Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 114 (1989), No. 3, 225--229

Persistent URL: `http://dml.cz/dmlcz/118370`

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114 (1989) ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY No. 3, 225—229

CENTERS OF DIRECTED CACTI

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(Received February 1, 1985)

Summary. A cactus is a graph in which each edge is contained in at most one circuit. The paper studies outcenters, incenters, outradii and inradii of strongly connected digraphs obtained from cacti by orienting their edges.

Keywords: directed cactus, distance, outcenter, incenter, outradius, inradius.

AMS classification: 05C20, 05C32.

A cactus was defined as a special type of an undirected graph. Here we study graphs obtained from cacti by introducing an orientation. All considered graphs are finite.

A cactus is an undirected graph G with the property that each edge of G is contained in at most one circuit of G.

Each block of a cactus either is a circuit, or consists of one edge with its end vertices. We admit also circuits of the length two, i.e. double edges. A directed cactus is, in general, any directed graph which is obtained from a cactus by orienting its edges. However, we will be interested only in such directed cacti which are strongly connected. It is easy to prove the following proposition.

Proposition 1. *A directed cactus is strongly connected if and only if each of its blocks is a cycle.*

By a cycle we mean a directed circuit. We again admit also cycles of the length two, i.e. pairs of oppositely directed edges joining the same pair of vertices.

Also the following proposition is easy to prove.

Proposition 2. *A directed graph G has the property that for any two distinct vertices u, v of G there exists a unique directed path from u to v in G if and only if G is a strongly connected directed cactus.*

Thus we see that in a certain sense strongly connected directed cacti are analogues of trees in the case of directed graphs. They form a particular case of the concept of unipathic digraph, as it was defined in [1]. A unipathic digraph is a digraph in which for any two vertices u, v there exists at most one directed path from u to v.

In [1] also the concepts of outradius, inradius, outcenter and incenter were defined.

We will study these concepts in strongly connected directed cacti, analogously as it was done in [2] fot radii and centers in (undirected) trees.

Let u, v be two vertices of a directed graph G; we define the distance $d(u, v)$ from u to v. If $u = v$, then $d(u, v) = 0$. If $u \neq v$ and there exists a directed path from u to v in G, then $d(u, v)$ is the minimum length of such a path; if there exists no directed path from u to v in G, then $d(u, v) = \infty$.

Note that in general $d(u, v) \neq d(v, u)$.

The vertex set of G will be denoted by $V(G)$. For each vertex $u \in V(G)$ put

$$e^+(u) = \max \{d(u, x) \mid x \in V(G)\},$$
$$e^-(u) = \max \{d(x, u) \mid x \in V(G)\}.$$

Further put

$$\varrho^+(G) = \min \{e^+(u) \mid u \in V(G)\},$$
$$\varrho^-(G) = \min \{e^-(u) \mid u \in V(G)\}.$$

The numbers $\varrho^+(G)$, $\varrho^-(G)$ will be called the outradius of G and the inradius of G, respectively. The vertex c of G with the property that $e^+(c) = \varrho^+(G)$ (or $e^-(c) = \varrho^-(G)$) will be called an outcentral (or incentral, respectively) vertex of G. The set of all outcentral (or incentral) vertices of G is called the outcenter (or the incenter, respectively) of G.

In [1] it is proved that all outcentral vertices of a strongly connected digraph lie in one block, and so do all incentral vertices. Also it is shown by an example that these blocks need not coincide. But this example is not a cactus; we will show an example of a cactus with this property.

Example. The block of a strongly connected directed cactus G containing all outcentral vertices need not coincide with the block of G containing all incentral vertices.

This is shown in Fig. 1. The cactus in this figure is strongly connected, has one outcentral vertex c^+ and one incentral vertex c^-; these two vertices lie in different blocks.

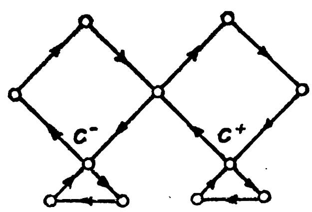

Fig. 1.

The following theorem shows that the number of outcentral vertices in strongly connected directed cacti has no upper bound.

Theorem 1. *Let C be a cycle of length n, where n is an arbitrary integer, $n \geqq 2$. Let A, B be two non-empty subsets of $V(C)$. Then there exists a strongly connected directed cactus G with the property that C is a block of G, the set of all outcentral vertices of G is A and the set of all incentral vertices of G is B.*

Proof. Denote $\bar{A} = V(C) - A$, $\bar{B} = V(C) - B$. Further, let A' be the set of all initial vertices of edges incoming into vertices of $\bar{A}$ and let B' be the set of all terminal vertices of edges outgoing from vertices of $\bar{B}$. In Figs. 2, 3, 4, 5 we see four graphs; in each of them there is one special vertex denoted by a black circle. To each vertex u of C we assign a graph $H(u)$ isomorphic to one of these graphs and identify its special vertex with the vertex u. If $u \in A' \cap B'$, then $H(u)$ is isomorphic to the graph in Fig. 2. If $u \in A' - B'$, then $H(u)$ is isomorphic to the graph in Fig. 3. If $u \in B' - A'$, then

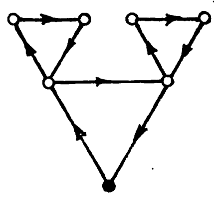

Fig. 2.

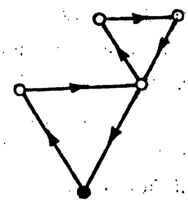

Fig. 3.

$H(u)$ is isomorphic to the graph in Fig. 4. If $u \in V(C) - (A' \cup B')$, then $H(u)$ is isomorphic to the graph in Fig. 5. In the graph G thus obtained we have $\varrho^+(G) =$

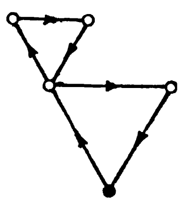

Fig. 4.

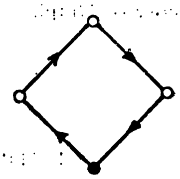

Fig. 5.

$= \varrho^-(G) = n - 2$. The set of all outcenters of G is A, the set of all its incenters is B. The graph G is a strongly connected directed cactus. □

The following two theorems concern radii of cacti.

Theorem 2. *Let k, n be integers, $1 \leqq k \leqq n - 1$. Then there exists a strongly connected directed cactus G with n vertices such that $\varrho^+(G) = \varrho^-(G) = k$.*

Proof. If $k = n - 1$, then the required cactus is a cycle of length n. If $k \leqq n - 2$, we take a cycle C_0 of length $k + 1$ and $n - k - 1$ cycles $C_1, \ldots, C_{n-k-1}$ of length 2. In each of the cycles $C_0, C_1, \ldots, C_{n-k-1}$ we choose one vertex and identify all the chosen vertices. The graph G thus obtained is a strongly connected directed cactus with n vertices and $\varrho^+(G) = \varrho^-(G) = k$. $\square$

Theorem 3. *Let q be an integer. Then there exists a strongly connected directed cactus G with the property that $\varrho^+(G) - \varrho^-(G) = q$.*

Proof. First let $q \geqq 0$. Put $k = 2q + 2$. Take a cycle C of length k. Let u, v be two vertices of C such that there exists an edge of C from u to v. Let H_1 (or H_2) be a digraph obtained from an undirected path of length $3k$ (or $k - 1$, respectively) by replacing each edge by a pair of oppositely directed edges joining the same pair of vertices. Identify u (or v) with the vertex of H_1 (or H_2, respectively) which was a terminal vertex of the above mentioned path. Let G be the directed cactus thus obtained; it is evidently strongly connected. Let c^+ (or c^-) be the vertex of H_1 whose distance from u is $\frac{1}{2}k + 1 = q + 2$ (or $k = 2q + 2$, respectively). The vertex c^+ is the unique outcenter of G and $\varrho^+(G) = \frac{5}{2}k - 1 = 5q + 4$. The vertex c^- is the unique incenter of G and $\varrho^-(G) = 2k = 4q + 4$. We have $\varrho^+(G) - \varrho^-(G) = q$. If $q < 0$, then we construct a cactus G' for which $\varrho^+(G') - \varrho^-(G') = -q$ holds and invert the orientation of all its edges. The graph G thus obtained has $\varrho^+(G) = \varrho^-(G')$, $\varrho^-(G) = \varrho^+(G')$ and the required equality holds.

References

[1] *F. Harary, R. Z. Norman, D. Cartwright:* Structural Models. John Wiley & Sons, Inc. New York—London—Sydney 1965.

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Souhrn

CENTRA ORIENTOVANÝCH KAKTUSŮ

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Silně souvislý orientovaný kaktus je orientovaný graf, jehož bloky jsou cykly. Zkoumají se vlastnosti center a poloměrů takovýchto grafů.

Резюме

ЦЕНТРЫ ОРИЕНТИРОВАННЫХ КАКТУСОВ

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Сильно связный ориентированный кактус есть ориентированный граф, блоки которого являются циклами. Исследуются свойства центров и радиусов таких графов.

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