Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 115 (1990), No. 3, 293--306

Persistent URL: http://dml.cz/dmlcz/118409

115 (1990) ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY No. 3, 293–306

MOMENTS OF ORDER STATISTICS

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(Received October 31, 1988)

Summary. Let $X_{1:n} \leqq X_{2:n} \leqq \ldots \leqq X_{n:n} = M_n$ denote the order statistics of a sample of size n. In this paper we investigate the asymptotic behaviour of $\mathsf{E}(M_n)$ and $\mathsf{E}(X_{n-k:n})$ as $n \to \infty$. We show that $\{\mathsf{E}(M_n)\}_N$ and all its differentials $(\Delta^i \mathsf{E}(M_n)\}_N$ are regularly varying sequences if the underlying d.f. has a regularly varying tail.

Keywords: Order statistics, asymptotic results, rate of convergence, regularly varying functions.

AMS Classification: 60F99.

INTRODUCTION

Let $X_{1:n} \leqq \ldots \leqq X_{n:n} = M_n$ denote the order statistics of a random sample $X_1, X_2, \ldots, X_n$ of size n from a distribution with a distribution function (d.f.) F which is concentrated on R_+. In this paper we shall be concerned with the asymptotic behaviour of $\mathsf{E}(M_n)$ and $\mathsf{E}(X_{n-k:n})$ as $n \to \infty$. In Section 1 we show that $\mathsf{E}(M_n)/n \to 0$ as $n \to \infty$ if $\mathsf{E}(X_1) < \infty$. If not only $\mathsf{E}(X_1)$ is finite but if X_1 belongs to the max-domain of attraction of a stable law, we show that $\mathsf{E}(M_n)$ and $\mathsf{E}(X_{n-k:n})$ have a very nice regularly varying behaviour. In Section 2 we discuss regularly varying and 0-regularly varying behaviour of $\mathsf{E}(X_{n-k:n})$, $k = 0, 1, \ldots$. Among others we show that $1 - F \in \mathrm{RV}_{-\alpha}$ implies that

$$\lim_{n \to \infty} \frac{\mathsf{E}(M_n)}{a_n} = \Gamma(1 - 1/\alpha)$$

for a (known) sequence $\{a_n\}$. In Section 3 some rate of convergence results are established.

1. MOMENT CONDITIONS

It is well-known that for i.i.d. non-negative random variables $X_1, X_2, \ldots$ we have $\mathsf{E}(M_n) < \infty$ if and only if $\mathsf{E}(X_1) < \infty$. In the next proposition we obtain some more precise information concerning $\mathsf{E}(M_n)$.

Proposition 1.1. *Assume that $X_1, \ldots, X_n$ are i.i.d. non-negative random variables and that $\mathsf{E}(X_1) < \infty$. Define $\mu_n := \mathsf{E}(M_n)/n$. Then*

(i) $\lim_{n\to\infty} \mu_n = 0$;

(ii) μ_n *is non-increasing and* $\lim_{n\to\infty} (\mu_n/\mu_{n-1}) = 1$;

(iii) $\lim_{n\to\infty} [\mathsf{E}(M_n) - \mathsf{E}(M_{n-1})] = \lim_{n\to\infty} n(\mu_n - \mu_{n-1}) = 0$.

Proof. (i) We have $0 \leqq \mathsf{P}(M_n > x) = 1 - F^n(x) \leqq n(1 - F(x))$. Since $\mathsf{E}(X_1) < \infty$ and $\lim_{n\to\infty} (1/n)$. $\mathsf{P}(M_n > x) = 0$, by Lebesgue's theorem on dominated convergence we obtain that

$$\lim_{n\to\infty} \frac{\mathsf{E}(M_n)}{n} = \lim_{n\to\infty} \int_0^\infty \frac{\mathsf{P}(M_n > x)}{n} \, dx = 0 .$$

(ii) and (iii) A general result in the theory of order statistics [3], p. 37 states that

$$(1.1) \qquad (n - r)\, \mathsf{E}(X_{r:n}) + r\mathsf{E}(X_{r+1:n}) = n\mathsf{E}(X_{r:n-1}) .$$

Applying (1.1) for $r = n - 1$ we obtain that

$$\mathsf{E}(X_{n-1:n}) = n\, \mathsf{E}(M_{n-1}) - (n - 1)\, \mathsf{E}(M_n) .$$

It follows that

$$\mu_n \leqq \mu_{n-1} = \mu_n + \frac{\mathsf{E}(X_{n-1:n})}{n(n-1)} \leqq \mu_n + \mu_n \frac{1}{n-1} .$$

Hence μ_n is non-increasing and $\mu_n \leqq \mu_{n-1} \leqq \mu_n(n/n - 1)$. Now the results (ii) and (iii) easily follow. ■

The following corollary follows immediately.

Corollary 1.2. *Assume that $X_1, \ldots, X_n$ are i.i.d. non-negative random variables and that $\mathsf{E}(X_1^\beta) < \infty$ for some $\beta \geqq 1$. Then*

$$\lim_{n\to\infty} \frac{\mathsf{E}(M_n)}{n^{1/\beta}} = 0 .$$

Proof. Let $Y_1 = X_1^\beta$; from Proposition 1.1 (i) we obtain that $\lim_{n\to\infty} (\mathsf{E}(M_n^\beta)/n) = 0$. Now apply Hölder's inequality. ■

As to real random variables, we have

Corollary 1.3. *Assume $X_1, \ldots, X_n$ are i.i.d. random variables and assume g is non-decreasing. If $\mathsf{E}|g(X_1)| < \infty$, then $\lim_{n\to\infty} \mathsf{E}(g(M_n))/n = 0$.*

Proof. Since $|\mathsf{E}(g(M_n))| \leqq \mathsf{E}(\max_{1 \leqq i \leqq n} |g(X_i)|)$ the result follows from Proposition 1.1 (i) with $Y_1 = |g(X_1)|$. ■

2. O-REGULARLY VARYING AND REGULARLY VARYING BEHAVIOUR

Recall that a measurable function $f\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is O-regularly varying (we write $f \in \mathrm{ORV}$) if for all $x > 0$,

$$\limsup_{t \to \infty} \frac{f(tx)}{f(t)} < \infty .$$

Further, a measurable function $f\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is regularly varying with an index $\alpha \in \boldsymbol{R}$ (we write $f \in \mathrm{RV}_\alpha$) if for all $x > 0$,

$$\lim_{t \to \infty} \frac{f(tx)}{f(t)} = x^\alpha .$$

It is well-known [1] that $f \in \mathrm{ORV}$ implies the existence of constants α, A, and t_0 such that

$$\frac{f(tx)}{f(t)} \leqq Ax^\alpha \quad \text{for all} \quad x \geqq 1 . \quad t \geqq t_0 .$$

We call α an upper index of f. For details we refer to [1], [2], [4], and [7].

Finally, we say that a sequence $\{a_n\}_{\boldsymbol{N}}$ ($\boldsymbol{N}$ is the set of all positive integers) of non-negative real numbers is regularly varying if the function f defined by $f(x) = a_{[x]}$ is a regularly varying function where as usual $[x]$ is the integer part of x.

In order to estimate $\mathsf{E}(M_n)$ we start with some auxiliary results.

Lemma 2.1. *Let F be a distribution function concentrated on $\boldsymbol{R}_+$ and let a_n be defined as $a_n := \inf\{x\colon 1 - F(x) \leqq 1/n\}$.*

(i) *If $1 - F \in \mathrm{ORV}$ is such that for some $\beta > 1$, $A > 0$ and $t_0 > 0$,*

$$\text{(2.1)} \qquad \frac{1 - F(tx)}{1 - F(t)} \leqq Ax^{-\beta} \quad \textit{for all} \quad x \geqq 1 \quad \textit{and} \quad t \geqq t_0$$

holds, then there exists a number $n_0 \in \boldsymbol{N}$ such that

$$\text{(2.2)} \qquad \mathsf{P}\{M_n > a_n x\} \leqq Ax^{-\beta} \quad \textit{for} \quad x \geqq 1 , \quad n \geqq n_0 .$$

(ii) *If $1 - F \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 1$, then $\{a_n\}_{\boldsymbol{N}} \in \mathrm{RV}_{1/\alpha}$ and for all $x > 0$, $\lim_{n \to \infty} n(1 - F(a_n x)) = x^{-\alpha}$ and*

$$\text{(2.3)} \qquad \lim_{n \to \infty} \mathsf{P}\{M_n \leqq a_n x\} = e^{-x^{-\alpha}}$$

hold. Moreover, (2.2) holds with $\beta = \alpha - \varepsilon > 1$.

Proof. (i) Obviously

$$\mathsf{P}\{M_n > a_n x\} = 1 - F^n(a_n x) \leqq n(1 - F(a_n x)) \leqq \frac{1 - F(a_n x)}{1 - F(a_n)}.$$

Since $a_n \to \infty (n \to \infty)$, the first part of (2.2) follows from (2.1). The second part of (2.2) is trivially true.

(ii) These results are well-known from the extreme value theory, see e.g. [5]. ∎

Now we estimate $\mathsf{E}(M_n)$ using the classes RV and ORV. As before we shall assume that X_1 is a nonnegative r.v. and that $\{a_n\}_N$ is defined as in Lemma 2.1.

Theorem 2.2. *Let F denote a d.f. on R_+.*

(i) *If $1 - F \in \mathrm{ORV}$ is such that (2.1) holds, then*

$$\limsup_{n\to\infty} \frac{\mathsf{E}(M_n)}{a_n} < \infty .$$

(ii) *If $1 - F \in \mathrm{RV}_{-\alpha}$, $\alpha > 1$, then $\{\mathsf{E}(M_n)\}_N \in \mathrm{RV}_{1/\alpha}$ and*

$$\lim_{n\to\infty} \frac{\mathsf{E}(M_n)}{a_n} = \Gamma(1 - 1/\alpha) \tag{2.4}$$

where $\Gamma(\cdot)$ denotes the gama function.

Proof. (i) Since

$$\frac{\mathsf{E}(M_n)}{a_n} = \int_0^\infty \mathsf{P}\{M_n > a_n x\}\, \mathrm{d}x ,$$

the result is a consequence of (2.2).

(ii) Using (2.2), (2.3) and Lebesgue's theorem on dominated convergence we have

$$\lim_{n\to\infty} \frac{\mathsf{E}(M_n)}{a_n} = \lim_{n\to\infty} \int_0^\infty \mathsf{P}\{M_n > a_n x\}\, \mathrm{d}x = \int_0^\infty (1 - e^{-x^{-\alpha}})\, \mathrm{d}x = \Gamma(1 - 1/\alpha).$$

Hence (2.4) follows; since $\{a_n\}_N \in \mathrm{RV}_{1/\alpha}$, also $\{\mathsf{E}(M_n)\}_N \in \mathrm{RV}_{1/\alpha}$. ∎

Corollary 2.3. *Let F denote a d.f. on R_+, and let a_n be defined as before. If $1 - F \in$ $\in \mathrm{RV}_{-\alpha}$ with $\alpha > \beta$, then*

$$\lim_{n\to\infty} \frac{\mathsf{E}(M_n^\beta)}{a_n^\beta} = \Gamma\left(1 - \frac{\beta}{\alpha}\right).$$

Proof. Use Theorem 2.2 (ii) with $Y_1 := X_1^\beta$. ∎

In our next result we show that for each $k \in N \cup \{0\}$ the k-th differential $\Delta^k\, \mathsf{E}(M_n)$

is regularly varying. To formulate the result we define

$$\Delta^0 \mathsf{E}(M_n) = \mathsf{E}(M_n)\,,$$

$$\Delta^{k+1} \mathsf{E}(M_n) = \Delta^k \mathsf{E}(M_{n+1}) - \Delta^k \mathsf{E}(M_n)\,.$$

Obviously, $\Delta^1 \mathsf{E}(M_n) = \int_0^\infty F^n(x)\,(1 - F(x))\,\mathrm{d}x$ and by induction over k it follows that

$$\Delta^k \mathsf{E}(M_n) = (-1)^{k+1} \int_0^\infty F^n(x)\,(1 - F(x))^k\,\mathrm{d}x$$

so that

$$\frac{(-1)^{k+1}\, n^k \Delta^k \mathsf{E}(M_n)}{a_n} = \int_0^\infty F^n(a_n x)\,[n(1 - F(a_n x)]^k\,\mathrm{d}x\,. \tag{2.5}$$

Now we prove

Theorem 2.4. *If* $1 - F \in \mathrm{RV}_{-\alpha}$, $\alpha > 1$, *then for each* $k \in N$

(i) $\displaystyle\lim_{n\to\infty} \frac{(-1)^{k+1}\, n^k \Delta^k \mathsf{E}(M_n)}{a_n} = \frac{1}{\alpha}\Gamma\left(k - \frac{1}{\alpha}\right)$;

(ii) $\{(-1)^{k+1}\, \Delta^k \mathsf{E}(M_n)\}_N \in \mathrm{RV}_{1/\alpha-k}$;

(iii) $\displaystyle\lim_{n\to\infty} \frac{n[\Delta^k \mathsf{E}(M_{n+1}) - \Delta^k \mathsf{E}(M_n)]}{\Delta^k \mathsf{E}(M_n)} = \frac{1}{\alpha} - k\,.$

Proof.

(i) Let $f(z, n, k) = z^n[n(1 - z)]^k$ $(0 \leqq z \leqq 1)$. It is easily seen that $0 \leqq f(z, n, k) \leqq$ $\leqq f(n/(n + k), n, k) \leqq k^k$ $(0 \leqq z \leqq 1)$; substituting $z = F(a_n x)$ we have

$$0 \leqq F^n(a_n x)\,(n(1 - F(a_n x))^k \leqq k^k \quad \text{for all} \quad x \geqq 0\,, \quad k \in N\,. \tag{2.6}$$

Also, from (2.1) with $\beta = \alpha - \varepsilon > 1$ we have

$$0 \leqq F^n(a_n x)\,[n(1 - F(a_n x)]^k \leqq A^k x^{-k\beta} \quad \text{for all} \quad x \geqq 1\,, \quad n \geqq n_0\,. \tag{2.7}$$

Now combine (2.5), (2.6) and (2.7) and Lebesgue's theorem on dominated convergence to obtain

$$\lim_{n\to\infty} \frac{(-1)^{k+1}\, n^k \Delta^k \mathsf{E}(M_n)}{a_n} = \int_0^\infty e^{-x^{-\alpha}} x^{-\alpha k}\,\mathrm{d}x\,,$$

which proves (i).

(ii) This assertion immediately follows from (i) and the regular variation of $\{a_n\}_N$.

(iii) Using (i) with k replaced by $k + 1$ we have

$$\lim_{n\to\infty} \frac{(-1)^{k+2}\, n^{k+1}(\Delta^k \mathsf{E}(M_{n+1}) - \Delta^k \mathsf{E}(M_n))}{a_n} = \frac{1}{\alpha}\Gamma\left(k + 1 - \frac{1}{\alpha}\right).$$

Using (i) once again, we obtain

$$\lim_{n\to\infty} \frac{(-1)\, n(\Delta^k \mathsf{E}(M_{n+1}) - \Delta^k \mathsf{E}(M_n))}{\Delta^k \mathsf{E}(M_n)} = \frac{\Gamma\left(k + 1 - \frac{1}{\alpha}\right)}{\Gamma\left(k - \frac{1}{\alpha}\right)},$$

from which the result (iii) follows. ∎

Remark. The previous result shows that the sequence $\{\mathsf{E}(M_n)\}_N$ is regularly varying together with all its "derivatives" $\Delta^k \mathsf{E}(M_n)$. This illustrates that the operations $(X_1, X_2, \ldots, X_n) \to M_n \to \mathsf{E}(M_n)$ have very smoothing character.

In our next result we estimate $\mathsf{E}(X_{n-k:n})$ for fixed k, as $n \to \infty$. We first express $\mathsf{E}(X_{n-k:n})$ in terms of $\Delta^i \mathsf{E}(M_j)$.

Lemma 2.5. *Let $n \in N$, $k \in N \cup \{0\}$, $k \leqq n$, and let $X_1, \ldots, X_n$ be i.i.d. random variables. Then*

$$\mathsf{E}(X_{n-k:n}) = \sum_{i=0}^{k} (-1)^i \binom{n}{i} \Delta^i \mathsf{E}(M_{n-i}) \tag{2.8}$$

and

$$\mathsf{E}(X_{n-k+1:n+1}) - \mathsf{E}(X_{n-k:n}) = (-1)^k \binom{n}{k} \Delta^{k+1} \mathsf{E}(M_{n-k}). \tag{2.9}$$

Proof. The relation (2.8) is obviously true for $k = 0$ and $n \in N$. Suppose it holds for all $k \leqq K$ and all $n \geqq K$. We prove that the relation holds for $k = K + 1$ and all $n > K$. By (1.1) with $r = n - K - 1$ we have

$$\mathsf{E}(X_{n-K-1:n}) = \frac{n}{K+1} \mathsf{E}(X_{n-1-K:n-1}) - \frac{n-K-1}{K+1} \mathsf{E}(X_{n-K:n}),$$

by (2.7) we obtain

$$\begin{aligned}
\mathsf{E}(X_{n-K-1:n}) &= \frac{1}{K+1} \sum_{0}^{K} (-1)^i \binom{n}{i} \{(n-i)\, \Delta^i \mathsf{E}(M_{n-1-i}) - \\
&\quad - (n-K-1)\, \Delta^i \mathsf{E}(M_{n-i})\} = \\
&= \frac{1}{K+1} \sum_{0}^{K} (-1)^i \binom{n}{i} \{(K+1-i)\, \Delta^i \mathsf{E}(M_{n-i}) - \\
&\quad - (n-i)\, \Delta^{i+1} \mathsf{E}(M_{n-i-1})\} = \\
&= \frac{1}{K+1} \sum_{i=0}^{K} (-1)^i \binom{n}{i} (K+1-i)\, \Delta^i \mathsf{E}(M_{n-i}) + \\
&\quad + \frac{1}{K+1} \sum_{j=1}^{K+1} (-1)^j \binom{n}{j-1} (n-j+1)\, \Delta^j \mathsf{E}(M_{n-j}) =
\end{aligned}$$

$$= \frac{1}{K+1}\sum_{i=1}^{K}(-1)^i\left[\binom{n}{i}(K+1-i)+\binom{n}{i-1}(n-i+1)\right].$$

$$.\,\Delta^i\,\mathsf{E}(M_{n-i}) + \Delta^0\,\mathsf{E}(M_n) + (-1)^{K+1}\binom{n}{K+1}\Delta^{K+1}\,\mathsf{E}(M_{n-K-1}) =$$

$$= \sum_0^{K+1}(-1)^i\binom{n}{i}\Delta^i\,\mathsf{E}(M_{n-i}).$$

This proves (2.8).

To prove (2.9) we use (2.8) twice to obtain

$$\mathsf{E}(X_{n+1-k:n+1}) - \mathsf{E}(X_{n-k:n}) =$$

$$= \sum_0^k(-1)^i\binom{n+1}{i}\Delta^i\,\mathsf{E}(M_{n+1-i}) - \sum_0^k(-1)^i\binom{n}{i}\Delta^i\,\mathsf{E}(M_{n-i}).$$

Using $\Delta^i\,\mathsf{E}(M_{n-i}) = \Delta^i\,\mathsf{E}(M_{n+1-i}) - \Delta^{i+1}\,\mathsf{E}(M_{n-i})$ we obtain

$$\mathsf{E}(X_{n+1-k:n+1}) - \mathsf{E}(X_{n-k:n}) =$$

$$= \sum_0^k(-1)^i\binom{n}{i}\Delta^{i+1}\,\mathsf{E}(M_{n-i}) + \sum_1^k(-1)^i\binom{n}{i-1}\Delta^i\,\mathsf{E}(M_{n+1-i}) =$$

$$= (-1)^k\binom{n}{k}\Delta^{k+1}\,\mathsf{E}(M_{n-k}).\quad \blacksquare$$

Now we prove

Theorem 2.6. *If* $1 - F \in \mathrm{RV}_{-\alpha}$ *with* $\alpha > 1$, *then for each* $k \in N \cup \{0\}$

$$(2.10)\qquad \lim_{n\to\infty}\frac{\mathsf{E}(X_{n-k:n})}{a_n} = \frac{\Gamma\left(k+1-\frac{1}{\alpha}\right)}{k!}$$

and

$$(2.11)\qquad \lim_{n\to\infty}\frac{\mathsf{E}(X_{n+1-k:n+1}) - \mathsf{E}(X_{n-k:n})}{a_n}\,n = \frac{\Gamma\left(k+1-\frac{1}{\alpha}\right)}{\alpha k!}.$$

Proof. From Theorem 2.2, Theorem 2.4 and the regular variation of $\{a_n\}_N$ we obtain that

$$\lim_{n\to\infty}\frac{(-1)^i\binom{n}{i}\Delta^i\,\mathsf{E}(M_{n-i})}{a_n} = \begin{cases}\Gamma\left(1-\frac{1}{\alpha}\right) & \text{if } i = 0,\\ -\frac{1}{\alpha}\Gamma\left(i-\frac{1}{\alpha}\right)\frac{1}{i!} & \text{if } i \geqq 1.\end{cases}$$

Using (2.8) we conclude that

$$\lim_{n\to\infty} \frac{\mathsf{E}(X_{n-k:n})}{a_n} = \Gamma\left(1 - \frac{1}{\alpha}\right) - \sum_{i=1}^{k} \frac{\Gamma\left(i - \frac{1}{\alpha}\right)}{i!\,\alpha} = \frac{1}{k!}\Gamma\left(k + 1 - \frac{1}{\alpha}\right).$$

To prove the second assertion, we use again Theorem 2.4 and (2.9). ∎

3. RATES OF CONVERGENCE

In Theorem 2.2 we proved that a regular variation of $1 - F$ implies that

$$\frac{\mathsf{E}(M_n)}{a_n} \to \Gamma\left(1 - \frac{1}{\alpha}\right).$$

In this section, we consider the rate of convergence of

$$\frac{\mathsf{E}(M_n)}{a_n} \quad \text{to} \quad \Gamma\left(1 - \frac{1}{\alpha}\right).$$

We shall start with the following lemma.

Lemma 3.1. *Let $\alpha > 1$ and let $\eta \in \mathrm{RV}_s$, where $s \geqq \alpha$. If X_1 has a distribution function F such that $\mathsf{P}(X_1 \geqq 0) = 1$ and*

(3.1) $$\varrho_{\eta,F,\alpha} = \sup_{x\geqq 0} \eta(x)\,|F(x) - e^{-x^{-\alpha}}| < \infty$$

then

(3.2) $$\limsup_{n\to\infty} \frac{\eta(n^{1/\alpha})}{n}\left|\mathsf{E}\left(\frac{M_n}{n^{1/\alpha}}\right) - \Gamma\left(1 - \frac{1}{\alpha}\right)\right| < \infty .$$

Proof. Introduce a function $\psi: R_+ \to R$ determined by

$$\psi(x) = \eta(x^{1/\alpha}) \quad \text{for all} \quad x \geqq 0 ,$$

and random variables $Y_i = X_i^\alpha$ for $i \in N$. We find that

$$\psi \in \mathrm{RV}_u , \quad \text{where} \quad u = \frac{s}{\alpha} \geqq 1$$

and

$$N_n = \max\{Y_1; \ldots; Y_n\} = M_n^\alpha .$$

Further, since $X_1, X_2, \ldots$ are i.i.d. with a common distribution function F, the random variables $Y_1, Y_2, \ldots$ are i.i.d. as well and their common distribution function denoted by G has the form

$$G(x) = \begin{cases} F(x^{1/\alpha}) & \text{for} \quad x \geqq 0\,, \\ 0 & \text{for} \quad x < 0\,. \end{cases}$$

It is useful to re-formulate the present theorem in terms of ψ, u, Y_i, G and N_n instead of η, s, X_i, F and M_n:

Let $\alpha > 1$ and let $\psi \in \mathrm{RV}_u$ where $u \geqq 1$. If Y_1 has a distribution function G such that $\mathsf{P}(Y_1 \geqq 0) = 1$ and

$$(3.1') \qquad \varrho_{\eta,F,\alpha} = \sup_{x \geqq 0} \psi(x^\alpha)\, |G(x^\alpha) - e^{-x^{-\alpha}}| = \\ = \sup_{x \geqq 0} \psi(x)\, |G(x) - e^{-1/x}| < \infty$$

then

$$(3.2') \qquad \limsup_{n \to \infty} \frac{\psi(n)}{n} \left| \mathsf{E}\left(\frac{N_n^{1/\alpha}}{n^{1/\alpha}}\right) - \Gamma\left(1 - \frac{1}{\alpha}\right) \right| < \infty\,.$$

Notice that $\Gamma(1 - 1/\alpha)$ is equal to the mean value of a random variable having the distribution function

$$J_\alpha(x) = \begin{cases} e^{-x^{-\alpha}} & \text{for} \quad x \geqq 0\,, \\ 0 & \text{for} \quad x < 0\,. \end{cases}$$

We have

$$\left| \mathsf{E}\left(\frac{N_n^{1/\alpha}}{n^{1/\alpha}}\right) - \Gamma\left(1 - \frac{1}{\alpha}\right) \right| \leqq \int_0^\infty \left| \mathsf{P}\left(\frac{N_n}{n} \leqq y^\alpha\right) - e^{-y^{-\alpha}} \right| \mathrm{d}y \leqq$$

$$\leqq \sup_{y \geqq 0} \left| \mathsf{P}\left(\frac{N_n}{n} \leqq y^\alpha\right) - e^{-y^{-\alpha}} \right| + \int_1^\infty \left| G^n(ny^\alpha) - [e^{-1/ny^\alpha}]^n \right| \mathrm{d}y \leqq$$

$$\leqq \sup_{x \geqq 0} \left| \mathsf{P}\left(\frac{N_n}{n} \leqq x\right) - e^{-1/x} \right| +$$

$$+ \int_1^\infty \frac{n}{\psi(ny^\alpha)} \left[\sup_{z \geqq 0} \psi(nz^\alpha)\, |G(nz^\alpha) - e^{-1/nz^\alpha}| \right] \mathrm{d}y\,.$$

Further, Rachev and Omey proved in [6] (Corollary 2.2) that

$$(3.3) \qquad \limsup_{n \to \infty} \frac{\psi(n)}{n} \sup_{x \geqq 0} \left| \mathsf{P}\left(\frac{N_n}{n} \leqq x\right) - e^{-1/x} \right| < \infty\,.$$

Since $\psi \in \mathrm{RV}_u$ with $u \geqq 1$ there exist positive constants n_0 and b such that

$$\frac{\psi(nx)}{\psi(n)} \geqq bx^{1-(\alpha-1)/2\alpha} = bx^{(\alpha+1)/2\alpha} \quad \text{for all} \quad x \geqq 1 \quad \text{and} \quad n \geqq n_0$$

(see [4]). Finally, we obtain

$$\limsup_{n\to\infty} \frac{\psi(n)}{n} \left| \mathsf{E}\left(\frac{N_n^{1/\alpha}}{n^{1/\alpha}}\right) - \Gamma\left(1 - \frac{1}{\alpha}\right) \right| \leqq$$

$$\leqq \limsup_{n\to\infty} \frac{\psi(n)}{n} \sup_{x\geqq 0} \left| \mathsf{P}\left(\frac{N_n}{n} \leqq x\right) - e^{-1/x} \right| +$$

$$+ \sup_{x\geqq 0} \psi(x) \left|G(x) - e^{-1/x}\right| \limsup_{n\to\infty} \int_1^\infty \frac{\psi(n)}{\psi(ny^\alpha)} \, \mathrm{d}y \leqq$$

$$\leqq \limsup_{n\to\infty} \frac{\psi(n)}{n} \sup_{x\geqq 0} \left| \mathsf{P}\left(\frac{N_n}{n} \leqq x\right) - e^{-1/x} \right| +$$

$$+ \varrho_{\eta,F,\alpha} \frac{1}{b} \int_1^\infty y^{-(\alpha+1)/2} \, \mathrm{d}y < \infty .$$

Thus, the proof of (3.2′) which is equivalent to (3.2) is completed.

We are now able to give the desired result concerning the rate of convergence of $\mathsf{E}(M_n)/a_n$ to $\Gamma(1 - 1/\alpha)$.

Theorem 3.2. *Let X_1 have a distribution function F such that $\mathsf{P}(X_1 \geqq 0) = 1$ and $1 - F \in \mathrm{RV}_{-\alpha}$ where $\alpha > 1$. Let $c > 0$ and let $\eta \in \mathrm{RV}_s$ where $s \geqq \alpha$. If*

$$\sup_{x\geqq 0} \eta(x) \left|F(cx) - e^{-x^{-\alpha}}\right| < \infty \tag{3.4}$$

and

$$\limsup_{n\to\infty} \frac{\eta(n^{1/\alpha})}{n} \left| \frac{a_n}{cn^{1/\alpha}} - 1 \right| < \infty \tag{3.5}$$

then

$$\limsup_{n\to\infty} \frac{\eta(n^{1/\alpha})}{n} \left| \mathsf{E}\left(\frac{M_n}{a_n}\right) - \Gamma\left(1 - \frac{1}{\alpha}\right) \right| < \infty . \tag{3.6}$$

Proof. Introduce auxiliary random variables $Z_i = (1/c) X_i$ for $i \in N$ and $K_n = \max\{Z_1; \ldots; Z_n\} = M_n/c$. We find that $Z_1, Z_2, \ldots$ are i.i.d. random variables with a common distribution function $H(x) = F(cx)$. Our aim is to apply Lemma 3.1 with X_i, F and M_n substituted by Z_i, H and K_n. To this end we need to verify the validity of (3.1). We have

$$\varrho_{\eta,H,\alpha} = \sup_{x\geqq 0} \eta(x) \left|H(x) - e^{-x^{-\alpha}}\right| =$$

$$= \sup_{x\geqq 0} \eta(x) \left|F(cx) - e^{-x^{-\alpha}}\right| < \infty$$

by (3.4). Thus, we know from Lemma 3.1 that

(3.7) $$\limsup_{n\to\infty} \frac{\eta(n^{1/\alpha})}{n} \left|\mathsf{E}\left(\frac{K_n}{n^{1/\alpha}}\right) - \Gamma\left(1 - \frac{1}{\alpha}\right)\right| =$$
$$= \limsup_{n\to\infty} \frac{\eta(n^{1/\alpha})}{n} \left|\mathsf{E}\left(\frac{M_n}{c\,.\,n^{1/\alpha}}\right) - \Gamma\left(1 - \frac{1}{\alpha}\right)\right| < \infty\,.$$

Finally, we obtain

$$\left|\frac{\mathsf{E}(M_n)}{a_n} - \Gamma\left(1 - \frac{1}{\alpha}\right)\right| \leqq$$
$$\leqq \left|\frac{\mathsf{E}(M_n)}{a_n} - \frac{\mathsf{E}(M_n)}{c\,.\,n^{1/\alpha}}\right| + \left|\mathsf{E}\left(\frac{M_n}{c\,.\,n^{1/\alpha}}\right) - \Gamma\left(1 - \frac{1}{\alpha}\right)\right|$$

so that

$$\limsup_{n\to\infty} \frac{\eta(n^{1/\alpha})}{n} \left|\frac{\mathsf{E}(M_n)}{a_n} - \Gamma\left(1 - \frac{1}{\alpha}\right)\right| \leqq$$
$$\leqq \limsup_{n\to\infty} \frac{\eta(n^{1/\alpha})}{n} \frac{\mathsf{E}(M_n)}{a_n} \left|1 - \frac{a_n}{c\,.\,n^{1/\alpha}}\right| +$$
$$+ \limsup_{n\to\infty} \frac{\eta(n^{1/\alpha})}{n} \left|\mathsf{E}\left(\frac{M_n}{c\,.\,n^{1/\alpha}}\right) - \Gamma\left(1 - \frac{1}{\alpha}\right)\right| < \infty$$

by (2.4), (3.5) and (3.7).

We conclude the present section by demonstrating the contribution of Theorem 3.2 by the following example.

Example. Let

$$F(x) = \begin{cases} 0 & \text{for } x < 1\,, \\ 1 - \dfrac{x^2 + 1}{2x^4} & \text{for } x \geqq 1\,. \end{cases}$$

We find that $1 - F \in \mathrm{RV}_{-2}$, i.e. $\alpha = 2$ in this case, and by Theorem 2.2 (ii)

$$\lim_{n\to\infty} \frac{\mathsf{E}(M_n)}{a_n} = \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\,.$$

The distribution function F is continuous on $\boldsymbol{R}$ and the solution of the equation

$$1 - F(x) = \frac{1}{n}$$

results in

$$a_n = \tfrac{1}{2}\sqrt{(n + \sqrt{(n^2 + 8n)})}\,.$$

Thus,

$$\lim_{n\to\infty} \frac{a_n}{n^{1/2}} = \lim_{n\to\infty} \frac{1}{2}\sqrt{\left[1 + \sqrt{\left(1 + \frac{8}{n}\right)}\right]} = \frac{\sqrt{2}}{2}\,,$$

i.e. we put $c = \frac{1}{2}\sqrt{2}$ in Theorem 3.2 – cf. (3.5). Further, $cx \geqq 1$ if and only if $x \geqq \sqrt{2}$ and

$$|F(cx) - e^{-x^{-\alpha}}| = \left|1 - x^{-2} - 2x^{-4} - \sum_{k=0}^{\infty} \frac{(-1)^k x^{-2k}}{k!}\right| =$$

$$= \tfrac{5}{2}x^{-4} + x^{-4} \sum_{k=3}^{\infty} x^{-2k+4} \leqq \tfrac{7}{2}x^{-4} \quad \text{for} \quad x \geqq \sqrt{2}$$

and

$$|F(cx) - e^{-x^{-\alpha}}| = e^{-x^{-2}} \leqq e^{-1/2} \quad \text{for} \quad x \in [0; \sqrt{2}] .$$

With respect to (3.4) and to the fact that $\alpha = 2$, we can take e.g. $\eta(x) = x^s$ where $s \in [2; 4]$. However, the greater is the exponent s the stronger will be the achieved result so that we choose $\eta(x) = x^4$. We find that

$$\sup_{x \geqq 0} \eta(x) |F(cx) - e^{-x^{-\alpha}}| \leqq \max \{\tfrac{7}{2}; 4e^{-1/2}\} < \infty ,$$

i.e. the assumption (3.4) is fulfilled. Finally, easy calculation yields

$$\limsup_{n \to \infty} \frac{\eta(n^{1/\alpha})}{n} \left|\frac{a_n}{c \, . \, n^{1/\alpha}} - 1\right| =$$

$$= \lim_{n \to \infty} n \left| - 1 + \frac{1}{\sqrt{2}} \sqrt{\left[1 + \sqrt{\left(1 + \frac{8}{n}\right)}\right]}\right| = 1 < \infty ,$$

i.e. the assumption (3.5) of Theorem 3.2 is fulfilled as well. Thus, its assertion reads

$$\limsup_{n \to \infty} n \left|\frac{\mathsf{E}(M_n)}{a_r} - \sqrt{\pi}\right| < \infty ,$$

i.e. the rate of convergence of $\mathsf{E}(M_n)/a_n$ to its limit $\sqrt{\pi}$ is at least the same as the rate of convergence of $1/n$ to 0.

Remark. From Theorem 3.2 we find that the choice of the normalizing sequence $\{a_n\}_N$ is very important. We made the choice $a_n = \inf \{x; 1 - F(x) \leqq 1/n\}$. On the other hand, if we make the choice $b_n = \inf \{x; -\log F(x) \leqq 1/n\}$ we obviously obtain all results of Section 2 with a_n replaced by b_n. This follows from the obvious asymptotic equality $-\log F(x) \sim 1 - F(x)$ for $x \to \infty$ and from the assumption that $1 - F \in \mathrm{RV}_{-\alpha}$ with the index $\alpha > 0$. Moreover, with the latter choice condition (3.4) alone implies

$$\limsup_{n \to \infty} \frac{\eta(n^{1/\alpha})}{n} \left|\mathsf{E}\left(\frac{M_n}{b_n}\right) - \Gamma\left(1 - \frac{1}{\alpha}\right)\right| < \infty .$$

To see this, we show that (3.5) holds for the sequence $\{b_n\}_N$. From (3.4) we obtain

$$\limsup_{x \to \infty} \eta(x) |\log F(cx) + x^{-\alpha}| < \infty .$$

With our choice of b_n we have

$$\limsup_{n\to\infty} \eta\left(\frac{b_n}{c}\right)\left|\frac{1}{n} - \left(\frac{b_n}{c}\right)^{-\alpha}\right| < \infty .$$

Since $b_n \sim c \,.\, n^{1/\alpha}$ for $n \to \infty$ and since $\eta(x)$ is regularly varying we find that

$$\limsup_{n\to\infty} \frac{\eta(n^{1/\alpha})}{n}\left|\frac{b_n^\alpha}{c^\alpha \,.\, n} - 1\right| < \infty$$

and hence (3.5) with a_n replaced by b_n follows.

Acknowledgement. The second author would like to thank the Mathematical Institute of the Czechoslovak Academy of Sciences for the hospitality he obtained in the period during which this paper was written.

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Souhrn

MOMENTY POŘÁDKOVÝCH STATISTIK

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Označme $X_{1:n} \leqq \ldots \leqq X_{n:n} = M_n$ pořádkové statistiky z náhodného výběru $X_1, \ldots, X_n$ o rozsahu n a nechť rozložení náhodné veličiny X_1 s distribuční funkcí F je soustředěno na $\boldsymbol{R}_+$. Článek pojednává o asymptotickém chování hodnot $\mathsf{E}(M_n)$ a $\mathsf{E}(X_{n-k:n})$ pro pevné k a $n \to \infty$. Speciální pozornost je věnována případu, že $1 - F$ je regulárně se měnící funkce.

Резюме

МОМЕНТЫ ПОРЯДКОВЫХ СТАТИСТИК

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Пусть $X_{1:n} \leqq \ldots \leqq X_{n:n} = M_n$ — порядковые статистики из случайной выборки $X_1, \ldots, X_n$ размера n и пусть распределение случайной величины X_1 с функцией распределения F сосредоточено на $\boldsymbol{R}_+$. Статья трактует ассимптотическое поведение математических ожиданий $\mathsf{E}(M_n)$ и $\mathsf{E}(X_{n-k:n})$ для постоянного k и $n \to \infty$. Специальное внимание уделяется случаю, что $(1 - F)$ — регулярно меняющаяся функция.

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