes T^*(TM) \to TM)$ is said to be connection admissible if $\alpha = \mathrm{id}_{TM} \otimes_{TM} v \,.\, A - A \,.\, v \otimes_{TM} \mathrm{id}_{TM} \in \in C^\infty((TM \otimes_{TM} T^*M) \otimes_{TM} (TM \otimes_{TM} T^*M) \to TM)$ is regular. Obviously, the map $\alpha_0 \colon C^\infty(TM \otimes_{TM} T^*M) \to C^\infty(TM \otimes_{TM} T^*M)$ is of the form

$$(1) \qquad \bar{y}_j^i = (\delta_s^i a_j^u - h_s^i \delta_j^u)\, y_u^s \,.$$

Let $C_\Gamma^\infty(TTM \otimes T^*TM)$ denote the space of all connection admissible forms. We will prove that the connection admissible $T(TM)$-valued 1-forms on TM determine connections on TM. First, we will construct a connection by means of a connection admissible form A. Denote $\bar{A} := A \otimes_{TM} \mathrm{id}_{TM} - \mathrm{id}_{TTM} \otimes v \,.\, A =$
$= (a_j^i \delta_s^k - \delta_j^i a_s^k)\, \mathrm{d}x^j \otimes \partial/\partial x^i \otimes \mathrm{d}x^s \otimes \partial/\partial x^k + e_j^i \delta_s^k\, \mathrm{d}x^j \otimes \partial/\partial x_1^i \otimes \mathrm{d}x^s \otimes \partial/\partial x^k +$
$+ (h_j^i \delta_s^k - \delta_j^i a_s^k)\, \mathrm{d}x_1^j \otimes \partial/\partial x_1^i \otimes \mathrm{d}x^s \otimes \partial/\partial x^k$. Then for the coordinate form of $\bar{A}_0 \colon T(TM) \otimes T^*M \to T(TM) \otimes T^*M$ we have

$$\bar{z}_s^i = (a_j^i \delta_s^k - \delta_j^i a_s^k)\, z_k^j \,, \quad Z_s^i = e_j^i \delta_s^k z_k^j + (h_j^i \delta_s^k - \delta_j^i a_s^k)\, Z_k^j \,.$$

Obviously, there exists a unique $Z_0 \in C^\infty(T(TM) \otimes_{TM} T^*M \to TM)$ such that $v \,.\, Z_0 = \mathrm{id}_{p^*_M TM}$ and $\bar{A}_0(Z_0) = 0$. The coordinate equations of $Z_0 = (z_k^i, Z_k^i)$ are as follows:

$$(2) \qquad z_k^i = \delta_k^i\,, \qquad (\delta_j^i a_s^k - h_j^i \delta_s^k)\, Z_k^j = e_s^i\,.$$

Since α_0 is regular, therefore the components Z_k^i of Z_0 are equal to those of $\alpha_0^{-1}(E)$, where E is a local $(1, 1)$-tensor determined by the components (e_s^i). By virtue of the property $v \,.\, Z_0 = \mathrm{id}_{p_M{}^*TM}$, Z_0 is the horizontal form of a connection on TM that will be denoted by Γ_A. If ϕ_{sj}^{iq} are local components of α_0^{-1} then $\Gamma_j^i = \phi_{sj}^{iq} e_q^s$ are the Christoffel functions of Γ_A.

As v, $v \,.\, A$, $A \,.\, v \in C^\infty(TM \otimes {}_{TM}T^*M \to TM)$ we have for example $\varphi = C_1 v + C_2 v \,.\, A + C_3 A \,.\, v + C_4(v \,.\, A) \,.\, (v \,.\, A) + C_5(A \,.\, v) \,.\, (v \,.\, A) \in C^\infty(TM \otimes {}_{TM}T^*M \to TM)$ and consequently $\Gamma_A + \varphi$ is a connection on TM. In general the following proposition holds

Proposition 1. *Let* $A \in C_\Gamma^\infty(T(TM) \otimes T^*TM \to TM)$. *Then the map* $A \mapsto \Gamma_A + \varphi(v \,.\, A,\ A \,.\, v)$, *where* φ *is a natural operator of zero order from* $C^\infty(TM \otimes {}_{TM}T^*M \times {}_{TM}TM \otimes {}_{TM}T^*M \to TM)$ *into* $C^\infty(TM \otimes {}_{TM}T^*M \to TM)$, *is a natural operator of zero order from* $C_\Gamma^\infty(TTM \otimes T^*TM)$ *into the space* ΓTM *of all connections on* TM.

It is possible to prove that every natural operator of zero order from $C^\infty(TTM \otimes T^*TM)$ into ΓTM is of the form $A \mapsto \Gamma_A + \varphi(v \,.\, A,\ A \,.\, v)$.

3. CONNECTIONS DETERMINED BY VECTOR FIELDS ON TM

For further use we introduce the notation $f_i := \partial f/\partial x^i$ and $f_{\bar k} := \partial f/\partial x_1^k$ for a function f on TM.

Let $Z = c^i(x, x_1)\, \partial/\partial x^i + b^i(x, x_1)\, \partial/\partial x_1^i$ be a vector field on TM, that is a section $Z\colon TM \to T(TM)$, $Z(x^i, x_1^i) = (x^i, x_1^i, c^i, b^i)$. Then $TZ(x^i, x_1^i, \mathrm{d}x^i, \mathrm{d}x_1^i) = (x^i, x_1^i, c^i, b^i, \mathrm{d}x^i, \mathrm{d}x_1^i, c_k^i\, \mathrm{d}x^k + c_{\bar k}^i\, \mathrm{d}x_1^k, b_k^i\, \mathrm{d}x^k + b_{\bar k}^i\, \mathrm{d}x^k)$.

Therefore

$$TZ \,.\, v(x^i, x_1^i, \mathrm{d}x^i, \mathrm{d}x_1^i) = (x^i, x_1^i, c^i, b^i, 0, dx_1^i, c_{\bar k}^i\, \mathrm{d}x^k, b_{\bar k}^i\, \mathrm{d}x^k)\,.$$

On the other hand, using the T-prolongation of the canonical involution i_2 we get

$$Ti_2 \,.\, TZ(x^i, x_1^i, \mathrm{d}x^i, \mathrm{d}x_1^i) = (x^i, c^i, x_1^i, b^i, \mathrm{d}x^j, c_k^i\, \mathrm{d}x^k + c_{\bar k}^i\, \mathrm{d}x_1^k\,, \mathrm{d}x_1^i, b_k^i\, \mathrm{d}x^k + b_{\bar k}^i\, \mathrm{d}x_1^k)\,.$$

Let $v_1\colon T(TTM) \to T(TTM)$ be the canonical morphism on $T(TM)$. Then $Ti_2 \,.\, v_1 \,.\, Ti_2 \,.\, TZ(x^i, x_1^i, \mathrm{d}x^i, \mathrm{d}x_1^i) = (x^i, x_1^i, c^i, b^i, 0, \mathrm{d}x^i, 0, c_k^i\, \mathrm{d}x^k + c_{\bar k}^i\, \mathrm{d}x_1^k)$.

In the case of a vector bundle $\pi\colon E \to M$ let $\mathrm{pr}_\pi\colon VE \to E$ denote the canonical projection on the second factor, $\mathrm{pr}_\pi(x^i, y^\alpha, 0, \mathrm{d}y^\alpha) = (x^i, \mathrm{d}y^\alpha)$. Then

$$\mathrm{pr}_{p_{TM}} \,.\, (TZ \,.\, v - Ti_2 \,.\, v_1 \,.\, Ti_2 \,.\, TZ) \,.\, (x^i, x_1^i, \mathrm{d}x^i, \mathrm{d}x_1^i) = = (x^i, x_1^i, c_j^i\, \mathrm{d}x^j, (b_{\bar k}^i - c_k^i)\, \mathrm{d}x^k - c_{\bar k}^i\, \mathrm{d}x_1^k)\,.$$

It means that $\mathrm{pr}_{pTM} . (TZ . v - Ti_2 . v_1 . Ti_2 . TZ) = c^i_j \, \mathrm{d}x^j \otimes \partial/\partial x^i +$ $+ [(b^i_k - c^i_k) \, \mathrm{d}x^k - c^i_k \, \mathrm{d}x^k_1] \otimes \partial/\partial x^i_1$ belongs to $C^\infty(T(TM) \otimes T^*TM \to TM)$. By a direct coordinate calculus we obtain

Lemma 1. *Let $L_Z(v)$ be the Lie derivative of the canonical morphism v by Z. Then*

$$L_Z v = -\mathrm{pr}_{pTM} . (TZ . v - Ti_2 . v_1 . Ti_2 . TZ) .$$

A vector field Z on TM will be said to be connection admissible if $-L_Z v$ is a connection admissible $T(TM)$-form on TM. In coordinates, $v . (-L_Z v) = L_Z v . v =$ $= c^i_k \, \mathrm{d}x^k \otimes \partial/\partial x^i_1$. Therefore Z is connection admissible if and only if

$$\mathrm{id}_{TM} \otimes {}_{TM} v . (-L_Z v) - v . L_Z v \otimes {}_{TM} \mathrm{id}_{TM} = (\delta^i_s c^u_k + c^i_s \delta^u_k)$$

is regular.

Lemma 2. *A vector field Z on TM is connection admissible if and only if there is only the zero solution $x = 0$ of the equation*

$$(v . L_Z v) . x = -x . (v . L_Z v)$$

in the algebra $TM \otimes {}_{TM} T^*M$.

Proof. The map α_0: $y^i_k = (\delta^i_s c^u_k + c^i_s \delta^u_k) \, x^s_u = x^i_u c^u_k + c^i_u x^u_k$ is regular if and only if the equation $x^i_u c^u_k + c^i_u x^u_k = 0$ has only the zero solution. This completes our proof.

Corollary. *If $v . L_Z v$ is regular then Z is connection admissible if and only if the linear operator $x \mapsto (v . L_Z v)^{-1} . x(v . L_Z v)$ in algebra $TM \otimes {}_{TM} T^*M$ does not have the eigenvalue -1.*

Remark. If Z is a differential equation of the second order on M, $c^i = x^i_1$, then $-v . L_Z v = \mathrm{id}_{pM^*TM}$. Therefore in virtue of Lemma 2 every differential equation of the second order is connection admissible.

If Z is connection admissible then the connection $\Gamma_{-L_Z v}$ determined by the connection admissible form $-L_Z v$ will be shortly denoted by Γ_Z. Let $C^\infty_\Gamma TM$ be the space of all connection admissible vector field on TM. With regard to Proposition 1 we obtain

Proposition 2. *Let $Z \in C^\infty_\Gamma TM$. Then any map $Z \mapsto \Gamma_Z + \varphi(v . L_Z v)$, where φ is a natural differential operator of zero order from $C^\infty(TM \otimes {}_{TM} T^*M \to TM)$ into itself, is a natural operator of the first order from $C^\infty_\Gamma TM$ into the space ΓTM of all connections on TM.*

Now, we will prove that every natural differential operator of the first order from $C^\infty_\Gamma TM$ into ΓTM over id_{TM} is of the form $Z \mapsto \Gamma_Z + \varphi(v . L_Z v)$.

Let Tf: $TM \to TN$, TTf: $TTM \to TTN$, JTf: $JTM \to JTN$, $JT(Tf)$: $J(T(TM) \to$ $\to TM) \to J(T(TN) \to TN)$ be the local diffeomorphisms determined by the tangent prolongation functor T and by the first-jet prolongation functor J applied to a local diffeomorphism $f\colon M \to N$. If X or Γ is a vector field or a connection on TM then $TTf(X)$ or $JTf(\Gamma)$ is a vector field or a connection on TN, respectively. Recall that the condition for an operator A from $C^\infty TM$ into ΓTM to be natural is

$$A_N(TTf(X)) = JTf(A_M(X))$$

for any local diffeomorphism f from M into N and every vector field X on TM. Then A is of the first order if

$$(j_h^1 X_1 = j_h^1 X_2) \Rightarrow (AX_1(h) = AX_2(h))$$

for any $X_1, X_2 \in C^\infty TM$, $h \in TM$.

By the theory of natural functors and operators, see [2], [4], [6], [8], the set of all natural operators of the first order from $C^\infty TM$ into ΓTM over TM is in a bijection with the space of all natural transformations Φ from $J(T(TM) \to TM)$ into JTM over id_{TM}. Recall that Φ is a family of maps from $JT(TM)$ into JTM such that $JTf \, . \, \Phi_M = \Phi_N \, . \, JT(Tf)$ for any local diffeomorhism f from M into N.

We will need the coordinate forms of $JT(Tf)$ and JTf. Let (x^i, x_1^i), (x^i, x_1^i, c^i, b^i), (x^i, x_1^i, x_j^i), $(x^i, x_1^i, c^i, b^i, c_k^i, c_{\bar{k}}^i, b_k^i, b_{\bar{k}}^i)$ be local charts on TM, TTM, JTM, $JT(TM)$, respectively. Then $\bar{x}^i = f^i(x^j)$ and

$$\bar{x}_1^i = f_j^i x_1^j \tag{3}$$

are the equations of Tf. Adding the equations

$$\bar{c}^i = f_j^i c^j \,, \quad \bar{b}^i = f_{jk}^i x_1^j c^k + f_j^i b^j \tag{4}$$

to those of Tf we get the local expression of TTf.

Let $\tilde{f}\colon N \to M$ be the inverse map to f. Let $g \in J(T(TM) \to TM)$, $g = j_h^1(u \mapsto \sigma(u)) = j_{(x,x_1)}^1((u^i, u_1^i) \mapsto (u^i, u_1^i, \gamma^i(u, u_1), \beta^i(u, u_1))$. Then $\bar{g} = JT(Tf)(g) = j_{\bar{h}}^1(\bar{u} \mapsto TTf.\, \sigma(T\tilde{f}(\bar{u})) = j_{(\bar{x},\bar{x}_1)}^1(\bar{u}, \bar{u}_1) \mapsto (\bar{u}, \bar{u}_1, f_j^i(\tilde{f}(\bar{u}))\, \gamma^j(\tilde{f}(\bar{u}), \tilde{f}_t^s(\bar{u})\, \bar{u}_1^t),$ $f_{jk}^i(\tilde{f}(\bar{u}))\, \tilde{f}_t^j(\bar{u})\, \bar{u}_1^t \gamma^k(\tilde{f}(\bar{u}), \tilde{f}_q^p(\bar{u})\, \bar{u}_1^q) + f_j^i(\tilde{f}(\bar{u}))\, \gamma^j(\tilde{f}(\bar{u}), f_q^p(\bar{u})\, \bar{u}_1^q))$. After some calculation we obtain

$$\begin{aligned}
\bar{c}_j^i &= f_{sp}^i \tilde{f}_j^p c^s + f_s^i c_p^s \tilde{f}_j^p + f_s^i c_{\bar{p}}^s \tilde{f}_{tj}^p f_q^t x_1^q \,, \\
\bar{c}_{\bar{k}}^i &= f_t^i c_{\bar{u}}^t \tilde{f}_k^u \,, \\
\bar{b}_j^i &= f_{qks}^i \tilde{f}_j^s x_1^q c^k + f_{qk}^i \tilde{f}_{tj}^q f_s^t x_1^s c^k + f_{qk}^i x_1^q (c_u^k f_j^u + \\
&\quad + c_{\bar{u}}^k \tilde{f}_{pj}^u f_v^p x_1^v) + f_{qu}^i \tilde{f}_j^u b^q + f_q^i (b_p^q \tilde{f}_j^p + b_{\bar{p}}^q \tilde{f}_{sj}^p \tilde{f}_v^s x_1^v) \,, \\
\bar{b}_{\bar{k}}^i &= f_{qs}^i \tilde{f}_k^q c^s + f_{qs}^i x_1^q c_{\bar{u}}^s \tilde{f}_k^u + f_q^i b_{\bar{s}}^q \tilde{f}_k^s
\end{aligned} \tag{5}$$

where $\tilde{f}_s^i f_j^s = \delta_j^i$ and $\tilde{f}_{us}^i f_j^s f_k^u + \tilde{f}_u^i f_{kj}^u = 0$. It means that the map $JT(Tf)$ is locally determined by (3), (4), (5). It remains to derive the coordinate form of JTf. Let $h = (x^i, x_1^i, x_j^i) = j_x^1((u^i) \mapsto (u^i, \sigma^i(u))) \in JTM$. Then $\bar{h} = JTf(h) = j_{\bar{x}}^1(\bar{u}) \mapsto$ $\mapsto (\bar{u}^i, f_t^i(\tilde{f}(\bar{u}))\, \sigma^t(\tilde{f}(\bar{u})))$, i.e.

$$\bar{x}_j^i = f_{tu}^i \tilde{f}_j^u x_1^t + f_t^i x_s^t \tilde{f}_j^s \,. \tag{6}$$

This equation together with (3) yields JTf.

Let $h = (x^i, x_1^i, c^i, b^i, c_k^i, c_{\bar{k}}^i, b_k^i, b_{\bar{k}}^i) = j_u^1\sigma \in J(T(TM) \to TM)$. Being a local vector field on TM, σ locally determines $v\,.\,(-L_\sigma v)$. Denote $v\,.\,(-L_\sigma v)(u) := h_\Gamma$, $h_\Gamma = c_k^i \, dx^k \otimes \partial/\partial x^i \in TM \otimes_{TM} T^*M$. We will say that h is a connection element if

$$\mathrm{id}_{T_{PM(u)}M} \otimes_{TM} h_\Gamma + h_\Gamma \otimes_{TM} \mathrm{id}_{T_{PM(u)}M}$$

is regular. A vector field Z on TM is connection admissible if its jet prolongation JZ states a connection element $JZ(u)$ at every $u \in TM$.

It is easy to see that $JT(TM) \to M$ is a fibred manifold associated with the principle fibre bundle (H^3M, L_m^3) of all frames of the third order on M the structure group of which is the group L_m^3 of all 3-jets $j_0^3 f$ of all local diffeomorphisms f from R^m into R^m such that $f(0) = 0$. The action of L_m^3 on the type fibre $(JT(TR^m))_0$ is given by (3), (4), (5).

Quite anologously, JTM is associated with (H^2M, L_m^2) and the action of L_m^2 on $(JTR^m)_0$ is described by the equations (3) and (6).

There is a bijection between the space of all natural transformations Φ from $JT(TM)$ into JTM over id_{TM} and the set of all L_m^3-equivariant maps ψ from $(JT(TR^m))_0$ into $(JTR^m)_0$ over $\mathrm{id}_{TR^m{}_0}$ such that $\pi_2^3 f\,.\,\psi = \psi\,.\,f$ for all $f \in L_m^3$, where $\pi_2^3 \colon L_m^3 \to L_m^2$ is the group homomorphism determined by the projection of a 3-jet onto its 2-subjet. This means that our goal consists in finding all functions $\Gamma_j^i = \psi_j^i(x_1^p, c^p, b^p, c_k^p, c_{\bar{k}}^p, b_k^p, b_{\bar{k}}^p)$ such that

$$(7) \qquad f_{tu}^i \tilde{f}_j^u x_1^u + f_t^i \psi_u^t \tilde{f}_j^u = \psi_j^i(\bar{x}_1^p, \bar{c}^p, \bar{b}^p, \bar{c}_k^p, \bar{c}_{\bar{k}}^p, \bar{b}_k^p, \bar{b}_{\bar{k}}^p)\,,$$

where $\bar{x}_1^p, \bar{c}^p, \bar{b}^p, \bar{c}_k^p, \bar{c}_{\bar{k}}^p, \bar{b}_k^p, \bar{b}_{\bar{k}}^p$ are given by (3), (4), (5).

For any homothety $(k\delta_j^i, f_{jp}^i = 0, f_{jkt}^i = 0) \in L_m^3$ the relation (7) is of the form

$$\psi_j^i(x_1^p, c^p, b^p, c_k^p, c_{\bar{k}}^p, b_k^p, b_{\bar{k}}^p) = \Phi_j^i(kx_1^p, kc^p, kb^p, c_k^p, c_{\bar{k}}^p, b_k^p, b_{\bar{k}}^p)\,.$$

It implies that the functions ψ_j^i do not depend on x_1^p, c^p, b^p. Now, (7) is satisfied for every $f = (f_j^i = \delta_j^i, f_{jk}^i = 0, f_{jks}^i) \in \operatorname{Ker} \pi_2^3$ if and only if $\psi_j^i(c_k^p, c_{\bar{k}}^p, b_k^p, b_{\bar{k}}^p) = \psi_j^i(c_k^p, c_{\bar{k}}^p, f_{qtk}^p x_1^q c^t + b_k^p, b_{\bar{k}}^p)$. Therefore ψ_j^i are independent of b_k^p. Let $\pi_1^3 \colon L_m^3 \to L_m^1$ be the group homomorphism under which $\pi_1^3(f)$ is the 1-subjet of a 3-jet f. With respect to $f = (f_j^i = \delta_j^i, f_{jk}^i, f_{jks}^i) \in \operatorname{Ker} \pi_1^3$ and for $x_1^i = 0$ the equation of equivariance is of the form

$$\psi_j^i(c_k^p, c_{\bar{k}}^p, b_{\bar{k}}^p) = \psi_j^i(c_k^p + t_k^p, c_{\bar{k}}^p, b_{\bar{k}}^p + t_k^p)\,, \quad t_k^p = f_{qk}^p c^q\,.$$

Consequently $\psi_j^i = \psi_j^i(d_k^p, c_{\bar{k}}^p)$, $d_k^p = c_k^p - b_{\bar{k}}^p$. Now, for $f \in \operatorname{Ket} \pi_1^3$ we have

$$(8) \qquad f_{tj}^i x_1^t + \psi_j^i(d_k^p, c_{\bar{k}}^p) = \psi_j^i(d_k^p - c_{\bar{s}}^p f_{kt}^s x_1^t - f_{sq}^p x_1^p c_{\bar{k}}^q, c_{\bar{k}}^p)\,.$$

Differentating by d_s^k we deduce that $\partial\psi_j^i/\partial d_k^p$ does not depend on d_k^p, i.e. $\psi_j^i = \Phi_{sj}^{iq}(c_{\bar{k}}^p)\, d_q^s + \varphi_j^i(c_{\bar{k}}^p)$. Now (8) implies

$$f_{tj}^i = -\Phi_{sj}^{iq}(c_{\bar{u}}^s \delta_q^p + \delta_u^s c_{\bar{q}}^p) f_{pt}^u\,.$$

It means that the functions $\phi^{iq}_{sj}(c^p_k)$ are defined at $h \in (JT(TR^m))_0$ if and only if h is a connection element. In this case Φ^{iq}_{sj} are the components of the tensor Φ which is determined by the inverse map to $\alpha_0 = (\mathrm{id}_{R^*} \otimes h_\Gamma + h_\Gamma \otimes \mathrm{id}_{R^m})_0$. It establishes an L^1_m-equivariant map $h_\Gamma \mapsto \Phi$ from $R^m \otimes R^{m*}$ into $(R^m \otimes R^{m*}) \otimes (R^m \otimes R^{m*})$, i.e. we have

$$\Phi^{iq}_{sj}(f^i_q c^q_k \tilde{f}^k_u) = f^i_p \Phi^{pv}_{ek}(c^r_{\bar{w}}) \tilde{f}^e_s \tilde{f}^k_j f^q_v .$$

Consequently, the equivariance with respect to the subgroup $L^1_m \subset L^3_m$ leads to the equation

$$f^i_q \varphi^q_r(c^p_k) \tilde{f}^i_j = \varphi^i_j(f^p_t c^t_r \tilde{f}^r_k) .$$

This implies that if an L^3_m-equivariant map from $(JT(TR^m))_0$ into $(JTR^m)_0$ exists then it is of the form $\Gamma^i_j = \Phi^{iq}_{sj}(c^s_q - b^s_q) + \varphi^i_j(c^p_k)$, where φ^i_j is an L^i_m-equivariant map from $R^m \otimes R^{m*}$ into itself. We have proved

Proposition 2. *Only in the case of a connection admissible vector field Z on TM there is a connection Γ_Z on TM naturally associated with Z in the first order. Every natural differential operator of the first order from $C^\infty_\Gamma TM$ into ΓTM is of the form $Z \mapsto \Gamma_Z + \varphi(v \,.\, L_Z v)$, where φ is a natural zero-order operator on $C^\infty(TM \otimes {}_{TM}T^*M)$ over* id_{TM}.

Remarks. 1. Let Z be a projectable vector field on TM, $c^i = c^i(x)$. Then $v \,.\, L_Z v = 0$. Therefore Z is not connection admissible.

2. Let Z be a vector field on TM such that $v \,.\, L_Z v$ is a homothety on $p^*_M TM$, $v \,.\, L_Z v = g(x, x_1)\, \delta^i_j \,\mathrm{d}x^j \otimes \partial/\partial x^i$. Then $\mathrm{id}_{TM} \otimes v \,.\, L_Z v + v \,.\, L_Z v \otimes \mathrm{id}_{TM} = (-2g(x, x_1)\, \delta^i_s \delta^u_j)$. Therefore Z is connection admissible iff $g(x, x_1) \neq 0$. Then $\Phi^{iq}_{sj} = -(1/(g(x, x_1)))\, \delta^i_s \delta^q_j$ and $\Gamma^i_k = -(1/(2g(x, x_1)))\,(c^i_k - b^i_k) + \varphi(x, x_1)\, \delta^i_k$ where $\varphi(x, x_1)$ is an element of the space $\langle g(x, x_1)\rangle$ of all real functions on TM generated by $g(x, x_1)$. If Z is a differential equation of the second order on M, $c^i = x^i_1$, $g(x, x_1) = 1$, then $\Gamma^i_k = \frac{1}{2} b^i_k + c\delta^i_k$, $c \in R$, see [1].

3. Let $C = x^i_1\, \partial/\partial x^i_1$ be the Liouville field on TM. Let Z be a homogeneous field on TM; $[C, Z] = Z$, $c^i = c^i_j(x)\, x^j_i$, $b^i = \frac{1}{2} b^i_{jk}(x)\, x^j_1 x^k_1$. Then $v \,.\, L_Z v$ is projectable and it is easy to see that if Z is also connection admissible then the connection Γ_Z is linear.

4. In this paper we have dealt with operators of the first order. Our considerations about the connection admissible forms on TM offer methods for construction of connections associated in higher orders with the vector on TM. We will introduce an example of the second order. Let $Z = c^i\, \partial/\partial x^i + b^i\, \partial/\partial x^i_1$ be a vector field on TM. Then $L_Z(-v \,.\, L_Z) = -c^i_s c^s_k \,\mathrm{d}x^k \otimes \partial/\partial x^i + [(c^i_s c^s_k + c^i_{ks} c^s + c^i_{\bar{k}\bar{s}} b^s - b^i_s c^s_k)\,\mathrm{d}x^k + c^i_s c^s_k \,\mathrm{d}x^k_1] \otimes \partial/\partial x^i_1$ is a $T(TM)$-valued 1-form on TM. By (1) it is connection admissible if and only if $(y^i_q c^u_k + c^i_q y^u_k)\, c^q_u = 0$ implies $y^i_q = 0$. This is true if $\det(c^q_u) \neq 0$. It means that if $v \,.\, L_Z v$ is regular then the map $Z \mapsto \Gamma_{L_Z(v.L_Z v)}$ is an operator

of the second order from $C^\infty(T(TM) \to TM)$ into ΓTM. Let us note that if Z is a differential equation of the second order then $-v \,.\, L_Z v = v$.

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Súhrn

VEKTOROVÉ POLIA A KONEXIE NA *TM*

ANTON DEKRÉT

V práci je charakterizovaná množina $C^\infty_\Gamma TM$ všetkých vektorových polí na TM, ktoré určujú konexie na TM. Sú zostrojené všetka prirodzené operátory prvého rádu z $C^\infty_\Gamma TM$ do priestoru všetkých konexií na TM.

Резюме

ВЕКТОРНЫЕ ПОЛЯ И СВЯЗНОСТИ НА *TM*

ANTON DEKRÉT

В настоящей статье характеризуется множество $C^\infty_\Gamma TM$ тех векторных полей на TM, которые определяют связности на пространстве TM. Построены все натуральные дифференциальные операторы первого класса из $C^\infty_\Gamma TM$ в множество всех связностей на TM.

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