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LINEAR DIFFERENTIAL EQUATIONS WITH MEASURES AS COEFFICIENTS AND THE CONTROL THEORY

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Summary. Systems of linear differential equations with measures as coefficients are studied, together with their applications in control theory. A complete classification of the attainable sets for one-dimensional control systems is given.

Keywords: linear differential equation with measures as coefficients, one-dimensional control system, attainable set.

AMS Classification: 34A10, 34A30, 49E15.

1. INTRODUCTION

In this paper we study the system of linear differential equations

(1) $$\dot{x} = A(t)\,x + f(t)\,, \quad x \in \mathbb{R}^n\,, \quad t \in (a, b)\,, \quad -\infty \leqq a < b \leqq \infty$$

where the coefficients of the matrix $A(t)$ are measures and the term $f(t)$ may be a locally integrable function or a measure. The solution of the equation (1) will be a function of locally bounded variation in (a, b).

The study of such equations was initiated by J. Kurzweil who proved in 1958 the existence and unicity of solutions of a linear integral equation with Perron-Stieltjes integral ([4]), and next in 1959 the existence and unicity of solutions of the homogeneous system

(2) $$\dot{x} = A(t)\,x$$

where $A(t)$ is a measure ([5]). Next, in H. Hildebrand's [3] and S. Stallard's [10] papers the equation (2) was studied in the class of functions of locally bounded variation as solutions. Recently, such a generalization of solutions of the system (1) may be found in the papers and books of Š. Schwabik, M. Tvrdý, O. Vejvoda [9], Š. Schwabik [8], S. G. Pandit, S. G. Deo [7], in the papers of J. Ligęza who has used the sequential theory of distributions, and in the papers by U. Sztaba and the author who apply the Lebesgue-Stieltjes integral approach. The case of $A(t)$ being an integrable function and $f(t)$ a measure was studied by A. Halanay and D. Wexler [2].

2. BASIC DEFINITIONS AND NOTATION

Let (a, b), $-\infty \leqq a < b \leqq \infty$, be an open interval. Denote by

$BV_{loc}(a, b)$ – the space of all right-continuous functions of locally bounded variation in (a, b),

$L^p_{loc}(a, b)$ – the space of all locally integrable with the p-th power functions in (a, b),

$C^0(a, b)$ – the space of all continuous functions in (a, b).

Let $\mathfrak{M}$ be the σ-field of subsets $C \subset (a, b)$ of the form

$$C = \bigcup_{i=1}^{N} (c_i, d_i] , \quad N \leqq \infty ,$$

and $\mathscr{B}$ the σ-field of Borel subsets of (a, b).

Every function $g(\cdot) \in BV_{loc}(a, b)$ determines a measure μ_g on $\mathfrak{M}$:

$$\mu_g((c, d]) := g(d) - g(c)$$

and if $(c_i, d_i] \cap (c_j, d_j] = \emptyset$ for $i \neq j$, $i, j \in \mathbb{N}$, then

$$\mu_g(\bigcup_i (c_i, d_i]) = \sum_i [g(d_i) - g(c_i)] .$$

In particular, $\mu_g(\{e\}) := g(e) - g(e-)$.

Definition 1. Lebesgue's extension of the measure μ_g to a σ-field $\mathscr{B}^*$ which contains $\mathscr{B}$ will be called the *Lebesgue-Stieltjes measure* ($L - S$ measure) generated by the function g. It will be denoted by g' or by $\mathrm{d}g$ and will be called the *derivative* of the function g. Conversely, the function $g(\cdot)$ will be called the *primitive* function for the measure $\mathrm{d}g$.

Examples. 1. If $H(t) = \begin{cases} 0 \text{ for } t < 0 \\ 1 \text{ for } t \geqq 0 \end{cases}$ is the Heaviside function then the corresponding measure $\mathrm{d}H(t) = \delta(t)$ is the Dirac measure:

$$\delta(t)(B) = 1 \quad \text{iff} \quad 0 \in B \quad \text{and} \quad \delta(t)(B) = 0 \quad \text{otherwise} .$$

2. If $g(t)$ is absolutely continuous with respect to the Lebesgue measure then $g'(t)$ is the measure which coincides with the usual derivative of $g(t)$ (for a.e. $t \in (a, b)$):

$$g'(B) = \int_B g'(t) \,\mathrm{d}t = \int_B \mathrm{d}g(t) .$$

3. If $g(x) = x$ then $g'(x)$ is the usual Lebesgue measure $\mathrm{d}x$. ∎

Definition 1 implies that this differentiation is a linear operation, while the right-continuity yields that the difference between any two primitives for the same measure $\mathrm{d}g$ is a constant function.

If we have a measure μ defined on $\mathscr{B}$ then the function

$$g(x) := \mu((a, x])$$

is a primitive for the measure μ.

If a function $f(t)$ is $g'(\cdot)$-measurable (i.e., f is measurable with respect to the σ-field $\mathscr{B}^*$ such that the triple $((a, b), \mathscr{B}^*, g')$ is a measure space) then we can define the $L-S$-integral of $f(\cdot)$ with respect to the measure $\mathrm{d}g$:

(3) $$\int_c^d f(t)\,\mathrm{d}g(t) := \int_{(c,d]} f(t)\,\mathrm{d}g(t)\,, \quad a < c < d < b\,.$$

(For $f(t) \equiv 1$ we have $\int_{(c,d]} 1\,\mathrm{d}g(t) = g'((c, d])$.)

In particular

(3′) $$\int_{\{e\}} f(t)\,\mathrm{d}g(t) := f(e)\,g'(\{e\})\,.$$

For $c \in (a, b)$ the function

$$k(x) := \int_c^x f(t)\,\mathrm{d}g(t)$$

is of locally bounded variation. More precisely, $k(\cdot)$ is continuous at all points of continuity of $g(\cdot)$ and right-continuous at the remaining points. Taking $f(t) \equiv 1$ we obtain one of the primitives for the measure $\mathrm{d}g(\cdot)$.

If one of the functions f, g is continuous then the integral (3) can be understood in the Riemann-Stieltjes sense.

If $f, g \in BV_{\mathrm{loc}}(a, b)$ and $(c, d] \subset (a, b)$ then – as follows from the Lebesgue partition of f – $f(\cdot)$ is $L-S$-integrable with respect to the measure $g'(\cdot)$ in $(c, d]$.

Now we give the definition of the product of a function $f(\cdot) \in BV_{\mathrm{loc}}(a, b)$ and a measure $g'(\cdot)$.

Definition 2. Let $f, g \in BV_{\mathrm{loc}}(a, b)$. The *product* fg' is a measure q' such that

$$q'(B) = \int_B f(s)\,\mathrm{d}g(s) \quad \text{for all} \quad B \in \mathscr{B}\,.$$

Examples. 1. $H(t - r)\,\delta(t - s) = \begin{cases} \delta(t - s) & \text{if } r \leqq s \\ 0 & \text{if } r > s \end{cases}.$

2. If $f(\cdot) \in BV_{\mathrm{loc}}(a, b)$ then $f(t)\,\delta(t - s) = f(s)\,\delta(t - s)$. The same is true for $f(\cdot) \in C^0(a, b)$. ∎

This product satisfies the jointness principle in the sense that the following generalization of the Radon-Nikodym theorem holds: If $f, g, h \in BV_{\mathrm{loc}}(a, b)$ and $q' := fg'$ then

$$\int_B h(s) f(s)\,\mathrm{d}g(s) = \int_B h(s)\,\mathrm{d}q(s) \quad \text{for every} \quad B \in \mathscr{B} \quad (\text{see [11]})\,.$$

If $f, g \in BV_{\mathrm{loc}}(a, b)$ then the measures g' and fg' have the same atomic points.

Now we define when two measures are equal.

Definition 3. Let $f, g \in BV_{\mathrm{loc}}(a, b)$. Two measures are *equal* in the interval $(c, d] \subset (a, b)$ iff the difference $f' - g'$ is the zero-measure, i.e. if $(f' - g')\,.\,((\alpha, \beta]) = 0$ for any $(\alpha, \beta] \subset (c, d]$. This equality shows that the difference $f(\cdot) - g(\cdot)$ is a constant function in $(c, d]$.

In the end we define when a function $x(\cdot) \in BV_{loc}(a, b)$ may be called a solution to the equation (1) in which the derivative, product and equality are understood as in Definitions 1, 2, 3, respectively.

Definition 4. A function $x(\cdot) \in BV_{loc}(a, b)$ is a *solution* to the equation (1) iff the $L - S$-measures generated by the left- and right-hand sides of (1) coincide.

Applying Definitions 1 – 3 we obtain an equivalent definition of the solution.

Definition 4'. A function $x(\cdot) \in BV_{loc}(a, b)$ is a *solution* to the equation (1) with the initial condition

$$x(t_0) = x_0\,, \quad t_0 \in (a, b) \tag{4}$$

iff $x(\cdot)$ is a solution to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} [\mathrm{d}\mathscr{A}(s)]\, x(s) + \int_{t_0}^{t} \mathrm{d}\mathscr{F}(s)\,, \quad t \in (t_0, b) \tag{5}$$

where $\mathscr{A}' = A$ and $\mathscr{F}' = f$.

Iff the measures $A(\cdot), f(\cdot)$ are absolutely continuous with respect to the Lebesgue measure then the above definitions of solution coincide with the Carathéodory concept of solution.

3. LINEAR DIFFERENTIAL EQUATIONS WITH MEASURES AS COEFFICIENTS

Our starting point is the existence of solution of the Cauchy problem (2), (4). In the remaining part of the paper we assume that the following hypothesis H_1 is fulfilled:

$$\det (E - C_k) \neq 0 \quad \text{for} \quad k = 1, 2, \ldots . \tag{H_1}$$

Theorem 1. *If* H_1 *holds then there exists a solution* $x(\cdot) \in BV_{loc}(a, b)$ *of the Cauchy problem* (2), (4).

This theorem can be proved as in [1] by using the successive approximation method (in [1] the integral in (5) is understood in the Riemann-Stieltjes sense but defined in a special way), or as in [11] by the Euler method.

Now we will construct the fundamental matrix for the equation (2) and obtain the Cauchy formula for the problem (2), (4). To do this, let us decompose the measure $A(\cdot)$ into its continuous and atomic parts, which follows from the Lebesgue decomposition of an arbitrary primitive for this measure. So, the measure $A(t)$ may be written in the form

$$A(t) = \hat{A}(t) + \sum_{k=1}^{\infty} C_k \delta(t - t_k)\,, \quad t_k \in (a, b) \tag{6}$$

where $\hat{A}(\cdot) = \hat{\mathscr{A}}'(\cdot)$, $\hat{\mathscr{A}}(\cdot) \in BV_{loc}(a, b) \cap C^0(a, b)$, and C_k are some matrices. Assume that the following Hypothesis H_0 is fulfilled:

(H_0) The points $\{t_k\}$ are ordered: $a < t_0 \leqq t_1 \ldots < t_n < \ldots < b$, and the unique accumulation point of the sequence $\{t_k\}$ may be b

By the product $C\,\delta(s)$ we understand the matrix whose all elements are equal to $C_{ij}\,\delta(s)$, $1 \leqq i, j \leqq n$

According to (6) the equation (2) may be written as

$$\dot{x} = \hat{A}(t)\,x + \sum_{k=1}^{\infty} C_k\,x(t_k)\,\delta(t - t_k)\,. \tag{7}$$

The auxiliary equation

$$\dot{x} = \hat{A}(t)\,x \tag{8}$$

with the initial condition (4) has a unique solution $x(\cdot)$ which is a continuous function of locally bounded variation in (a, b) and may be written in the usual Cauchy form

$$x(t) = \hat{\Phi}(t)\,x_0$$

where $\hat{\Phi}(t) \in BV_{loc}(a, b) \cap C^0(a, b)$ is the fundamental matrix of (8).

Returning to the equation (2) we see that in every interval (t_k, t_{k+1}) it reduces to the equation (8). Therefore we are looking for the solution of (2) in the form

$$x(t) = \hat{\Phi}(t)\,\hat{\Phi}^{-1}(t_k)\,s_k\,, \quad t \in [t_k, t_{k+1})\,, \quad k = 0, 1, \ldots \tag{9}$$

where $s_k = x(t_k)$. Obviously, $s_0 = x_0$. The sequence $\{s_k\}$ must be constructed in such a way that the piecewise continuous function $x(t)$ is a solution to the equation (2) in (t_0, b). Differentiating (9) we obtain

$$\begin{aligned} \dot{x} &= \hat{A}(t)\,x + \sum_{k=1}^{\infty} [x(t_k+) - x(t_k-)]\,\delta(t - t_k) = \\ &= \hat{A}(t)\,x + \sum_{k=1}^{\infty} [s_k - \hat{\Phi}(t_k)\,\hat{\Phi}^{-1}(t_{k-1})\,s_{k-1}]\,\delta(t - t_k) = \\ &= \hat{A}(t)\,x + \sum_{k=1}^{\infty} \varepsilon_k \delta(t - t_k) \end{aligned} \tag{10}$$

where $\varepsilon_k := s_k - x(t_k-)$ is the jump of the solution $x(\cdot)$ at the instant t_k. Comparing (10) with the right-hand side of (7) we conclude that $x(\cdot)$ is a solution of (2) iff

$$s_k - \hat{\Phi}(t_k)\,\hat{\Phi}^{-1}(t_{k-1})\,s_{k-1} = C_k s_k\,, \quad k = 1, 2, \ldots\,. \tag{11}$$

Therefore the sequence $\{s_k\}$ satisfies the following reccurence equation of the first order:

$$(E - C_k)\,s_k = \hat{\Phi}(t_k)\,\hat{\Phi}^{-1}(t_{k-1})\,s_{k-1}\,, \quad k = 1, 2, \ldots\,, \quad s_0 = x_0 \tag{12}$$

(E denotes the unit matrix).

Consequently, knowing s_{k-1} (and thus also $x(t)$ in the interval (t_{k-1}, t_k)) we can compute s_k from (12) in a unique manner by the hypothesis H_1, then we can extend this solution to the next interval (t_k, t_{k+1}), and so on.

Thus we obtain

Theorem 2. *Under the hypothesis* H_1 *the problem* (2), (4) *has a unique solution* $x(\cdot) \in BV_{loc}(a, b)$. *This solution may be written in the form*

(13) $$x(t) = \hat{\Phi}(t)\,\hat{\Phi}^{-1}(t_0)\,x_0 + \sum_{k:t_k \leqq t} \hat{\Phi}(t)\,\hat{\Phi}^{-1}(t_k)\,\varepsilon_k H(t - t_k)\,.$$

If the hypothesis H_1 is not fulfilled then the problem (2), (4) may have more than one solution as is shown in the following

Example. The equation

$$\dot{x} = \delta(t)\,x\,, \quad x(-1) = 0\,, \quad x \in \mathbb{R}^1$$

has a continuum of solutions: every function $x(t) = H(t)\,c$, c an arbitrary number is a solution of this equation. ∎

The formula (12) under the hypothesis H_1 may be rewritten as

(14) $$s_k = (E - C_k)^{-1}\,\hat{\Phi}(t_k)\,\hat{\Phi}^{-1}(t_{k-1})\,s_{k-1} = (E - C_k)^{-1}\,x(t_k-)\,.$$

By (14) we have that if $x_0 = 0$ then $s_k = 0$ for all $k \in \mathbb{N}$, thus the problem (2), (4) with $x_0 = 0$ has only the null-solution. Hence the problem (2), (4) has a unique solution for an arbitrary x_0.

Now we deduce some interrelations between the sequences $\{s_k\}$ and $\{\varepsilon_k\}$. If we compare the last term of (10) with (7) we obtain the equality

$$\varepsilon_k = C_k\,x(t_k) = C_k s_k = C_k[\hat{\Phi}(t_k)\,\hat{\Phi}^{-1}(t_{k-1})\,s_{k-1} + \varepsilon_k]\,, \quad k = 1, 2, \ldots\,.$$

Therefore, if the hypothesis H_1 is fulfilled then

(15) $$\varepsilon_k = (E - C_k)^{-1}\,C_k\,\hat{\Phi}(t_k)\,\hat{\Phi}^{-1}(t_{k-1})\,s_{k-1} = (E - C_k)^{-1}\,C_k\,x(t_k-)\,.$$

Under an additional hypothesis

(H_2) $$\det C_k \neq 0 \quad \text{for} \quad k = 1, 2, \ldots$$

we prove that the sequence $\{\varepsilon_k\}$ satisfies the reccurence equation

(16) $$\varepsilon_{k+1} = (E - C_{k+1})^{-1}\,C_{k+1}\,\hat{\Phi}(t_{k+1})\,\hat{\Phi}^{-1}(t_k)\,C_k^{-1}\varepsilon_k\,, \quad k = 1, 2, \ldots$$

where $\varepsilon_1 = (E - C_1)^{-1}\,C_1\,\hat{\Phi}(t_1)\,\hat{\Phi}^{-1}(t_0)\,x_0$ is calculated from (15). Indeed, from (15) we have

(17) $$x(t_k-) = C_k^{-1}(E - C_k)\,\varepsilon_k\,,$$

and (14) implies the equality

(18) $$\begin{aligned}\varepsilon_{k+1} &= s_{k+1} - (E - C_{k+1})\,s_{k+1} = \\ &= C_{k+1}\{\varepsilon_{k+1} + \hat{\Phi}(t_{k+1})\,\hat{\Phi}^{-1}(t_k)\,[x(t_k-) + \varepsilon_k]\}\,.\end{aligned}$$

Multiplying (18) by C_{k+1}^{-1} we obtain

$$(C_{k+1}^{-1} - E)\,\varepsilon_{k+1} = \hat{\Phi}(t_{k+1})\,\hat{\Phi}^{-1}(t_k)\,[x(t_k-) + \varepsilon_k]\,,$$

so

$$\varepsilon_{k+1} = (C_{k+1}^{-1} - E)^{-1} \hat{\Phi}(t_{k+1}) \hat{\Phi}^{-1}(t_k) [x(t_k-) + \varepsilon_k] .$$

If we substitute (17) into the last equality then after simple calculations we obtain (16). ∎

From (14) and (16) we deduce by induction that all elements of the sequences $\{s_k\}$ and $\{\varepsilon_k\}$ depend only on x_0, and these dependences are linear and continuous:

$$s_k = T_k x_0 , \quad \varepsilon_k = Q_k x_0$$

where T_k, Q_k are some matrices. Therefore, substituting these relations into (13) and eliminating x_0 outside the brackets we obtain the usual Cauchy form of the solution to the equation (2):

$$x(t) = \Phi(t)\, x_0 , \quad t \in (t_0, b)$$

where the matrix $\Phi(t)$, normed at t_0, has all elements belonging to $BV_{\mathrm{loc}}(a, b)$. This matrix has the following properties (analogous to those occurring in the classical case; the proofs are identical and follow from the construction):

P. 1 $\Phi(t_0) = E$;

P. 2 $\dfrac{\mathrm{d}}{\mathrm{d}t} \Phi(t) = A(t)\, \Phi(t)$;

P. 3 $\Phi(t_2)\, \Phi(t_1) = \Phi(t_1 + t_2)$;

P. 4 $\Phi(t)$ is non-singular for all t and $\Phi^{-1}(\cdot) \in BV_{\mathrm{loc}}(a, b)$.

The last property is not quite obvious, but can be proved by using (14) and the hypothesis H_1. We have

$$s_k = x(t_k-) + \varepsilon_k = (E - C_k)^{-1}\, x(t_k-) \quad \text{for} \quad k = 1, 2, \dots ,$$

and consequently, H_1 implies

$$s_k \neq 0 \quad \text{iff} \quad x(t_k-) \neq 0$$

which follows by induction from P. 1. Thus $\Phi(t) \neq 0$ on every interval $t \in [t_k, t_{k+1})$, and so on. Therefore the inverse matrix $\Phi^{-1}(t)$ exists for every $t \in (a, b)$. The second part of P. 4 follows from the construction of the inverse matrix and from the properties of functions of bounded variation. ∎

The above properties enable us to solve the equation (1) by applying the variation-of-constants method. We are looking for the solution to (1) in the form

$$x(t) = \Phi(t)\, z(t) \tag{19}$$

where $z(\cdot) \in BV_{\mathrm{loc}}(a, b)$, so $x(\cdot) \in BV_{\mathrm{loc}}(a, b)$ as well. Substituting (19) into (1) we obtain

$$\dot{x} = \dot{\Phi}(t)\, z(t) + \Phi(t)\, \dot{z}(t) = A(t)\, \Phi(t)\, z(t) + \Phi(t)\, \dot{z}(t) =$$
$$= A(t)\, x(t) + \Phi(t)\, \dot{z}(t) = A(t)\, x(t) + f(t) .$$

Then (19) is a solution to (1) iff

$$\dot{z}(t) = \Phi^{-1}(t) f(t) .$$

Therefore $z(\cdot)$ must be a primitive function for the measure $\Phi^{-1}(t) f(t)$:

$$z(t) = c + \int_{t_0}^{t} \Phi^{-1}(s) f(s) \, ds \quad \text{if} \quad f \in L^1_{\text{loc}}(a, b)$$

or

$$z(t) = c + \int_{t_0}^{t} \Phi^{-1}(s) \, d\mathcal{F}(s) \quad \text{if} \quad f = \mathcal{F}' \quad \text{is a measure} .$$

We conclude that the solution to the problem (1), (4) may be written by the following Cauchy formula:

$$x(t) = \Phi(t) x_0 + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) f(s) \, ds , \quad f \in L^1_{\text{loc}}(a, b)$$

or

$$x(t) = \Phi(t) x_0 + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) \, d\mathcal{F}(s) , \quad f = \mathcal{F}' \quad \text{is a measure} .$$

4. ATTAINABLE SETS AND THE TIME-OPTIMAL CONTROL PROBLEM

In this part we introduce the concept of the attainable set for the control system

(20) $\dot{x} = A(t) x + f(t, u) , \quad x(t_0) = x_0 , \quad x \in \mathbb{R}^n , \quad u \in \mathbb{R}^m , \quad t_0 \in (a, b)$

(u is called the control). The time-optimal control problem will be also studied. The following assumptions will be made:

Z. 1 All elements of the matrix $A(\cdot)$ are measures fulfilling the hypotheses H_0, H_1.

Z. 2 $f(t, \cdot)$ is continuous and $f(\cdot, x)$ is a measurable function.

Z. 3 The set of admissible controls is

$$\mathcal{U} = \{u(\cdot) : (a, b) \ni t \to U(t); \; u(\cdot) \text{ is measurable}\}$$

where for every $t \in (a, b)$, $U(t)$ is a non-empty, compact set and the multifunction $t \to U(t)$ is measurable in the sense that for every closed $D \subset \mathbb{R}^m$ the set $\{t \in (a, b) : U(t) \cap D \neq \emptyset\}$ is measurable.

Z. 4 There is a function $\mu(\cdot) \in L^1_{\text{loc}}(a, b)$ such that for arbitrary $u(\cdot) \in \mathcal{U}$,

$$|f_i(t, u(t))| \leqq \mu_i(t) \quad \text{a.e.} \quad t \in (a, b) , \quad i = 1, \ldots, n .$$

These assumptions guarantee that for every $u(\cdot) \in \mathcal{U}$ the composition $f(\cdot, u(\cdot))$ belongs to $L^1_{\text{loc}}(a, b)$, therefore for every admissible control $u(\cdot)$ there exists a unique solution to the problem (20) which may be written by the Cauchy formula

(21) $$x(t) = \Phi(t) x_0 + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) f(s, u(s)) \, ds .$$

Fix an arbitrary T, $T \in (t_0, b)$ and consider the set $\mathcal{K}(T, U) = \{x(T) : x(\cdot) \text{ is given by (21)}, u(\cdot) \in \mathcal{U}\} \subset \mathbb{R}^n$.

Definition 5. The set $\mathscr{K}(T, U)$ is called the *attainable set* for the system (20) at the instant T.

This set may be written in the form

$$\mathscr{K}(T, U) = \Phi(T)\{x_0 + \int_{t_0}^{T} \Phi^{-1}(s)f(s, U(s))\,\mathrm{d}s\}$$

where the last integral is understood in the sense of Aumann. In the author's paper [13] the following properties of the attainable set were proved:

Theorem 3. *The set $\mathscr{K}(T, U)$ is non-empty, compact and convex, and the following bang-bang principle holds:*

$$\mathscr{K}(T, U) = \Phi(T)\{x_0 + \int_{t_0}^{T} \operatorname{extr}[\operatorname{conv} \Phi^{-1}(s)f(s, U(s))]\,\mathrm{d}s$$

where extr (Z) *denotes the set of all extremal points of the set Z and* conv Z *is the convex hull of the set Z. The attainability multifunction*

$$h\colon (t_0, b) \ni T \to \mathscr{K}(T, U)$$

is right-continuous in the Hausdorff metric; more precisely, if $T \neq t_k$ $(k = 1, 2, \ldots)$ then h is continuous at the instant T.

Now we formulate the time-optimal control problem. Fix an arbitrary final state $x_1 \in \mathbb{R}^n$ called the target of control. The problem reads as follows:

Find the control $\bar{u}(\cdot) \in \mathscr{U}$ and the instant $\bar{t}$ such that the corresponding trajectory $\bar{x}(\cdot)$ of the system (20) satisfies the condition $\bar{x}(\bar{t}) = x_1$ with the smallest $\bar{t}$ possible.

First we formulate the existence theorem.

Theorem 4. *If $x_1 \in \mathscr{K}(t_1, U)$ for some $t_1 > t_0$ then there exists $\bar{t} \in (t_0, t_1]$ such that $x_1 \in \mathscr{K}(\bar{t}, U)$ and $x_1 \notin \mathscr{K}(t, U)$ for $t < \bar{t}$.*

Proof. The set

$$S := \{t \in (t_0, b)\colon\ x_1 \in \mathscr{K}(t, U)\}$$

is a sum of at most countable number of intervals and every summand of S is closed from the left (this follows from Theorem 3). Moreover, S is bounded from below by t_0. Consequently, in the set S there exists an infimum t which is the left end of a summand of S and – by the definition of the attainable set – there exists a time-optimal control. ∎

In the classical situation (i.e., in the case of a non-atomic measure $A(\cdot)$) the optimal control is an extremal one in the sense that x_1 is a boundary point of the set $\mathscr{K}(\bar{t}, U)$. This property plays an important part in the proof and makes it possible to obtain a necessary condition of optimality in the form of the Pontryagin maximum principle. If the set of atomic points of the measure $A(\cdot)$ is non-empty then the target x_1 may be a boundary or an interior point of the set $\mathscr{K}(\bar{t}, U)$ as is illustrated by the following example.

Example. Let us consider a one-dimensional system

$$\dot{x} = [1 + \tfrac{2}{3}\delta(t-1)]\, x + u\,, \quad x(0) = 1\,, \quad -1 \leqq u \leqq 0\,.$$

The emission zone of the initial point is bounded from below by the curve

$$x_b(t) = \begin{cases} 1 & \text{if} \quad t < 1 \\ 1 + \frac{2}{3}e^t & \text{if} \quad t \geqq 1 \end{cases} \qquad (\text{for } u(t) \equiv -1)$$

and from above by the curve

$$x_a(t) = \begin{cases} e^t & \text{if} \quad t < 1 \\ 3e^t & \text{if} \quad t \geqq 1 \end{cases} \qquad (\text{for } u(t) \equiv 0)$$

Since $\varepsilon_1 = 2e$, we have $\Phi(t) = x_a(t)$. If $x_1 \in (1 + \frac{2}{3}e, 3e)$ then $\bar{t} = 1$ and $x_1 \in \operatorname{int} \mathscr{K}(1, U)$. ∎

5. COMPLETE CLASSIFICATION OF THE BEHAVIOUR OF THE ATTAINABILITY MULTIFUNCTION FOR ONE-DIMENSIONAL CONTROL SYSTEMS

In this part we present the complete classification of the behaviour of multifunction h introduced in the previous part for one – dimensional control systems. As will be shown, there is 16 different situations (possibilities) of such behaviour. In higher dimensions such a classification is – in my opinion – impossible. This classification will be demonstrated by a very simple example.

Example. Let us consider the system

$$\dot{x} = A\delta(t-1) + u\,, \quad x(0) = e \geqq 0\,, \quad t \geqq 0\,, \quad x, u \in \mathbb{R}^1\,, \quad B \leqq u \leqq C$$

where

$$A \neq 1\,, \quad B < C\,.$$

Now

$$\hat{\Phi}(t) = E = 1\,, \quad \varepsilon_1 = \frac{A}{1-A}\,, \quad s_1 = \frac{1}{1-A}\,, \quad \Phi(t) = 1 + \frac{A}{1-A} H(t-1)\,.$$

The emission zone of the point $(0, e)$ is bounded by the curves

$$x_B(t) = \begin{cases} e + Bt & \text{if} \quad t < 1 \\ \dfrac{1}{1-A}\,[e + B + B(1-A)(t-1)] & \text{if} \quad t \geqq 1 \end{cases} \qquad (\text{for } u(t) \equiv B)$$

$$x_C(t) = \begin{cases} e + Ct & \text{if} \quad t < 1 \\ \dfrac{1}{1-A}\,[e + C + C(1-A)(t-1)] & \text{if} \quad t \geqq 1 \end{cases} \qquad (\text{for } u(t) \equiv C)\,.$$

Obviously, $x_B(t) < x_C(t)$ for $t \in (0, 1)$.

For arbitrary $A \neq 1$ and $B < C$ we have 35 possible situations (22 of which are "qualitatively" different) which we describe below. We arrange these situations in some groups.

0. If $A = 0$ then we have the classical situation, i.e., the multifunction h is continuous in the Hausdorff metric:

$$h(t) = [e + Bt,\ e + Ct] \quad \text{for all} \quad t > 0$$

I. $C > B > -e$.

1. $A < 0$.

α) If $0 > A > \dfrac{B - C}{e + B}$ then $x_C(1-) > x_C(1) > x_B(1-) > x_B(1) > 0$.

β) If $A = \dfrac{B - C}{e + B}$ then $x_C(1-) > x_C(1) = x_B(1-) > x_B(1) > 0$.

γ) If $A < \dfrac{B - C}{e + B}$ then $x_C(1-) > x_B(1-) > x_C(1) > x_B(1) > 0$.

2. $A \in (0, 1)$.

α) If $0 < A < \dfrac{C - B}{e + C}$ then $x_C(1) > x_C(1-) > x_B(1) > x_B(1-) > 0$.

β) If $A = \dfrac{C - B}{e + C}$ then $x_C(1) > x_C(1-) = x_B(1) > x_B(1-) > 0$.

γ) If $A > \dfrac{C - B}{e + C}$ then $x_C(1) > x_B(1) > x_C(1-) > x_B(1-) > 0$.

3. $A > 1$.

Then $0 > x_C(1-) > x_B(1-) > x_B(1) > x_C(1)$.

II. $C > B = -e$.

In this case $x_B(\cdot)$ is continuous and $x_B(1) = 0$.

1. If $A < 0$ then $x_C(1-) > x_C(1) > x_B(1)$.
2. If $A \in (0, 1)$ then $x_C(1) > x_C(1-) > x_B(1)$.
3. If $A > 1$ then $x_C(1-) > x_B(1) > x_C(1)$.

III. $C > -e > B$.

1. If $A < 0$ then $x_C(1-) > x_C(1) > x_B(1) > x_B(1-)$.
2. If $A \in (0, 1)$ then $x_C(1) > x_C(1-) > 0 > x_B(1-) > x_B(1)$.
3. If $A > 1$ then

α) $B + C > -2e$.

α₁) If $1 < A < \dfrac{B - C}{e + B}$ then $x_B(1) > x_C(1-) > 0 > x_B(1-) > x_C(1)$.

α_2) If $A = \dfrac{B - C}{e + B}$ then $x_C(1-) > x_B(1) > 0 > x_B(1-) = x_C(1)$.

α_3) If $\dfrac{B - C}{e + B} < A < \dfrac{C - B}{e + C}$ then $x_C(1-) > x_B(1) > 0 > x_B(1-) > x_C(1)$.

α_4) If $A = \dfrac{C - B}{e + C}$ then $x_C(1-) = x_B(1) > 0 > x_B(1-) > x_C(1)$.

α_5) If $A > \dfrac{C - B}{e + C}$ then $x_C(1-) > x_B(1) > 0 > x_C(1) > x_B(1-)$.

β) $B + C = -2e$.

β_1) If $1 < A < 2$ then $x_C(1-) > x_B(1) > 0 > x_C(1) > x_B(1-)$.

β_2) If $A = 2$ then $x_C(1-) = x_B(1) > 0 > x_B(1-) = x_C(1)$.

β_3) If $A > 2$ then $x_B(1) > x_C(1-) > 0 > x_B(1-) > x_C(1)$.

γ) $B + C < -2e$.

γ_1) If $1 < A < \dfrac{B - C}{e + B}$ then $x_B(1) > x_C(1-) > 0 > x_B(1-) > x_C(1)$.

γ_2) If $A = \dfrac{B - C}{e + B}$ then $x_B(1) > x_C(1-) > 0 > x_C(1) = x_B(1-)$.

γ_3) If $\dfrac{B - C}{e + B} < A < \dfrac{C - B}{e + C}$ then $x_B(1) > x_C(1-) > 0 > x_C(1) > x_B(1-)$.

γ_4) If $A = \dfrac{C - B}{e + C}$ then $x_C(1-) = x_B(1) > 0 > x_C(1) > x_B(1-)$.

γ_5) If $A > \dfrac{C - B}{e + C}$ then $x_C(1-) > x_B(1) > 0 > x_C(1) > x_B(1-)$.

IV. $-e = C > B$.

In this case $x_C(\cdot)$ is continuous and $x_C(1) = 0$.

1. If $A < 0$ then $x_C(1) > x_B(1) > x_B(1-)$.
2. If $A \in (0, 1)$ then $x_C(1) > x_B(1-) > x_B(1)$.
3. If $A > 1$ then $x_B(1) > x_C(1) > x_B(1-)$.

V. $-e > C > B$.

1. $A < 0$.

α) If $0 > A > \dfrac{B - C}{e + C}$ then $0 > x_C(1) > x_C(1-) > x_B(1) > x_B(1-)$.

β) If $A = \dfrac{B - C}{e + C}$ then $0 > x_C(1) > x_C(1-) = x_B(1) > x_B(1-)$.

γ) If $A < \dfrac{B - C}{e + C}$ then $0 > x_C(1) > x_B(1) > x_C(1-) > x_B(1-)$.

2. $A \in (0, 1)$.

α) If $0 < A < \dfrac{C - B}{e + C}$ then $0 > x_C(1-) > x_C(1) > x_B(1-) > x_B(1)$.

β) If $A = \dfrac{C - B}{e + C}$ then $0 > x_C(1-) > x_C(1) = x_B(1-) > x_B(1)$.

γ) If $A > \dfrac{C - B}{e + C}$ then $0 > x_C(1-) > x_B(1-) > x_C(1) > x_B(1)$.

3. If $A > 1$ then $x_C(1) > x_B(1) > 0 > x_C(1-) > x_B(1-)$.

Some of these situations may be "qualitatively" identified, namely: I. 1 with V. 2, I. 2 with V. 1, α_1 with β_1 and with γ_5; α_2 with γ_4, α_3 with γ_3, α_4 with γ_2, α_5 with β_3 and with γ_1.

In I. 1, I. 2, II. 1, II. 2, III. 1, III. 2, IV. 1, IV. 2, V. 1, V. 2 we have $x_B(t) < x_C(t)$ for all $t > 0$, therefore in these cases the attainable set is $h(T) = [x_B(T), x_C(T)]$ while in the remaining situations we have the following inequalities: $x_B(t) > x_C(t)$ for $t \in (1, \bar{t})$, $x_B(t) < x_C(t)$ for $t \in (\bar{t}, 1)$ and $x_B(\bar{t}) = x_C(\bar{t}) = e/(1 - A)$ where $\bar{t} = A/(A - 1)$.

In these last cases we have

$$h(T) = \begin{cases} [x_B(T),\ x_C(T)] & \text{if} \quad 0 \leqq T < 1 \quad \text{or} \quad T > \bar{t} \\ [x_C(T), x_B T)] & \text{if} \quad 1 \leqq T < \bar{t} \\ \left\{\dfrac{e}{1 - A}\right\} & \text{if} \quad T = \bar{t}\,. \end{cases}$$

The multifunction h has a closed graph in the following situations (apart from the classical situation 0): II. 2, III. 2, III. 3, α_4, α_5, β_2, γ_1, γ_2, IV. 2, One of these situations is of special interest, namely, III. 3, β_2 because in this situation h is also continuous in the Hausdorff metric.

From some of the situations described we deduce that the following proposition is not generally true (contrary to the classical case of a non-atomic measure $A(\cdot)$):

If $u(t) \leqq v(t)$ for a.e. $t \in (t_0, t_1)$ then $x_u(t) \leqq x_v(t)$ for $t \in (t_0, t_1)$. For example, if we put $A = \frac{3}{2}$, $e = 1$, $u(t) \equiv 1$, $v(t) \equiv 2$ then $x_u(t) < x_v(t)$ for $t \in (0, 1)$ while $x_u(t) > x_v(t)$ for $t \in (1, 3)$.

In the conclusion we solve the time-optimal control problem for the system

$$\dot{x} = \tfrac{3}{2}\delta(t - 1)\,x + u\,, \quad x(0) = 1\,, \quad -2 \leqq u \leqq 1$$

with the target $x_1 = 2$ (cf. III. 3. α_4).

For any $\varepsilon > 0$ there exists an ε-suboptimal control, namely, $\bar{u}_\varepsilon(t) \equiv 1$ which transfers our system from the initial state to the final state $x(1 - \varepsilon) = 2 - \varepsilon$ so that

$$|x(1 - \varepsilon) - x_1| = \varepsilon,$$

but the optimal control for this problem is

$$\bar{u}(t) \equiv -2$$

because for this control $\bar{x}(1) = 2$ and $x(t) < 2$ for every t and every admissible control different from $\bar{u}(t)$. ∎

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Souhrn

LINEÁRNÍ DIFERENCIÁLNÍ ROVNICE, JEJICHŽ KOEFICIENTY JSOU MÍRY, A TEORIE ŘÍZENÍ

ZDZISŁAW WYDERKA

V článku se studují soustavy lineárních diferenciálních rovnic, jejichž koeficienty jsou míry, a jejich aplikace v teorii řízení. Je podána úplná klasifikace dosažitelných množin pro jednodimenzionální soustavu optimální regulace.

Резюме

ЛИНЕЙНЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ, КОЭФФИЦИЕНТАМИ КОТОРЫХ ЯВЛЯЮТСЯ МЕРЫ, И ТЕОРИЯ УПРАВЛЕНИЯ

ZDZISŁAW WYDERKA

В статье изучаются системы линейных дифференциальных уравнений, коэффициентами которых являются меры, и их приложения в теории управления. Приводится полная классификация достижимых множеств для одномерной системы оптимального управления.

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