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REPRESENTATIONS OF CYCLICALLY ORDERED GROUPS

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Summary. Theorem on the representation of cyclically ordered groups was proved by S. Swierczkowski (by using a result of L. Rieger). In the present note sufficient conditions are found for such a representation to be determined uniquely up to isomorphism.

Keywords: cyclic order, cyclically ordered group, linearly ordered group, linearly ordered kernel.

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1. INTRODUCTION

Cyclically ordered groups were investigated by L. Rieger [3], S. Swierczkowski [4], A. I. Zabarina [5], [6], A. I. Zabarina and G. G. Pestov [7], Š. Černák and the first author [2]. For the basic notions, cf. also L. Fuchs [1].

By applying a result of Rieger [3], Swierczkowski [5] proved a theorem on the representation of a cyclically ordered group. (For a thorough formulation cf. Definition 2.4 and Theorem 5 below.) A natural question arises whether this representation is unique up to isomorphism.

Let G be a cyclically ordered group and let $\varphi: G \to K_1 \otimes L$ be a representation in the sense of [4]. Here K_1 is a subgroup of the cyclically ordered group K from Example 2.2 below and L is a linearly ordered group. We denote by φ_1 the natural homomorphism of G onto K_1 which is induced by the representation φ. Similarly, let φ_2 be the natural homomorphism of G onto L which is induced by the representation φ.

Let $\psi: G \to K_1' \otimes L'$ be another such representation. In the present paper it will be proved that, with regard to the above question, the situation concerning the first subdirect factor (i.e., the subgroup of K) essentially differs from the situation concerning the second subdirect factor (the linearly ordered group). Namely, it will be shown that

(i) *there exists an isomorphism i_1 of K_1 onto K_1' such that the diagram*

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, dash] & K_1 \arrow[d, "i_1"] \\ {} \arrow[r, "\psi_1"'] & K_1' \end{tikzcd}$$

is commutative;

(ii) *the linearly ordered groups* L *and* L' *need not be isomorphic in general.*

In fact, there exists a cyclically ordered group G having the representations

$$\psi_n: G \to K_n \otimes L_n \quad (n = 1, 2, 3, \ldots)$$

such that, whenever m and n are distinct positive integers, then L_m fails to be isomorphic to L_n.

Let G_0 be the linearly ordered kernel of G (cf. Section 3 below). It will be proved that

(iii) *if* G_0 *is divisible and if for each* $g \in G$ *there is a positive integer* n *with* $ng \in G_0$, *then there exists an isomorphism* i_2 *of* L *onto* L' *such that the diagram*

$$\begin{tikzcd} G \arrow[r, "\varphi_2"] \arrow[d, dash] & L \arrow[d, "i_2"] \\ {} \arrow[r, "\psi_2"'] & L' \end{tikzcd}$$

is commutative.

Section 2 contains the basic definitions; also, Swierczkowski's theorem is recalled here. Two characterizations of G_0 are given in Section 3 and Section 4. The above assertion (i) is proved in Section 5. The proofs of the assertions (ii) and (iii) are presented in Section 6.

2. PRELIMINARIES

Let G be a group. The group operation will be denoted additively, the commutativity of this operation will not be assumed.

Suppose that a ternary relation $[x, y, z]$ is defined on G such that the following conditions are satisfied for each $x, y, z, a, b \in G$:

I. If $[x, y, z]$ holds, then x, y and z are distinct; if x, y and z are distinct, then either $[x, y, z]$ or $[z, y, x]$.

II. $[x, y, z]$ implies $[y, z, x]$.

III. $[x, y, z]$ and $[y, u, z]$ imply $[x, u, z]$.

IV. $[x, y, z]$ implies $[a + x + b, a + y + b, a + z + b]$.

Then G is said to be a cyclically ordered group. The ternary relation under consideration is called the cyclic order on G.

If H is a subgroup of G, then H is considered as cyclically ordered by the cyclic order reduced to H.

The notion of isomorphism for cyclically ordered groups is defined in a natural way. Let G and G' be cyclically ordered groups and let f be a mapping of G into G' such that the following conditions are satisfied:

(i) f is a homomorphism with respect to the group operation;

(ii) if x, y and z are elements of G such that $[x, y, z]$ is valid in G and if the elements $f(x), f(y)$ and $f(z)$ are distinct, then $[f(x), f(y), f(z)]$ is valid in G'.

Under these suppositions f is said to be a homomorphism of G into G'.

2.1. Example. Let L be a linearly ordered group. For distinct elements x, y and z of L we pout $[x, y, z]$ if

$$(1) \qquad x < y < z \quad \text{or} \quad z < x < y \quad \text{or} \quad y < z < x$$

is valid. Then L turns out to be a cyclically ordered group. The cyclic order defined in this way is said to be induced by the linear order. Each linearly ordered group will be considered as cyclically ordered by the induced cyclic order.

2.2. Example. Let K be the set of all real numbers x with $0 \leqq x < 1$; the operation $+$ on K is defined to be the addition mod 1. For distinct elements x, y and z of K we put $[x, y, z]$ if the relation (1) is valid. Then K is a cyclically ordered group.

2.3. Example. Let L be as in 2.1 and let K be as in 2.2. Let K_1 be a subgroup of K. Let $K_1 \times L$ be the direct product of the groups K_1 and L. For distinct elements $u = (a, x)$, $v = (b, y)$ and $w = (c, z)$ of $K_1 \times L$ we put $[u, v, w]$ if some of the following conditions is satisfied:

(i) $[a, b, c]$;

(ii) $a = b \neq c$ and $x < y$;

(iii) $b = c \neq a$ and $y < z$;

(iv) $c = a \neq b$ and $z < x$;

(v) $a = b = c$ and $[x, y, z]$.

The group $K_1 \times L$ with this ternary relation is a cyclically ordered group; it will be denoted by $K_1 \otimes L$.

For $M \subseteq K_1 \times L$ we denote by $M(K_1)$ the set of all elements $a \in K_1$ having the property that there exists $x \in L$ with $(a, x) \in M$. The set $M(L)$ is defined analogously.

2.4. Definition. *Let G be a cyclically ordered group. Let $K_1 \otimes L$ be as in 2.3. A mapping φ of G into $K_1 \otimes L$ is said to be a representation of G if the following conditions are satisfied:*

(i) *φ is an isomorphism of G into $K_1 \otimes L$;*

(ii) *$\varphi(G)(K_1) = K_1$ and $\varphi(G)(L) = L$.*

Let G, K_1, L and φ be as in 2.4. Let $g \in G$ and $\varphi(g) = (a, x)$. Then we put $\varphi_1(g) = a$, $\varphi_2(g) = x$.

The following result is due to Swierczkowski [4].

2.5. Theorem. *Each cyclically ordered group possesses a representation.*

From 2.5 we obtain by a routine calculation:

2.6. Lemma. *Let a and b be elements of a cyclically ordered group such that $[0, a, b]$ is valid. Then $[0, -b, -a]$ holds.*

3. LINEARLY ORDERED SUBGROUPS OF A CYCLICALLY ORDERED GROUP

If G is a cyclically ordered group, then by the expression "G is linearly ordered" we always mean the fact that there exists a linear order $\leqq$ on G such that $(G; \leqq)$ is a linearly ordered group and that the given cyclic order on G is induced by this linear order.

The relation of cyclic order on G will often be denoted by $[\ ]$.

3.1. Lemma. *Let $(G; \leqq)$ be a linearly ordered group and let $[\ ]$ be the corresponding induced cyclic order on G. Then the linear order on G can be uniquely reconstructed from the cyclically ordered group $(G; [\ ])$.*

Proof. In view of the definition of the induced cyclic order (cf. Example 2.1) we have

$$x > 0 \Leftrightarrow [-x, 0, x] .$$

3.2. Corollary. *Let $(G; +, \leqq_1)$ and $(G; +, \leqq_2)$ be linearly ordered groups. Let $[\ ]_1$ and $[\ ]_2$ be the corresponding induced cyclic orders. If $[\ ]_1$ coincides with $[\ ]_2$, then $\leqq_1$ coincides with $\leqq_2$.*

Example 2.1 and Lemma 3.1 show that the notion of cyclically ordered group is a generalization of the notion of linearly ordered group. Moreover, the class of all linearly ordered groups as a subclass of all cyclically ordered groups can be determined by using merely the properties of the corresponding cyclic orders; namely, the following assertion is valid:

3.3. Lemma. *Let G be a cyclically ordered group. Then the following conditions are equivalent:*

(i) *G is linearly ordered.*

(ii) *Each nonzero subgroup of G is infinite and for each $g \in G$ and each positive integer n the relation*

$$[-g, 0, g] \Rightarrow [-g, 0, ng]$$

is valid.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. Assume that (ii) holds. By way of contradiction, suppose that G fails to be linearly ordered. According to 2.5, there exists a representation

$$\varphi: G \to K_1 \otimes L$$

of the cyclically·ordered group G. Then without loss of generality we can suppose that $G \subseteq K_1 \times L$ and that φ is the identity on G. Because G is not linearly ordered, there exists $g \in G$, $g = (a, x)$ such that $a \neq 0$.

First we shall verify that $na \neq 0$ for each positive integer n. In fact, suppose that there exists a positive integer n with

(1) $$na = 0\,;$$

let n be the least positive integer with the property mentioned. Clearly $n \geqq 2$ and there exists a positive integer $m < n$ such that $a = m/n$, and the positive integers m and n are relatively prime.

Since the subgroup of G generated by the element g is infinite, the relation (1) yields $x \neq 0$. There exists an integer k such that $kg = (1/n, kx)$ and $kx < 0$. Then $-kg = (1 - 1/n, -kx)$. According to the definition of the cyclic order in $K_1 \otimes L$ we have

$$[0, kg, -kg]\,,$$

hence $[-kg, 0, kg]$. Therefore in view of (ii), the relation

(2) $$[-kg, 0, nkg]$$

holds. According to (1) we have $nkg = (0, nkx)$. Since $nkx < 0$, we obtain $[0, -kg, nkg]$. Thus $[nkg, 0, -kg]$, contradicting (2).

Let b be the inverse element of a in K_1, hence $b = 1 - a$. Since $a \neq \frac{1}{2}$, one of the elements a or b is contained in the open interval of reals with the endpoints 0 and $\frac{1}{2}$. Without loss of generality we can assume that $0 < a < \frac{1}{2}$. Then $\frac{1}{2} < b < 1$, whence $[0, g, -g]$ and thus $[-g, 0, g]$. In view of (ii) the relation

(3) $$[-g, 0, ng]$$

is valid for each positive integer n. Because $0 < a < b$, there is a positive integer t such that

(4) $$ta < b \quad \text{and} \quad (t + 1)\,a \geqq b\,.$$

(In the relations (4) the addition and the multiplication is in the field of reals, not in K_1.) If $(t + 1)\,a = b$, then in K_1 we would have $(t + 2)\,a = 0$, which contradicts the result proved above. Hence $(t + 1) a > b$. From the first inequality in (4) we obtain

$$(t + 1) a < 1 .$$

Thus $0 < b < (t + 1) a < 1$, whence

$$[0, (b, -x), ((t + 1) a, (t + 1) x)] ,$$

and thus $[0, -g, (k + 1) g]$. Therefore $[(k + 1) g, 0, -g]$, contradicting the relation (3).

3.4. Corollary. *Let G be a cyclically ordered group. Let G_1 be a subgroup of G. Then G_1 is linearly ordered if and only if it fulfils the condition* (ii) *of* 3.3.

Let G be a cyclically ordered group and let $\varphi: G \to K_1 \otimes L$ be its representation. We denote by $G_0(\varphi)$ the set of all $g \in G$ having the property that there exists $x_g \in L$ with $\varphi(g) = (0, x_g)$. Then $G_0(\varphi)$ is a subgroup of G.

In view of the facts mentioned in the proof of 3.3 we obviously have

3.5. Lemma. *Let H be a subgroup of a cyclically ordered subgroup of G. Let φ be as above. Then the following conditions are equivalent*:

(a) *$H \subseteq G_0(\varphi)$.*

(b) *H is linearly ordered.*

Proof. The implication (a) $\Rightarrow$ (b) is obvious. The proof of the implication (b) $\Rightarrow$ (a) is the same as the proof of the relation (ii) $\Rightarrow$ (i) in 3.3.

For a related result, cf. [7], Theorem 2.7.

3.6. Corollary. (i) *$G_0(\varphi)$ is the largest linearly ordered subgroup of G.* (ii) *$G_0(\varphi)$ is a normal subgroup of G.* (iii) *If $\psi: G \to K_1' \otimes L'$ is another representation of G, then $G_0(\psi) = G_0(\varphi)$.*

In view of part (iii) of 3.6 we shall often write G_0 rather than $G_0(\varphi)$.

4. c-CONVEX SUBGROUPS

It is well known that the kernel of a homomorphism of a linearly ordered group G into a linearly ordered group is a convex subgroup of G.

In this section the notion of a c-convex subgroup of a cyclically ordered group is introduced. The definition of c-convexity is formulated in such a way that in the particular case of linearly ordered groups it coincides with the notion of convexity.

4.1. Definition. *Let G be a cyclically ordered group. A subgroup H of G is said to be c-convex if one of the following conditions is fulfilled*:

(i) *$H = G$;*

(ii) *for each* $h \in H$ *with* $h \neq 0$ *we have* $2h \neq 0$; *if* $h \in H$, $g \in G$, $[-h, 0, h]$, $[-h, g, h]$, *then* $g \in H$.

Another notion of convexity in cyclically ordered groups (called z-convexity) was studied in [7].

From 4.1 and from the fact that for a nonzero element h of a linearly ordered group we always have $2h \neq 0$ we immediately obtain

4.2. Lemma. *Let* H *be a subgroup of a linearly ordered group* G. *Then the following conditions are equivalent*:

(i) H *is a convex subgroup of the linearly ordered group* G.

(ii) H *is a c-convex subgroup of the cyclically ordered group* G.

For a subgroup H of a linearly ordered group G the condition (i) from 4.2 is equivalent to each of the following conditions:

(α_1) If $h_1, h_2 \in H$, $g \in G$, $h_1 < g < h_2$, then $g \in H$.

(α_2) If $h \in H$, $g \in G$, $0 < g < h$, then $g \in H$.

By analogy to (α_1) and (α_2) we can consider the following conditions for a subgroup H of a cyclically ordered group G:

(β_1) If $h_1, h_2 \in H$, $g \in G$, $[h_1, g, h_2]$, then $g \in H$.

(β_2) If $h \in H$, $g \in G$, $[0, g, h]$, then $g \in H$.

The question arises whether (β_1) is equivalent to the c-convexity of H (and similarly for the condition (β_2)). The following lemma (and the fact that there exists a cyclically ordered group G having a nonzero c-convex subgroup distinct from G) shows that the answer is negative.

4.3. Lemma. *Let* H *be a nonzero subgroup of a cyclically ordered group* G. *Assume that* H *satisfies the condition* (β_2). *Then* $H = G$.

Proof. There exists $h \in H$ with $h \neq 0$. Let $g \in G$, $g \neq 0$, $-h \neq g \neq h$. If $2h = 0$, then we have either $[0, g, h]$ or $[0, -g, h]$; hence in both cases, g belongs to H. Thus $H = G$.

Now suppose that $2h \neq 0$. Without loss of generality we may suppose that $[-h, 0, h]$ (if not, then we take $-h$ instead of h). Thus one of the following possibilities is valid: (i) $[0, g, h]$; (ii) $[h, g, -h]$; (iii) $[-h, g, 0]$. In the case (i) we have $g \in H$. If (ii) is valid, then $[0, g - h, -2h]$, hence $g - h \in H$ and thus $g \in H$. In the case (iii) the relation $[0, -g, h]$ holds (in view of 2.6), whence $g \in H$.

Since (β_1) $\Rightarrow$ (β_2), we infer that in 4.3 the condition (β_2) can be replaced by (β_1).

The following two assertions are immediate consequences of 4.1.

4.4. Lemma. *Let* G *be a cyclically ordered group. Let* H *be a c-convex subgroup of* G *and let* H_1 *be a c-convex subgroup of* H. *Then* H_1 *is a c-convex subgroup of* G.

4.5. Lemma. G_0 *is a c-convex subgroup of* G.

4.6. Lemma. *Let G be a cyclically ordered group. Let H be a subgroup of G. Then the following conditions are equivalent:*

(i) *H is a c-convex subgroup of G.*

(ii) *Either $H = G$ or H is a c-convex subgroup of G_0.*

Proof. The assertion (ii) $\Rightarrow$ (i) follows from 4.4 and 4.5. Assume that (i) is valid. If $H \subseteq G_0$, then H is obviously a c-convex subgroup of G_0. Suppose that H fails to be a subset of G_0. Hence there is $h \in H$ such that, under a representation $\varphi: G \to K_1 \otimes L$ of G, we have $\varphi(h) = (a, x)$, $a \neq 0$. Let $g \in G$, $\varphi(g) = (b, y)$.

Let us first consider the case $a = \frac{1}{2}$. Then $x \neq 0$ (because $2h \neq 0$). Without loss of generality we can suppose that $x < 0$ (in the case $x > 0$ we take $-h$ instead of h). Thus $[-h, 0, h]$ holds. If $b \neq \frac{1}{2}$, then $[-h, g, h]$ is valid, hence $g \in H$. In the case $b = \frac{1}{2}$ denote $g' = g - h$. We have $[-h, g', h]$, thus $g' \in H$ and therefore $g \in H$. We obtain $H = G$.

Next, let us consider the case $a \neq \frac{1}{2}$. Then without loss of generality we may suppose that $0 < a < \frac{1}{2}$, hence $[-h, 0, h]$. We distinguish the following subcases:

(α) If $0 \leqq b < a$ or $1 - a < b < 1$, then $[-h, g, h]$, hence $g \in H$.

(β) If $b = a$ or $b = 1 - a$, then for the element $g' = g - h$ or $g' = g + h$, respectively, we have $g' \in H$ (according to (α)), hence $g \in H$.

(γ) Let $a < b < 1 - a$. First suppose that there exists a positive integer n such that $na = \frac{1}{2}$. Put $h_1 = nh \in H$. Then $\varphi(h_1) = (\frac{1}{2}, nx)$. Hence in view to the result proved above (for the case $a = \frac{1}{2}$) we infer that $g \in G$. Now suppose that $na \neq \frac{1}{2}$ for each positive integer n. There exists the least positive integer n with $na < \frac{1}{2} < (n + 1) a$. Let $\varphi(-g) = (b_1, g_1)$ and assume that $g \neq 0$. It suffices to verify that $g \in H$ or $-g \in H$. We have either $0 \leqq b \leqq \frac{1}{2}$ or $0 \leqq b_1 \leqq \frac{1}{2}$. Without loss of generality we may suppose that $0 \leqq b \leqq \frac{1}{2}$. If $b = na$, then according to (α) or (β), we obtain $g \in H$ (we take now $h' = nh$ instead of h). Let $na < b$. Then $na < b < (n + 1) a$, hence $0 < b - na < a$. In view of (α) we have $g - nh \in H$, which implies that $g \in H$. Therefore $H = G$.

4.7. Corollary. *Let G be a nonzero cyclically ordered group, $G \neq G_0$. Then G_0 is the largest c-convex proper subgroup of G.*

For a related result (concerning z-convexity) cf. [7], Theorem 2.6.

From 3.6 and 4.7 we obtain

4.8. Corollary. *Let H be a subgroup of a cyclically ordered group G. Then the following conditions are equivalent:*

(i) *H is the largest linearly ordered subgroup of G;*

(ii) *H is the largest c-convex proper subgroup of G.*

5. THE FIRST SUBDIRECT FACTORS

In this section the assertion (i) from the introduction will be proved.

Let $\varphi: G \to K_1 \otimes L$ be a representation of a cyclically ordered group G. Then K_1 of L is said to be the first subdirect factor and the second subdirect factor, respectively, with regard to the given representation of G. Let $g \in G$ and $\varphi(g) = (a, x)$. Denote $\varphi_1(g) = a$. The mapping φ_1 is a homomorphism of the group G into the group K_1. For each $a_1 \in K_1$ there exists $g_1 \in G$ such that $\varphi_1(g) = a_1$. Hence φ_1 is an epimorphism. The kernel of φ_1 is $G_0(\varphi)$. For each element $g + G_0$ of the factor group $G/G_0(\varphi)$ we put

$$\varphi_1'(g + G_0) = a\,.$$

Then φ_1' is correctly defined on $G/G_0(\varphi)$ and φ_1' is an isomorphism of the group $G/G_0(\varphi)$ onto the group K_1.

Let $g_i + G_0$ $(i = 1, 2, 3)$ be distinct elements of $G/G_0(\varphi)$, $\varphi(g_i) = (a_i, x_i)$ $(i = 1, 2, 3)$. Then we have

$$(1) \qquad [g_1, g_2, g_3] \Leftrightarrow [a_1, a_2, a_3]\,.$$

Thus if $g_i' \in g_i + G_0$ $(i = 1, 2, 3)$, then

$$(2) \qquad [g_1, g_2, g_3] \Leftrightarrow [g_1', g_2', g_3']\,.$$

Under the above notation we put $[g_1 + G_0, g_2 + G_0, g_3 + G_0]$ if $[g_1, g_2, g_3]$ is valid. In view of (2) the relation $[\,]$ on $G/G_0(\varphi)$ is correctly defined. From (1) we obtain

5.1. Lemma. *$(G/G_0(\varphi), [\,])$ is a cyclically ordered group. The mapping φ_1' is an isomorphism of this cyclically ordered group onto the cyclically ordered group K_1. The mapping φ_1 is a homomorphism of the cyclically ordered group G onto the cyclically ordered group K_1.*

The mapping φ_1 will be said to be the natural homomorphism of G onto K_1.

Let ν_1 be the natural mapping of G onto $G/G_0(\varphi)$; i.e., for each $g \in G$ we have $\nu_1(g) = g + G_0$. We obviously have

5.2. Lemma. *ν_1 is a homomorphism of the cyclically ordered group G onto the cyclically ordered group $G/G_0(\varphi)$ and the diagram*

$$\begin{tikzcd}
G \arrow[r, "\varphi_1"] \arrow[d] & K_1 \\
{} \arrow[r, "\nu_1"'] & G/G_0(\varphi) \arrow[u, "\varphi_1'"']
\end{tikzcd}$$

is commutative.

Now let $\psi: G \to K_1' \otimes L'$ be another representation of the cyclically ordered group G. Let ψ_1, ψ_1' and ν_2 be defined analogously as φ_1, φ_1' and ν_1. As $G_0(\psi) =$

$= G_0(\varphi)$ (cf. 3.6), we obtain $v_2 = v_1$. For each $a \in K_1$ we put

$$i_1(a) = \psi_1'((\varphi_1')^{-1}(a)).$$

According to 5.1 (and the corresponding assertion which concerns ψ) i_1 is an isomorphism of the cyclically ordered group K_1 onto the cyclically ordered group K_1'. From 5.2 and from the corresponding assertion concerning ψ we obtain

5.3. Theorem. *Let $\varphi: G \to K \otimes L$ and $\psi: G \to K' \otimes L'$ be representations of a cyclically ordered group G. Let $\varphi_1: G \to K$ and $\psi_1: G \to K'$ be the corresponding natural homomorphisms. Then there exists an isomorphism i_1 of the cyclically ordered group K onto K' such that the diagram*

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d] & K_1 \arrow[d, "i_1"] \\ {} \arrow[r, "\psi_1"'] & K_1' \end{tikzcd}$$

is commutative.

6. THE SECOND SUBDIRECT FACTORS

In this section it will be proved that the assertions (ii) and (iii) from the introduction are valid.

Let us denote by R the additive group of all reals with the natural linear order. Let Q be the subgroup of R consisting of all rationals. Let y be an irrational number with $0 < y < 1$. We denote by A the set of all elements r of R which can be expressed in the form

$$r = q + ny, \tag{1}$$

where $q \in Q$ and n is an integer. Then A is a subgroup of R and each element of A can be uniquely expressed in the form (1).

Let N be the set of all positive integers with the natural linear order. For each $n \in N$ let $B_n = Q$. Denote $L = \Gamma_{n \in N} B_n$. (Cf. [1].) Let K_1 be the subgroup of K generated by the element y. Put

$$G = K_1 \otimes L.$$

Let $m \in N$. Put $C_m = A$, and $C_n = Q$ for each $n \in N \setminus \{m\}$. Denote $L(m) = \Gamma_{n \in N} C_n$. Then we obviously have

6.1. Lemma. *If m_1 and m_2 are distinct positive integers, then $L(m_1)$ fails to be isomorphic to $L(m_2)$.*

The elements of G will be denoted by $g = (ty, b_n)_{n\in N}$, where t is an integer and $b_n \in B_n$. Similarly, the elements of the cyclically ordered group $G(m) = K_1 \otimes L(m)$ can be denoted by $(t'y, c_n)_{n\in N}$, where t' is an integer and $c_n \in C_n$ for each $n \in N$. Let us put

$$\varphi_n(g) = (ty, c_n)_{n\in N}\,,$$

where

(i) $c_n = b_n$ for each $n \in N \setminus \{m\}$,

(ii) $c_m = b_n + ty$.

The following two assertions are easy to verify.

6.2. Lemma. *φ_m is an isomorphism of the cyclically ordered group G into $G(m)$.*

6.3. Lemma. *Let $(c_n)_{n\in N} \in L(m)$. Then there exists $g = (ty, b_n) \in G$ such that $\varphi_m(g) = (ty, c_n)_{n\in N}$.*

From 6.2 and 6.3 we obtain

6.4. Corollary. *For each $m \in N$, the mapping φ_m: $G \to K_1 \otimes L(m)$ is a representation of the cyclically ordered group G.*

Now, 6.1 and 6.4 yield that the assertion (ii) from Introduction holds.

Let us investigate the following conditions for a cyclically ordered group G:

(α) G_0 is divisible.

(β) For each $g \in G$ there exists a positive integer n such that $ng \in G_0$.

Let φ: $G \to K_1 \otimes L$ be a representation of G. If $g \in G$ and $\varphi(g) = (a, x)$, then we put $\varphi_2(g) = x$. Then we obviously have

6.5. Lemma. *φ_2 is a homomorphism of the group G onto the group L.*

φ_2 will be said to be the natural homomorphism of the group G onto L.

6.6. Lemma. *Let (α) and (β) be valid. Let φ be as above, $x \in L$. There exists $g_1 \in G_0$ such that $\varphi(g_1) = (0, x)$.*

Proof. There exist $g \in G$ and $a \in K_1$ such that $\varphi(g) = (a, x)$. In view of (β) there exists a positive integer n such that $ng = g_0 \in G_0$. Hence $\varphi(ng) = (na, nx)$ and $na = 0$ in K. Next, according to (α) there is $g_1 \in G_0$ with $ng_1 = g_0$. Let $\varphi(g_1) = (0, z)$. Hence $\varphi(ng_1) = (0, nz) = \varphi(g_0) = (0, nx)$. We infer that $nz = nx$. Because L is linearly ordered, we obtain that $z = x$.

Under the notation as in 6.6 put $\varphi_2'(x) = g_1$. Then we have

6.7. Lemma. *φ_2' is an isomorphism of the linearly ordered group L onto the linearly ordered group G_0.*

Let ψ: $G \to K_1' \otimes L'$ be another representation of G. Let ψ_2 be defined analogously to φ_2. The elements g_0 and g_1 are as above (we apply 3.6). If $x \in L$, $g_0 \in G_0$, $\varphi(g_0) =$

$= (0, x)$, then there is a uniquely determined element $x' \in L'$ with $\psi(g_0) = (0, x')$. Put $i_2(x) = x'$. In view of 6.7 (and of the analogous assertion concerning ψ'_2) we obtain

6.8. Theorem. *Let G be a cyclically ordered group such that the conditions (α) and (β) are satisfied. Let $\varphi\colon G \to K_1 \otimes L$ and $\psi\colon G \to K'_1 \otimes L'$ be representations of G. Then i_2 is an isomorphism of L onto L' such that the diagram*

$$\begin{tikzcd} G \arrow[r, "\varphi_2"] \arrow[d] & L \arrow[d, "i_2"] \\ {} \arrow[r, "\psi_2"'] & L' \end{tikzcd}$$

is commutative.

Let Q_0 be the set of all rational numbers a with $0 \leqq a < 1$. By a slight modification of the above example which was used for proving the assertion (ii) from Introduction we obtain the following result (its proof will be omitted):

6.9. Proposition. *Let K_0 be a subgroup of K such that K_0 fails to be a subset of Q_0*

(i) *Put $G_1 = K_0 \otimes Q$. There exist representations $\psi_1\colon G_1 \to K_1 \otimes L_1$ and $\psi_2\colon G_1 \to K_2 \otimes L_2$ such that L_1 is not isomorphic to L_2.*

(ii) *Let α be an infinite cardinal. There is a linearly ordered group L such that for the cyclically ordered group $G_2 = K_0 \otimes L$ there exist representations $\psi_i\colon G_2 \to K_i \otimes L_i$ $(i \in I)$, card $I = \alpha$ having the property that whenever $i(1)$ and $i(2)$ are distinct elements of I, then $L_{i(1)}$ is not isomorphic to $L_{i(2)}$.*

References

[1] *L. Fuchs:* Partially ordered algebraic systems (Russian). Moskva 1972.

[2] *J. Jakubík, Š. Černák:* Completion of a cyclically ordered group Czechoslovak Math. J. 37, 1987, 157—174.

[3] *L. Rieger:* On ordered and cyclically ordered groups I, II, III, (Czech). Věstník král. české spol. nauk 1946, 1—31; 1947, 1—33; 1948, 1—26.

[4] *S. Swierczkowski:* On cyclically ordered groups. Fundam. Math. *47* (1959), 161—166.

[5] *A. I. Zabarina:* To the theory of cyclically ordered groups (Russian). Matem. zametki *31* (1982), 3—12.

[6] *A. I. Zabarina:* On linear and cyclical orders in groups (Russian). Sibir. matem. žurn. *26* (1985), 204—207.

[7] *A. I. Zabarina, G. G. Pestov:* As to Swierczkowski's theorem (Russian). Sibir. matem. žurn. *25* (1984), 46—53.

Súhrn

REPREZENTÁCIE CYKLICKY USPORIADANÝCH GRÚP

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S. Swierczkowski dokázal vetu o reprezantácii cyklicky usporiadaných grúp (pritom použil starší výsledok L. Riegra). V článku sú nájdené postačujúce podmienky, za ktorých je takáto reprezentácia určená jednoznačne (až na izomorfizmus).

Резюме

ПРЕДСТАВЛЕНИЯ ЦИКЛИЧЕСКИ УПОРЯДОЧЕННЫХ ГРУПП

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Теорема о представлениях циклически упорядоченных групп была доказана С. Сверчковским (при использовании одного результата Л. Ругера). В предлагаемой статье найдены достаточные условия, при которых это представление однозначно с точностью до изоморфизма.

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