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REFLECTION AND THE DIRICHLET PROBLEM ON DOUBLY CONNECTED REGIONS

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Summary. In the paper the Dirichlet problem on a doubly connected region in the plane bounded by two Jordan curves is solved. The "exterior" curve is supposed to be analytic and to have a reflection function (the "interior" curve is not smooth, in general). In that case the corresponding system of two integral equations can be reduced to a single integral equation considered on the "interior" curve.

Keywords: Laplace's equation, Dirichlet problem, integral equations.

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J. M. Sloss proved in 1974 ([15]) that for an analytic Jordan curve in the plane there is a "global reflection function". This reflection function has quite analogous properties with respect to the given curve as the function $1/\bar{z}$ has with respect to the unit circle. Further, J. M. Sloss showed ([16]) that the reflection function can be used in connection with solving the Dirichlet problem by means of integral equations in the case of multiply connected regions if the given region is bounded by finitely many Jordan curves, where the "exterior" one is an analytic curve having the reflection function. The other boundary curves are supposed to be smooth and to have continuous curvature. Then, if the boundary of the region consists of $n + 1$ Jordan curves, the system of $n + 1$ integral equations corresponding to the Dirichlet problem can be reduced to a system of n integral equations not involving the "exterior" curve. In the case $n = 1$, that is in the case of a doubly connected region, the system of two integral equations is reduced to a single integral equation with one unknown. Numerical examples concerning the last case are given in [17].

In this paper we will consider the case of doubly connected regions in the plane bounded by two Jordan curves. The "exterior" curve will be again analytic with a reflection function. As concerns the "interior" boundary curve the assumption of smoothness will be dropped. When investigating the corresponding integral equation we use the approach described in [7], which enables us to deal with non-smooth boundary curves.

In a subsequent paper it will be shown that the reflection function can be used to the reduction of the number of integral equations also in solving the Neumann problem. We investigate the Dirichlet and the Neumann problems separately since if the reflection function is used, the corresponding integral equations are no longer adjoint to each other.

Throughout the paper we deal with the plane R^2. The real plane R^2 will be identified with the complex plane C, that is, a point $[x, y] \in R^2$ will be identified with the point $z \in C$, $z = x + iy$ as usual; similarly we shall write $[\xi, \eta] = \zeta = \xi + i\eta$ etc. By $\bar{z}$ we mean the conjugate of z, that is

$$\bar{z} = \overline{x + iy} = x - iy .$$

If f is a real function defined on a subset of R^2, we may consider f as a real function of the complex variable z, but usually we view it as a real function of two real variables $[x, y]$. The partial derivatives of f with respect to the real variables x, y will be denoted by $\partial_x f$, $\partial_y f$, respectively. If g is a mapping of a subset of R^2 to R^2 we shall usually consider g as a complex function of the complex variable. The derivative of g with respect to the complex variable (if it exists) is denoted by g'. By $\bar{g}$ we mean the conjugate of the complex function g, that is $\bar{g} = g_1 - ig_2$ if $g = g_1 + ig_2$, where g_1, g_2 are real functions.

1. THE REFLECTION FUNCTION

In this part we recall some results of J. M. Sloss from the paper [15] concerning the reflection function. We will also show a simple assertion concerning the normal derivative of a function reflected by the reflection function we shall need in the sequel.

Throughout the paper we suppose that L is a Jordan analytic curve with the parametrization $\Phi(\theta) = \Phi_1(\theta) + i\Phi_2(\theta)$, $\theta \in \langle 0, 2\pi \rangle$, of the form

$$\Phi_1(\theta) = x(\theta) = \sum_{k=0}^{n} (a_k \cos k\theta + b_k \sin k\theta) , \tag{1.1}$$

$$\Phi_2(\theta) = y(\theta) = \sum_{k=0}^{m} (\alpha_k \cos k\theta + \beta_k \sin k\theta) . \tag{1.2}$$

Further, we suppose that $n \geqq m$,

$$(\Phi_1'(\theta))^2 + (\Phi_2'(\theta))^2 \neq 0 ,$$

$$(a_n, b_n) \neq (0, 0) \neq (\alpha_m, \beta_m) \tag{1.3}$$

and if $m = n$ and L is not a circle then, in addition, either

$$\alpha_n^2 + \beta_n^2 \neq a_n^2 + b_n^2 \tag{1.4}$$

or

$$\alpha_n a_n + \beta_n b_n \neq 0 . \tag{1.5}$$

Putting $t = e^{i\theta} = \cos \theta + i \sin \theta$ one can write (1.1), (1.2) in the form

$$2x = 2a_0 + \bar{c}_1 t + c_1 \bar{t} + \bar{c}_2 t^2 + c_2 \bar{t}^2 + \ldots + \bar{c}_n t^n + c_n \bar{t}^n , \tag{1.6}$$

(1.7) $$2y = 2\alpha_0 + \bar{\gamma}_1 t + \gamma_1 \bar{t} + \bar{\gamma}_2 t^2 + \gamma_2 \bar{t}^2 + \ldots + \bar{\gamma}_m t^m + \gamma_m \bar{t}^m ,$$
where
$$c_k = a_k + ib_k , \quad \gamma_k = \alpha_k + i\beta_k .$$
Multiplying (1.6) by $\bar{t}^n$ and (1.7) by $\bar{t}^m$ and putting

(1.8) $$\alpha = \alpha(x) = 2(a_0 - x) , \quad \beta = \beta(y) = 2(\alpha_0 - y)$$

we see that (1.6), (1.7) are equivalent respectively to

(1.9) $$f(\bar{t}) = 0 , \quad g(\bar{t}) = 0 ,$$
where
$$f(\bar{t}) = \bar{c}_n + \bar{c}_{n-1}\bar{t} + \ldots + \bar{c}_1 \bar{t}^{n-1} + \alpha \bar{t}^n + c_1 \bar{t}^{n+1} + \ldots + c_n \bar{t}^{2n} ,$$
$$g(\bar{t}) = \bar{\gamma}_m + \bar{\gamma}_{m-1}\bar{t} + \ldots + \bar{\gamma}_1 \bar{t}^{m-1} + \beta \bar{t}^m + \gamma_1 \bar{t}^{m+1} + \ldots + \gamma_m \bar{t}^{2m} .$$
This means that L is given by exactly those $[x, y]$ for which there is a $\bar{t}$ with $|\bar{t}| = 1$ satisfying (1.9), that is, the polynomials f, g have a common root $\bar{t}$ with $|\bar{t}| = 1$. However, a necessary and sufficient condition for f and g to have common roots is that the resultant (Sylvester's determinant) $R(f, g)$ vanishes (for the definition and the properties of the resultant of two polynomials see for example [18]). Put
$$\Delta[\alpha(x), \beta(y)] = R(f, g)$$
(α, β are given by (1.8)). Then we see that all the points $[x, y] \in L$ satisfy the algebraic equation

(1.10) $$\Delta[\alpha(x), \beta(y)] = 0 .$$

Note that J. M. Sloss asserts in [15] that the curve L is given by the equation (1.10) which is not quite right. Generally, it may happen that f, g have common roots $\bar{t}$ for which $|\bar{t}| \neq 1$ for some $[x, y]$, that is, it may happen that the equation is fulfilled for some $[x, y] \notin L$.

Further, for $z, \zeta \in R^2$ put
$$M(z, \zeta) = \Delta\left[\alpha\left(\frac{z + \zeta}{2}\right), \ \beta\left(\frac{z - \zeta}{2i}\right)\right].$$
The equation

(1.11) $$M(z, \zeta) = 0$$

is an algebraic equation of order $2n$ in ζ and ζ is determined by (1.11) as an algebraic function of the variable z. The equation (1.11) can be written in the form (see [15])

(1.12) $$g_{2n}(z)\, \zeta^{2n} + g_{2n-1}(z)\, \zeta^{2n-1} + \ldots + g_1(z)\, \zeta + g_0(z) = 0$$

($g_{2n}(z), \ldots, g_0(z)$ are polynomials in z).

We shall say that $e \in R^2$ is a critical point of $M(z, \zeta)$ (also called a critical point with respect to the curve L) if either $g_{2n}(e) = 0$ or the equation $M(e, \zeta) = 0$ (with the unknown ζ) has a multiple root. Let $e_1, \ldots, e_r$ be those critical points of $M(z, \zeta)$

which lie in

$$R = \operatorname{Int} L$$

(the interior region bounded by L). Finally, let L_i denote a Jordan arc lying in R which joins all the points $e_1, \ldots, e_r$.

Now we are in position to formulate the following theorem due to J. M. Sloss ([15]).

1.1. Theorem. *The equation* (1.11) *together with the condition* $z_0 = \bar{g}(z_0)$ *for some* $z_0 \in L$ *define a function*

$$\zeta = g(z)$$

with the following properties: There is a neighbourhood R_0 *of* L *such that*

(1.13) g *is defined and analytic on* $(R - \{e_1, \ldots, e_r\}) \cup R_0$;

(1.14) g *is single-valued on* $(R - L_i) \cup R_0$;

(1.15) $g'(z) \neq 0$ *for* $z \in (R - L_i) \cup R_0$;

(1.16) $\bar{g}(z) = z$ *for* $z \in L$;

(1.17) $\bar{g}(R - L_i) \cap (R - L_i) = \emptyset$;

(1.18) g *can be uniquely extended onto*

$$R_g = (R - L_i) \cup L \cup \bar{g}(R - L_i)$$

to be holomorphic there;

(1.19) $\bar{g}(\bar{g}(z)) = z$ *for* $z \in R_g$.

In [15] J. M. Sloss also investigated the behaviour of g near the critical points $e_1, \ldots, e_r$; we do not need those properties in the sequel.

The function (mapping) $\bar{g}$ will be called the reflection function (mapping) with respect to the curve L. It follows from Theorem 1.1 that $\bar{g}$ is one-to-one on R_g, $\bar{g}(R_g) = R_g$, $\bar{g}(R_g \cap R) = R_g \cap \operatorname{Ext} L$, $\bar{g}(R_g \cap \operatorname{Ext} L) = R_g \cap R$.

If h is a (real or complex) function defined on $M \subset R_g$ then by $h * \bar{g}$ we mean the composition of h and $\bar{g}$ defined on $(\bar{g})^{-1}(M)$, that is, for $z \in (\bar{g})^{-1}(M)$ we have $h * \bar{g}(z) = h(\bar{g}(z))$.

As L is analytic there is a normal vector to L at any point of L. We denote by $\boldsymbol{n}_e = \boldsymbol{n}_e^L$, $\boldsymbol{n}_i = \boldsymbol{n}_i^L$ ($\|\boldsymbol{n}_e\| = \|\boldsymbol{n}_i\| = 1$) the exterior and the interior normal to L, respectively.

1.2. Lemma. *Let* h *be a real function defined and continuously differentiable on a neighbourhood of* L *and let* $\bar{g}$ *be the reflection function with respect to* L. *Then* (*on* L)

$$\frac{\partial h}{\partial \boldsymbol{n}_e} = \frac{\partial}{\partial \boldsymbol{n}_i}(h * \bar{g}) = -\frac{\partial}{\partial \boldsymbol{n}_e}(h * \bar{g}) \,. \tag{1.20}$$

Proof of this lemma is simple – we include it here for completeness.

Recall that L has a parametrization of the form $z = \Phi(t) = \Phi_1(t) + i\,\Phi_2(t)$. Put

$$\boldsymbol{n} = \frac{1}{|\Phi'|}\left[-\Phi_2', \Phi_1'\right].$$

Then $\boldsymbol{n}$ is the normal to L (we have either $\boldsymbol{n} = \boldsymbol{n}_e$ or $\boldsymbol{n} = \boldsymbol{n}_i$ depending on the orientation of the given parametrization). If $g = g_1 + ig_2$ then

$$\operatorname{grad}(h * \bar{g}) = \left[\partial_x h\, \partial_x g_1 - \partial_y h\, \partial_x g_2,\ \partial_x h\, \partial_y g_1 - \partial_y h\, \partial_y g_2\right]$$

and

$$\frac{\partial}{\partial \boldsymbol{n}}(h * \bar{g}) = \boldsymbol{n} \,.\, \operatorname{grad}(h * \bar{g}) =$$

$$= \frac{1}{|\Phi'|}\left\{-\partial_x h\left[\partial_x g_1 \Phi_2' - \partial_y g_1 \Phi_1'\right] - \partial_y h\left[-\partial_x g_2 \Phi_2' + \partial_y g_2 \Phi_1'\right]\right\}.$$

As g is holomorphic we have

$$\partial_x g_1 = \partial_y g_2\,, \quad \partial_y g_1 = -\partial_x g_2$$

and thus

$$\frac{\partial}{\partial \boldsymbol{n}}(h * \bar{g}) = \tag{1.21}$$

$$= \frac{1}{|\Phi'|}\left\{-\partial_x h\left[\partial_y g_2 \Phi_2' + \partial_x g_2 \Phi_1'\right] - \partial_y h\left[\partial_y g_1 \Phi_2' + \partial_x g_1 \Phi_1'\right]\right\}.$$

For $z \in L$ we have $\bar{g}(z) = z$, which means

$$\Phi_1(t) = g_1(\Phi(t))\,, \quad \Phi_2(t) = -g_2(\Phi(t))$$

and we obtain

$$\Phi_1' = \partial_x g_1 \Phi_1' + \partial_y g_1 \Phi_2'\,, \quad \Phi_2' = -\partial_x g_2 \Phi_1' - \partial_y g_2 \Phi_2'\,.$$

This together with (1.21) yields

$$\frac{\partial}{\partial \boldsymbol{n}}(h * \bar{g}) = \frac{1}{|\Phi'|}\left[\partial_x h \Phi_2' - \partial_y h \Phi_1'\right] = -\frac{\partial h}{\partial \boldsymbol{n}}$$

which is the equality (1.20).

1.3. Remark. Let h be a real function defined and continuously differentiable on a "symmetric" neighbourhood of L, that is, on such neighbourhood U of L that $\bar{g}(z) \in U$ for any $z \in U$. Suppose that $h(\bar{g}(z)) = h(z)$ for each $z \in U$. Then for $\zeta \in L$ we have

$$\frac{\partial h}{\partial \boldsymbol{n}}(\zeta) = 0$$

since

$$\frac{\partial}{\partial \mathbf{n}}(h * \bar{g}) = -\frac{\partial h}{\partial \mathbf{n}}$$

by Lemma 1.2, and

$$\frac{\partial}{\partial \mathbf{n}}(h * \bar{g}) = \frac{\partial h}{\partial \mathbf{n}}$$

by the assumption $h * \bar{g} = h$.

2. THE DIRICHLET PROBLEM

We will find the solution of the Dirichlet problem on some doubly connected regions in the form of a combination of a double layer potential and the same potential reflected by the reflection function – that is, we shall use the method of J. M. Sloss [16]. However, if the boundary curves are not smooth the method of proving that the corresponding integral equations possess solutions are different. To deal with the double layer potential on non-smooth domain we use the method described in [7] (see also [8]). Nevertheless, it will be more convenient for our purposes to keep the notation and to use some assertions from [11], [12] (see also [9], [10]) which concern the case of plane regions bounded by curves. For the convenience of the reader we recall some notions and assertions we shall use.

By a path (or curve) in the plane we mean a continuous mapping $\psi : \langle a, b \rangle \to R^2$ of a compact interval $\langle a, b \rangle$ into R^2. A simple closed path (Jordan curve) is a path $\psi : \langle a, b \rangle \to R^2$ such that $\psi(a) = \psi(b)$ and $\psi(t_1) \neq \psi(t_2)$ for any $t_1, t_2 \in \langle a, b \rangle$, $|t_1 - t_2| < b - a$. The variation of the vector function ψ on $I \subset \langle a, b \rangle$ is denoted by var $[\psi; I]$ (the variation of a scalar function is denoted analogously). The path ψ has a finite length if var $[\psi; \langle a, b \rangle] < \infty$ (for the definition of the variation of a vector function, of the curvilinear integral and so on see for instance [11]).

Throughout the paper $\langle a, b \rangle$ will be a fixed compact interval, $\psi : \langle a, b \rangle \to R^2$ a simple closed path of finite length. Putting

$$K = \psi(\langle a, b \rangle) \tag{2.1}$$

we also speak about the curve K. The symbol $\mathscr{C}(K)$ stands for the space of all continuous (real) functions on K endowed with the supremum-norm; $\mathscr{C}(K)$ is then a Banach space.

For $z \in R^2$ let ϑ_z be a single-valued continuous branch of arg $[\psi - z]$ on $\langle a, b \rangle - \psi^{-1}(z)$ (for the existence of a single-valued continuous branch of the argument see for example [11]). For $0 < r \leqq +\infty$ we denote by $\gamma_{z,r}$ the family of all components of the set

$$\{t \in \langle a, b \rangle;\ 0 < |\psi(t) - z| < r\}\,.$$

For $\alpha \in R^1$ the number (finite or $+\infty$) of points in

$$\{t \in \langle a, b \rangle;\ \psi(t) - z = |\psi(t) - z|\, e^{i\alpha},\ 0 < |\psi(t) - z| < r\}$$

is denoted by $n_r^\psi(\alpha, z)$. The following assertion is valid (see [11], [9]).

2.1. Proposition. *For any $z \in R^2$, $r > 0$ the function $n_r^\psi(\alpha, z)$ of the variable $\alpha \in R^1$ is Lebesgue measurable. If we define*

$$v_r^\psi(z) = \int_0^{2\pi} n_r^\psi(\alpha, z)\, d\alpha \tag{2.2}$$

then

$$v_r^\psi(z) = \sum_{I \in \gamma_{z,r}} \operatorname{var}\,[\vartheta_z; I]\,. \tag{2.3}$$

Further, we shall shortly denote

$$n^\psi(\alpha, z) = n_\infty^\psi(\alpha, z)\,, \quad \gamma_z = \gamma_{z,\infty}\,, \quad v^\psi(z) = v_\infty^\psi(z)\,.$$

The symbol $v^\psi(z)$ is called the cyclic variation of the curve K (of the path ψ) at the point z.

In what follows we shal deal with the (logaritmic) double layer potential. The double layer potential can be defined in different ways. From possible definitions we choose the following (see [11], [12] or [9]). Let $z \in R^2$ be such that $v^\psi(z) < \infty$. Then for $f \in \mathscr{C}(K)$ the value of the double layer potential $W(z, f) = W_\psi(z, f) = W_K(z, f)$ is defined by the equality

$$W(z, f) = \frac{1}{\pi} \sum_{I \in \gamma_z} \int_I f(\psi(t))\, d\vartheta_z(t)\,. \tag{2.4}$$

Note that our definition of $W(z, f)$ differs from that given in [11], [12] by the multiplicative constant $1/\pi$.

Under the assumption $\operatorname{var}\,[\psi; \langle a, b \rangle] < \infty$ the following assertion holds (see [11], [9]). If $z = x + iy \in R^2 - K$, then $v^\psi(z) < \infty$ and for any $f \in \mathscr{C}(K)$

$$W(z, f) = \operatorname{Im} \frac{1}{\pi} \int_\psi \frac{f(\zeta)}{\zeta - z}\, d\zeta = \tag{2.5}$$

$$= \frac{1}{\pi} \int_\psi - \frac{f(\xi + i\eta)(\eta - y)}{|\xi + i\eta - z|^2}\, d\xi + \frac{f(\xi + i\eta)(\xi - x)}{|\xi + i\eta - z|^2}\, d\eta\,.$$

Thus $W(\cdot, f)$ as a function of two real variables $[x, y]$ $(z = x + iy)$ is harmonic on $R^2 - K$ and

$$\lim_{|z| \to +\infty} W(z, f) = 0\,. \tag{2.6}$$

Given $\zeta \in K$ choose $t_1 \in \langle a, b)$, $t_2 \in (a, b \rangle$ such that $\zeta = \psi(t_1) = \psi(t_2)$. If $v^\psi(\zeta) < \infty$ then there exist limits ([11])

$$(2.7) \qquad \tau_+(\zeta) = \tau_+^K(\zeta) = \lim_{t\to t_1+} \frac{\psi(t)-\zeta}{|\psi(t)-\zeta|},$$

$$(2.8) \qquad \tau_-(\zeta) = \tau_-^K(\zeta) = \lim_{t\to t_2-} \frac{\psi(t)-\zeta}{|\psi(t)-\zeta|}.$$

Further, let G denote the interior and E the exterior of the path ψ (i.e. $G = \operatorname{Int} K$, $E = \operatorname{Ext} K$). Let ι stand for the constant value of the index of a point with respect to ψ on G (that is $\iota = 1$ if ψ is positively (counterclockwise) oriented and $\iota = -1$ if ψ negatively (clockwise) oriented).

To a point $\zeta \in K$ with $v^\psi(\zeta) < \infty$ we assign two numbers $\alpha_+(\zeta), \alpha_-(\zeta)$ such that

$$(2.9) \qquad \tau_+(\zeta) = e^{i\alpha_+(\zeta)}, \quad \tau_-(\zeta) = e^{i\alpha_-(\zeta)}$$

and such that

(a) $\alpha_+(\zeta) < \alpha_-(\zeta) < \alpha_+(\zeta) + 2\pi$
if $\tau_+(\zeta) \neq \tau_-(\zeta)$,

(b) $\alpha_-(\zeta) = \alpha_+(\zeta) + (1 - \iota)\pi$
if $\tau_+(\zeta) = \tau_-(\zeta)$ and the vector $e^{i(\alpha_+(\zeta)+\pi)}$ is directed at ζ to E,

(c) $\alpha_-(\zeta) = \alpha_+(\zeta) + (1 + \iota)\pi$
if $\tau_+(\zeta) = \tau_-(\zeta)$ and the vector $e^{i(\alpha_+(\zeta)+\pi)}$ is directed at ζ to G.

(We say that a vector $v \in R^2 - \{0\}$ is directed at $z \in R^2$ to $A \subset R^2$ if there is an $r > 0$ such that $\{z + \varrho v;\ 0 < \varrho < r\} \subset A$.) Then we put

$$(2.10) \qquad \Delta(\zeta) = \pi - (\alpha_-(\zeta) - \alpha_+(\zeta)).$$

From now on we shall suppose that

$$(2.11) \qquad V_K = \sup_{z\in K} v^\psi(z) < \infty.$$

Then $\Delta(\zeta)$ is defined for each $\zeta \in K$. For $f \in \mathscr{C}(K)$, $\zeta \in K$ we define $\overline{W} f(\zeta)$ by

$$(2.12) \qquad \overline{W} f(\zeta) = W(\zeta, f) + \frac{1}{\pi} \Delta(\zeta) f(\zeta).$$

Under the assumption (2.11), for each $f \in \mathscr{C}(K)$, $\zeta \in K$ there exist finite limits (see [11], [12], [9], [10])

$$(2.13) \qquad W^i(\zeta, f) = \lim_{\substack{z\to\zeta \\ z\in G}} W(z, f) = W(\zeta, f) + \iota f(\zeta)\left(1 + \frac{1}{\pi}\iota\Delta(\zeta)\right),$$

$$(2.14) \qquad W^e(\zeta, f) = \lim_{\substack{z\to\zeta \\ z\in E}} W(z, f) = W(\zeta, f) - \iota f(\zeta)\left(1 - \frac{1}{\pi}\iota\Delta(\zeta)\right).$$

Note that in the case ψ is smooth we have $\tau_-(\zeta) = -\tau_+(\zeta)$ for each $\zeta \in K$, that is

$\Delta(\zeta) = 0$ and the equalities (2.13), (2.14) can be written in the form ((2.11) is supposed all the time)

$$W^i(\zeta, f) = W(\zeta, f) + \iota f(\zeta), \quad W^e(\zeta, f) = W(\zeta, f) - \iota f(\zeta)$$

which are the classical jump formulas for the double layer potential.

Keeping the given notation we have (under the assumption (2.11) only) for every $\zeta \in K$

$$W^i(\zeta, f) = \overline{W} f(\zeta) + \iota f(\zeta), \tag{2.15}$$

$$W^e(\zeta, f) = \overline{W} f(\zeta) - \iota f(\zeta), \tag{2.16}$$

$$\overline{W} f(\zeta) = W^i(\zeta, f) - \iota f(\zeta) = W^e(\zeta, f) + \iota f(\zeta). \tag{2.17}$$

It is seen from (2.17) that $\overline{W} f \in \mathscr{C}(K)$ for each $f \in \mathscr{C}(K)$ and $\overline{W}$ can be thus considered as a (linear) operator acting on $\mathscr{C}(K)$ ($\overline{W} : f \to \overline{W} f$, $\overline{W} : \mathscr{C}(K) \to \mathscr{C}(K)$).

Further, let L be a fixed analytic Jordan curve with the parametrization $z = \Phi(t)$, $\Phi = \Phi_1 + i\Phi_2$ of the form (1.1), (1.2). We always suppose that either L is a circle of L satisfies all the assumptions of Part I, especially the conditions (1.3) to (1.5) (the conditions (1.4), (1.5) are not fulfilled in the case of the circle). We denote

$$R = \operatorname{Int} L, \quad P = \operatorname{Ext} L.$$

Let $e_1, \ldots, e_r$ be all critical points with respect to L lying in R; if L is a circle then we consider its centre as the only critical point.

We shall also always suppose that the curves L, K are related as follows: Suppose that

$$K \subset R = \operatorname{Int} L \tag{2.18}$$

and that

$$\{e_1, \ldots, e_r\} \subset G = \operatorname{Int} K. \tag{2.19}$$

Let L_i be a Jordan arc joining all the points $e_1, \ldots, e_r$; according to (2.19) we can suppose that $L_i \subset G$ (if L is a circle then L_i is a singleton, of course). Let $\bar{g}$ be the reflection function determined by L (see Theorem 1.1; if L is the circle $|z - z_0| = r$ then $\bar{g}(z) = z_0 + r^2/\overline{(z - z_0)}$). Further denote

$$S^+ = R \cap E = \operatorname{Int} L \cap \operatorname{Ext} K, \quad S^- = \bar{g}(S^+). \tag{2.20}$$

As $L_i \cap (S^+ \cup K) = \emptyset$ we see that the reflection function $\bar{g}$ is defined on a neighbourhood of the set

$$K \cup S^+ \cup L \cup S^- \cup \bar{g}(K)$$

and g is holomorphic there.

Our aim is to solve the Dirichlet problem on S^+. In order to achieve it we first investigate an operator $\overline{H}$ defined as follows.

For $f \in \mathscr{C}(K)$ we define a function Hf on S^+ by

$$H f(z) = W(z, f) - W(\bar{g}(z), f) \quad (z \in S^+). \tag{2.21}$$

It is seen that for any $f \in \mathscr{C}(K)$ the function Hf is harmonic on S^+. If $\zeta \in L$ then $\bar{g}(\zeta) = \zeta$ and as $W(\cdot, f)$ is continuous on $R^2 - K$ we have for each $\zeta \in L$ (and for each $f \in \mathscr{C}(K)$)

(2.22) $$\lim_{\substack{z \to \zeta \\ z \in S^+}} Hf(z) = W(\zeta, f) - W(\zeta, f) = 0 .$$

It follows from (2.14), (2.16) that for $\zeta \in K$

(2.23) $$H^e f(\zeta) = \lim_{\substack{z \to \zeta \\ z \in S^+}} Hf(z) = W^e(\zeta, f) - W(\bar{g}(\zeta), f) = $$
$$= \overline{W} f(\zeta) - \iota f(\zeta) - W(\bar{g}(\zeta), f) .$$

Given $u \in \mathscr{C}(K)$, we look for the solution (in $\mathscr{C}(K)$) of the equation

(2.24) $$H^e f(\zeta) = u(\zeta) , \quad \zeta \in K .$$

Note that if $f \in \mathscr{C}(K)$ is a solution of (2.24) then according to (2.22), (2.23) the harmonic function Hf defined on S^+ is the solution of the Dirichlet problem on S^+ with the boundary condition u on K and with the zero boundary condition on L.

For $f \in \mathscr{C}(K)$, $\zeta \in K$ denote

(2.25) $$\hat{W} f(\zeta) = W(\bar{g}(\zeta), f) ;$$

then the equation (2.24) can be written in the form

(2.26) $$(I - \iota(\overline{W} - \hat{W})) f = -\iota u ,$$

where I stands for the identity operator on $\mathscr{C}(K)$. Finally, for $f \in \mathscr{C}(K)$, $\zeta \in K$ we denote

(2.27) $$\bar{H} f(\zeta) = (\overline{W} - \hat{W}) f(\zeta) = \overline{W} f(\zeta) - W(\bar{g}(\zeta), f) .$$

Since $\overline{W}: \mathscr{C}(K) \to \mathscr{C}(K)$, $\hat{W}: \mathscr{C}(K) \to \mathscr{C}(K)$ are linear operators on $\mathscr{C}(K)$ then also $\bar{H}$ is a linear operator acting on $\mathscr{C}(K)$. Now the equation (2.24) (that is, the equation (2.26)) can be written in the form

(2.28) $$(I - \iota \bar{H}) f = -\iota u .$$

Investigating the equation (2.28) we shall first determine the Fredholm radius of the operator $\bar{H}$ and then study the question of unicity of the solution of the homogeneous equation

(2.29) $$(I - \iota \bar{H}) f = 0 .$$

In this connection we shall make use of some results from [12].

However, let us show now the following auxiliary assertion.

2.2. Lemma. *Suppose that* var $[\psi; \langle a, b \rangle] < \infty$. *Then the operator* $\hat{W}: \mathscr{C}(K) \to \mathscr{C}(K)$ $(\hat{W}: f \to \hat{W} f)$ *is compact.*

Proof. Let $B = \{f \in \mathscr{C}(K);\ \|f\| \leqq 1\}$. It suffices to show that $\hat{W}(B)$ is a set of equicontinuous and uniformly bounded functions on K.

By [11], Lemma 9.2,

$$\operatorname{var}[\vartheta_z; \langle a, b\rangle] \leqq (\operatorname{dist}(z, K))^{-1} \operatorname{var}[\psi; \langle a, b\rangle]$$

for $z \notin K$, where for $A \subset R^2$ the term $\operatorname{dist}(z, A)$ stands for the distance of z and A, that is

$$\operatorname{dist}(z, A) = \inf\{|z - z_1|;\ z_1 \in A\}\ ;$$

similarly we put for $A, C \subset R^2$

$$\operatorname{dist}(A, C) = \inf\{|z_1 - z_2|;\ z_1 \in A,\ z_2 \in C\}\ .$$

Since $\bar{g}(\zeta) \in \operatorname{Ext} L$ for $\zeta \in K$, we have for such ζ

$$\operatorname{dist}(\bar{g}(\zeta), K) \geqq \operatorname{dist}(K, L) = c > 0\ .$$

Now it is seen that for $\zeta \in K, f \in \mathscr{C}(K)$

$$|\hat{W}f(\zeta)| = |W(\bar{g}(\zeta), f)| = \left|\frac{1}{\pi}\int_{\langle a,b\rangle} f(\psi(t))\, \mathrm{d}\vartheta_{\bar{g}(\zeta)}(t)\right| \leqq$$

$$\leqq \frac{1}{\pi}\|f\| \operatorname{var}[\vartheta_{\bar{g}(\zeta)}; \langle a, b\rangle] \leqq \frac{1}{\pi}\|f\|\, c^{-1} \operatorname{var}[\psi; \langle a, b\rangle]\ ,$$

which implies that $W(B)$ is a set of uniformly bounded functions.

By [11], Lemma 9.2

$$\operatorname{var}[\vartheta_{z_1} - \vartheta_{z_2}; \langle a, b\rangle] \leqq \tag{2.30}$$

$$\leqq |z_1 - z_2|\,(\operatorname{dist}(z_1, K))^{-1}\,(\operatorname{dist}(z_2, K))^{-1} \operatorname{var}[\psi; \langle a, b\rangle]$$

for any $z_1, z_2 \in R^2 - K$. If we put

$$\varrho_0 = \operatorname{dist}(K, \bar{g}(K))$$

then surely $\varrho_0 > 0$ and it follows from (2.30) that for $\zeta_1, \zeta_2 \in K$

$$\operatorname{var}[\vartheta_{\bar{g}(\zeta_1)} - \vartheta_{\bar{g}(\zeta_2)}; \langle a, b\rangle] \leqq |\bar{g}(\zeta_1) - \bar{g}(\zeta_2)|\, \varrho_0^{-2} \operatorname{var}[\psi; \langle a, b\rangle]\ . \tag{2.31}$$

From (2.31) ti follows that for $f \in B$, $\zeta_1, \zeta_2 \in K$

$$|\hat{W}f(\zeta_1) - \hat{W}f(\zeta_2)| = |W(\bar{g}(\zeta_1), f) - W(\bar{g}(\zeta_2), f)| =$$

$$= \left|\frac{1}{\pi}\int_{\langle a,b\rangle} f(\psi(t))\, \mathrm{d}\vartheta_{\bar{g}(\zeta_1)}(t) - \frac{1}{\pi}\int_{\langle a,b\rangle} f(\psi(t))\, \mathrm{d}\vartheta_{\bar{g}(\zeta_2)}(t)\right| =$$

$$= \frac{1}{\pi}\left|\int_{\langle a,b\rangle} f(\psi(t))\, \mathrm{d}(\vartheta_{\bar{g}(\zeta_1)}(t) - \vartheta_{\bar{g}(\zeta_2)}(t))\right| \leqq$$

$$\leqq \frac{1}{\pi}\|f\| \operatorname{var}[\vartheta_{\bar{g}(\zeta_1)} - \vartheta_{\bar{g}(\zeta_2)}; \langle a, b\rangle] \leqq$$

$$\leqq \frac{1}{\pi}|\bar{g}(\zeta_1) - \bar{g}(\zeta_2)|\, \varrho_0^{-2} \operatorname{var}[\psi; \langle a, b\rangle]\ .$$

Since $\bar{g}$ is continuous on K it is seen now that $W(B)$ is a set of equicontinuous functions.

2.3. Notation. Let $\mathscr{K}$ stand for the set of all compact (linear) operators acting on $\mathscr{C}(K)$. Given a linear continuous operator $A: \mathscr{C}(K) \to \mathscr{C}(K)$, denote

$$\omega A = \inf_{D \in \mathscr{K}} \|A - D\| .$$

Note that ωA is the reciprocal value of the Fredholm radius of A.

2.4. Lemma.

$$\omega \bar{H} = \frac{1}{\pi} \lim_{r \to 0+} \sup_{\zeta \in K} v_r^{\psi}(\zeta) = \frac{1}{\pi} \lim_{r \to 0+} \sup_{\zeta \in K} (v_r^{\psi}(\zeta) + |\Delta(\zeta)|) .$$

Proof. As $\bar{H} = \bar{W} - \hat{W}$ and $\hat{W}$ is compact, we have

$$\omega \bar{H} = \omega \bar{W} .$$

Now it suffices to note that by [12], Theorem 12.47 (see also [10])

$$\omega \bar{W} = \frac{1}{\pi} \lim_{r \to 0+} \sup_{\zeta \in K} v_r^{\psi}(\zeta) = \frac{1}{\pi} \lim_{r \to 0+} \sup_{\zeta \in K} (v_r^{\psi}(\zeta) + |\Delta(\zeta)|) .$$

2.5. Lemma. *Suppose that the condition* (2.11) *is fulfilled and let $f \in \mathscr{C}(K)$ be such that $(I - \iota \bar{H}) f$ is constant on K. Then $(I - \iota \bar{H}) f = 0$ on K and $W(z, f) = 0$ for every $z \in E$* $(= \operatorname{Ext} K)$.

Proof. For $\zeta \in K$ we have (see (2.23))

$$\begin{aligned} -\iota(I - \iota \bar{H}) f(\zeta) &= H^e f(\zeta) = W^e(\zeta, f) - W(\bar{g}(\zeta), f) = \\ &= \lim_{\substack{z \to \zeta \\ z \in E}} W(z, f) - W(\bar{g}(\zeta), f) . \end{aligned}$$

The function $W(\cdot, f)$ is harmonic on E,

$$\lim_{|z| \to +\infty} W(z, f) = 0$$

and $W(\cdot, f)$ has a continuous extension from E to $\bar{E}$ (the values of this extension on K are of course equal, to the values of $W^e(\cdot, f)$).

First, let us observe that $H^e f = c$ cannot hold on K with c constant, $c \neq 0$. Suppose that $H^e f = c$, $c \neq 0$. One can take $c = 1$ (it suffices to multiply f by a suitable constant). Then for $\zeta \in K$ we have

$$W^e(\zeta, f) - W(\bar{g}(\zeta), f) = 1 .$$

Since $W^e(\cdot, f)$ is continuous on K there is a $\zeta_0 \in K$ such that

$$W^e(\zeta_0, f) = \inf_{\zeta \in K} W^e(\zeta, f) .$$

By the maximum principle for harmonic functions in R^2 (see for instance [12], Lemma 14.3)

$$W(z,f) \geqq W^e(\zeta_0,f)$$

for any $z \in E$. But

$$W(\bar{g}(\zeta_0),f) = W^e(\zeta_0,f) - 1 < W^e(\zeta_0,f)$$

by the above, which is a contradiction as $\bar{g}(\zeta_0) \in E$.

Now let $f \in \mathscr{C}(K)$ be such that $(I - \iota\bar{H})f = 0$. There is a $\zeta_1 \in K$ such that

$$|W^e(\zeta_1,f)| = \sup_{\zeta \in K} |W^e(\zeta,f)| = \sup_{z \in E} |W(z,f)|$$

(the second equality follows from the maximum principle). By the assumption

$$0 = H^e f(\zeta_1) = W^e(\zeta_1,f) - W(\bar{g}(\zeta_1),f)\,,$$

that is $|W(\bar{g}(\zeta_1),f)| = |W^e(\zeta_1,f)|$. But this means that $W(\cdot,f)$ attains its extremal value on E at a point from E (for $\bar{g}(\zeta_1) \in E$) and by the sharp maximum principle $W(\cdot,f)$ is constant on E. Since $\lim_{|z| \to +\infty} W(z,f) = 0$, we have $W(z,f) = 0$ for every $z \in E$.

The following assertion is known (cf. [10] or [12], Lemma 12.55, for example).

2.6. Lemma. *Suppose that the condition*

$$(2.32) \qquad (\omega\bar{W} =) \frac{1}{\pi} \lim_{r \to 0+} \sup_{\zeta \in K} v_r^{\psi}(\zeta) < 1$$

is fulfilled and let $f \in \mathscr{C}(K)$ be such that $(I - \iota\bar{W})f$ is constant on K. Then f is constant on K.

2.7. Corollary. *Suppose that the condition*

$$(2.33) \qquad (\omega\bar{H} =) \frac{1}{\pi} \lim_{r \to 0+} \sup_{\zeta \in K} v_r^{\psi}(\zeta) < 1$$

is fulfilled. Then the space of all $f \in \mathscr{C}(K)$ which are solutions of the homogeneous equation $(I - \iota\bar{H})f = 0$ is just the space of all constant functions on K.

Proof. Let $f \in \mathscr{C}(K)$ be such that $(I - \iota\bar{H})f = 0$. Then, by Lemma 2.5, $W(z,f) = 0$ for every $z \in E$ which implies (cf. (2.16)) that

$$-(\iota I - \bar{W})f(\zeta) = W^e(\zeta,f) = \lim_{\substack{z \to \zeta \\ z \in E}} W(z,f) = 0$$

for each $\zeta \in K$. It follows immediately from Lemma 2.6 that f is constant on K.

Suppose now that f is constant on K, $f = c$. Then for $z \in E$

$$W(z,f) = \frac{1}{\pi} \int_{\langle a,b \rangle} f(\psi(t))\, \mathrm{d}\vartheta_z(t) = \frac{c}{\pi} \int_{\langle a,b \rangle} d\vartheta_z(t) = 0$$

since $\vartheta_z(t)$ is a single-valued continuous branch of $\arg[\psi(t) - z]$ on $\langle a, b\rangle$, ψ is closed (simple) and $z \in \operatorname{Ext} K$. It follows that for $\zeta \in K$ (note that $\bar{g}(\zeta) \in E$ for $\zeta \in K$)

$$-\iota(I - \iota\bar{H}) f(\zeta) = W^e(\zeta, f) - W(\bar{g}(\zeta), f) = 0 .$$

2.8. Some other notation. Let $\mathscr{C}'(K)$ stand for the space of all (real) signed Borel measures on K, that is, measures on R^2 with supports contained in K (measures from $\mathscr{C}'(K)$ are also called charges). For $\mu \in \mathscr{C}'(K)$ we denote by $\mu^+, \mu^-, |\mu|$ respectively the positive, the negative and the total variation of μ. We have then $\mu = \mu^+ - \mu^-$, $|\mu| = \mu^+ + \mu^-$, and $\mu^+, \mu^-, |\mu|$ are non-negative measures. The norm on $\mathscr{C}'(K)$ is defined by

$$\|\mu\| = |\mu|(K) .$$

$\mathscr{C}'(K)$ is then a Banach space — the dual space to $\mathscr{C}(K)$. Given $z \in R^2$, a function h_z is defined on R^2 by

$$h_z(\zeta) = \frac{1}{\pi} \ln \frac{1}{|\zeta - z|} , \quad \zeta \in R^2 - z , \tag{2.34}$$

and $h_z(z) = +\infty$.

For $\mu \in \mathscr{C}'(K)$ (generally for a signed measure μ with a compact support in R^2) the logaritmic potential U_μ is defined by

$$U_\mu(z) = \int_K h_z(\zeta) \, \mathrm{d}\mu(\zeta) \tag{2.35}$$

for all such $z \in R^2$ for which the integral on the right-hand side exists. Note that in any case U_μ is defined on $R^2 - K$ (if $\mu \in \mathscr{C}'(K)$; generally on $R^2 - \operatorname{spt} \mu$, where $\operatorname{spt} \mu$ denotes the support of μ) and is harmonic there.

For $r = 1, 2$ we denote by $\mathscr{H}_r$ the r-dimensional Hausdorff measure on R^2. $\mathscr{H}_r$ is supposed to be normalized in such a way that $\mathscr{H}_2$ coincides on R^2 with the outer 2-dimensional Lebesgue measure, while $\mathscr{H}_1$ coincides on lines R^2 with the outer linear (1-dimensional) Lebesgue measure on those lines.

The space of all infinitely differentiable (real) functions with compact supports in R^2 will be denoted by $\mathscr{D}$. The support of $\varphi \in \mathscr{D}$ is denoted by $\operatorname{spt} \varphi$ and $\varphi|_M$ stands for the restriction of φ to $M \subset R^2$.

Let h be a harmonic function on G (recall that $G = \operatorname{Int} K$) such that

$$\int_G |\operatorname{grad} h| \, \mathrm{d}\mathscr{H}_2 < \infty .$$

Then we define a functional (a distribution) $N_G h$ on $\mathscr{D}$ by (see [12], def. 13.12; cf. also [7])

$$\langle \varphi, N_G h \rangle = \int_G \operatorname{grad} \varphi \operatorname{grad} h \, \mathrm{d}\mathscr{H}_2 , \quad \varphi \in \mathscr{D} .$$

$N_G h$ may be considered as a generalized normal derivative of h on $\partial G = K$ (with respect to G). If $\mu \in \mathscr{C}'(K)$ then

$$\int_G |\text{grad } U_\mu| \, \mathrm{d}\mathscr{H}_2 < \infty$$

and thus $N_G U_\mu$ is always defined. The following assertion is valid (see [12], Theorem 13.33; cf. also [7]).

2.9. Proposition. *The distribution $N_G U_\mu$ can be represented by a charge $\nu_\mu \in \mathscr{C}'(K)$ for each $\mu \in \mathscr{C}'(K)$ (in the sense that*

$$\langle \varphi, N_G U_\mu \rangle = \int_K \varphi \, \mathrm{d}\nu_\mu$$

for each $\varphi \in \mathscr{D}$) if and only if

$$\sup_{\zeta \in K} v^\psi(\zeta) < \infty \,. \tag{2.36}$$

If the condition (2.36) is fulfilled then for each $\mu \in \mathscr{C}'(K)$ the charge $\nu_\mu \in \mathscr{C}'(K)$ is uniquely determined and

$$\|\nu_\mu\| \leqq \left(2 + \frac{1}{\pi} \sup_{\zeta \in K} v^\psi(\zeta)\right) \|\mu\| \,. \tag{2.37}$$

2.10. Remark. Suppose that the condition (2.36) is fulfilled. Then we can and will identify the functional $N_G U_\mu$ with the charge ν_μ. $N_G U$ can be regarded as an operator (linear operator) acting on $\mathscr{C}'(K)$:

$$N_G U: \mu \to N_G U_\mu \,, \quad N_G U: \mathscr{C}'(K) \to \mathscr{C}'(K) \,.$$

It follows from (2.37) that the operator $N_G U$ is bounded.

The following assertion (see [12], Theorem 13.36; cf. also [7], [10]) is important for our purposes.

2.10. Proposition. *Suppose that the condition (2.36) is fulfilled. Then the operators $N_G U$ and $(I - \iota\overline{W})$ are adjoint to each other.*

2.11. It will be useful in the following to know also the operator adjoint to $\hat{W}$. First let us briefly recall another expression of the double layer potential W and some related notions.

Let $z \in R^2 - K$ and let $f \in \mathscr{C}(K)$ be such that there is a $\varphi^f \in \mathscr{D}$ with $z \notin \operatorname{spt} \varphi^f$, $\varphi^f|_K = f$. The value of the double layer potential at z is then defined in [7] by

$$W^G f(z) = \int_G \text{grad } \varphi^f \text{ grad } h_z \, \mathrm{d}\mathscr{H}_2 \,.$$

It is also shown in [7] how to define the double layer potential for a general $f \in \mathscr{C}(K)$

provided (2.36) is fulfilled. Note that the definition of the cyclic variation in [7] is slightly different from the definition given here (it uses the notion of the so-called hits of a set) but the values of the cyclic variation in both definitions are equal to each other (up to a multiplicative constant) – see [4], Theorem 3.3.

If $M \subset R^2$ is a Borel set then put

$$P(M) = \sup_w \int_M \operatorname{div} w \, \mathrm{d}\mathscr{H}_2 ,$$

where $w = [w_1, w_2]$ ranges over all vector-valued functions with components $w_1, w_2 \in \mathscr{D}$ such that $\|w\|^2 = w_1^2 + w_2^2 \leqq 1$. $P(M)$ is called the perimeter of M. In the case $M = G$ we have

$$P(G) = \operatorname{var}[\varphi; \langle a, b\rangle] = \mathscr{H}_1(K) .$$

For the basic properties of sets with finite perimeter see [7]; more information can be found in [2], [3], [5], [6].

The term $\boldsymbol{n}^K(z)$ is used in the sequel to denote the exterior normal in Federer's sense of G at $z \in R^2$ (for the definition of the normal in Federer's sense see, for example, also [7]). The following assertion is valid (divergence theorem):

Suppose that $P(G) < \infty$. If $w = [w_1, w_2]$, where w_1, w_2 are continuously differentiable function on some neighbourhood of $\bar{G}$, then

$$\int_K w(\zeta)\, \boldsymbol{n}^K(\zeta) \, \mathrm{d}\mathscr{H}_1(\zeta) = \int_G \operatorname{div} w(z) \, \mathrm{d}\mathscr{H}_2(z) .$$

We shall use this assertion in the following situation. Let $\varphi \in \mathscr{D}$ and let u be a function harmonic on some neighbourhood of $\bar{G} \cap \operatorname{spt} \varphi$. Then clearly

$$\operatorname{div}(\varphi \operatorname{grad} u) = \operatorname{grad} \varphi \operatorname{grad} u$$

(on R^2) and the divergence theorem yields (if $P(G) < \infty$)

$$\text{(2.38)} \qquad \int_G \operatorname{grad} \varphi \operatorname{grad} u \, \mathrm{d}\mathscr{H}_2 = \int_K \varphi(\zeta)\, \boldsymbol{n}^K(\zeta) \operatorname{grad} u(\zeta) \, \mathrm{d}\mathscr{H}_1(\zeta) .$$

If $z \in R^2$, $\varphi \in \mathscr{D}$, $z \notin \operatorname{spt} \varphi$ then in particular

$$\int_G \operatorname{grad} \varphi \operatorname{grad} h_z \, \mathrm{d}\mathscr{H}_2 = \int_K \varphi(\zeta)\, \boldsymbol{n}^K(\zeta) \operatorname{grad}_\zeta h_z(\zeta) \, \mathrm{d}\mathscr{H}_1(\zeta)$$

($\operatorname{grad}_\zeta$ indicates that the gradient is taken with respect to the variable ζ).

It is shown in [4], Part 3 that the the values of the double layer potential in both definitions (the definition from [7] and the definition (2.4)) coincide up to the sign which depends on the orientation of ψ (and up to a multiplicative constant). Precisely, we obtain that for $z \in R^2 - K, f \in \mathscr{C}(K)$ such that $f = \varphi^f|_K$, where $\varphi^f \in \mathscr{D}$, $z \notin \operatorname{spt} \varphi^f$, the following identity holds:

$$(2.39) \qquad W(z,f) = -\iota \int_G \operatorname{grad} \varphi^f \operatorname{grad} h_z \, d\mathscr{H}_2 =$$

$$= -\iota \int_K f(\zeta)\, n^K(\zeta) \operatorname{grad}_\zeta h_z(\zeta) \, d\mathscr{H}_1(\zeta) \, .$$

Further, we denote

$$\hat{K} = \bar{g}(K) \, .$$

It follows from the properties of $\bar{g}$ that $\hat{K}$ is a Jordan curve, $\hat{K} \subset \operatorname{Ext} L$. In what follows it suffices to know that $\hat{K}$ is a compact set, dist $(G, \hat{K}) > 0$.

Analogously to $\mathscr{C}'(K)$ we denote by $\mathscr{C}'(\hat{K})$ the space of all signed Borel measures on $\hat{K}$. The following relation between $\mathscr{C}'(K)$ and $\mathscr{C}'(\hat{K})$ can be defined. Let $\mu \in \mathscr{C}'(K)$. We assign to this measure a measure $\hat{\mu} \in \mathscr{C}'(\hat{K})$ such that

$$(2.40) \qquad \hat{\mu}(M) = \mu(\bar{g}(M))$$

for any Borel set $M \subset \hat{K}$. Conversely, if $\hat{\mu} \in \mathscr{C}'(\hat{K})$ then we define $\mu \in \mathscr{C}'(K)$ such that for each Borel $M \subset K$ we put

$$\mu(M) = \hat{\mu}(\bar{g}(M)) \, .$$

Since $\bar{g}$ is one-to-one on $\hat{K}$ and the inverse mapping of $\bar{g}|_K$ is equal to $\bar{g}|_{\hat{K}}$, it is easily seen that the given correspondence is an isometric isomorphism of the spaces $\mathscr{C}'(K)$, $C'(\hat{K})$. We shall write $\mu \approx \hat{\mu}$. For $f \in \mathscr{C}(K)$, $\mu \in \mathscr{C}'(K)$, $\hat{\mu} \in \mathscr{C}'(\hat{K})$, $\mu \approx \hat{\mu}$ we then have

$$(2.41) \qquad \int_K f(\zeta) \, d\mu(\zeta) = \int_{\hat{K}} f(\bar{g}(\zeta)) \, d\hat{\mu}(\zeta) \, ,$$

and similarly for $f \in \mathscr{C}(\hat{K})$

$$(2.42) \qquad \int_{\hat{K}} f(\zeta) \, d\hat{\mu}(\zeta) = \int_K f(\bar{g}(\zeta)) \, d\mu(\zeta) \, .$$

For $\hat{\mu} \in \mathscr{C}'(\hat{K})$ the (logarithmic) potential is, of course, defined by

$$(2.43) \qquad U_{\hat{\mu}}(z) = \int_{\hat{K}} h_z(\zeta) \, d\hat{\mu}(\zeta)$$

for those $z \in R^2$ for which the integral on the right-hand side exists. If $\mu \in \mathscr{C}'(K)$, $\mu \approx \hat{\mu}$ then it follows from (2.42) that

$$U_{\hat{\mu}}(z) = \int_K h_z(\bar{g}(\zeta)) \, d\mu(\zeta) \, .$$

As $h_z(\zeta) = h_\zeta(z)$ we can also write

$$(2.44) \qquad U_{\hat{\mu}}(z) = \int_K h_{\bar{g}(\zeta)}(z) \, d\mu(\zeta) \, .$$

Taking into consideration that dist $(G, \hat{K}) > 0$ we see that for each $\hat{\mu} \in \mathscr{C}'(\hat{K})$ the potential $U_{\hat{\mu}}$ has bounded partial derivatives on G and thus the generalized normal derivative $N_G U_{\hat\mu}$ is defined by

$$\langle \varphi, N_G U_{\hat\mu} \rangle = \int_G \operatorname{grad} \varphi \operatorname{grad} U_{\hat\mu} \, d\mathscr{H}_2 , \quad \varphi \in \mathscr{D} .$$

If $\operatorname{var} [\psi; \langle a, b \rangle] < \infty$ then (2.38) yields

$$\langle \varphi, N_G U_{\hat\mu} \rangle = \int_K \varphi(y) \, \boldsymbol{n}^K(y) \operatorname{grad} U_{\hat\mu}(y) \, d\mathscr{H}_1(y) , \quad (\varphi \in \mathscr{D}) .$$

Since $\operatorname{grad} U_{\hat\mu}$ is bounded on K (for $\operatorname{dist}(K, \hat K) > 0$) we see that for each $\hat\mu \in \mathscr{C}'(\hat K)$ the (linear) functional $N_G U_{\hat\mu}$ is bounded on $\mathscr{D}$, the support of $N_G U_{\hat\mu}$ is contained in K and $\langle f, N_G U_{\hat\mu} \rangle$ can be defined for any $f \in \mathscr{C}(K)$ by

$$\langle f, N_G U_{\hat\mu} \rangle = \int_K f(y) \, \boldsymbol{n}^K(y) \operatorname{grad} U_{\hat\mu}(y) \, d\mathscr{H}_1(y) . \tag{2.45}$$

Then, of course, $N_G U_{\hat\mu}$ can be considered as a charge on K (that is $N_G U_{\hat\mu} \in \mathscr{C}'(K)$) and we can define an operator $^\wedge N_G U$ on $\mathscr{C}'(K)$ by

$$^\wedge N_G U \colon \mu \to N_G U_{\hat\mu} , \tag{2.46}$$

where $\mu \in \mathscr{C}'(K)$, $\hat\mu \in \mathscr{C}'(\hat K)$, $\mu \approx \hat\mu$. We have $^\wedge N_G U \colon \mathscr{C}'(K) \to \mathscr{C}'(K)$ and $^\wedge N_G U$ is a bounded linear operator. Indeed,

$$\sup_{y \in K} |\operatorname{grad} U_{\hat\mu}(y)| \leqq \int_{\hat K} \sup_{y \in K} |\operatorname{grad}_y h_y(\zeta)| \, d|\hat\mu(\zeta)| \leqq \frac{1}{\pi} (\operatorname{dist}(K, \hat K))^{-1} \|\hat\mu\| ,$$

and thus (as $\|\mu\| = \|\hat\mu\|$)

$$\begin{aligned} \|^\wedge N_G U \mu\| &= \|N_G U_{\hat\mu}\| = \sup \{ \langle f, N_G U_{\hat\mu} \rangle ; \, f \in \mathscr{C}(K), \, \|f\| \leqq 1 \} = \\ &= \sup \left\{ \int_K f(y) \, \boldsymbol{n}^K(y) \operatorname{grad} U_{\hat\mu}(y) \, d\mathscr{H}_1(y) ; \, f \in \mathscr{C}(K), \, \|f\| \leqq 1 \right\} \leqq \\ &\leqq \frac{1}{\pi} (\operatorname{dist}(K, \hat K))^{-1} \|\mu\| \operatorname{var} [\psi; \langle a, b \rangle] . \end{aligned}$$

Now we are in position to prove the following assertion.

2.12. Lemma. *Suppose that* $\operatorname{var} [\psi; \langle a, b \rangle] < \infty$. *Then the operators* $\hat W$ *and* $-\iota^\wedge N_G U$ *are adjoint to each other.*

Proof. We should show that for each $f \in \mathscr{C}(K)$, $\mu \in \mathscr{C}'(K)$

$$\langle \mu, \hat W f \rangle = \langle f, -\iota^\wedge N_G U \mu \rangle . \tag{2.47}$$

For $f \in \mathscr{C}(K)$, $\zeta \in K$ we have $\hat W f(\zeta) = W(\bar g(\zeta), f)$ by definition. Now let $\varphi \in \mathscr{D}$, $\operatorname{spt} \varphi \cap \hat K = \emptyset$, $f = \varphi|_K$. Then $\bar g(\zeta) \notin \operatorname{spt} \varphi$ for $\zeta \in K$ and by (2.39),

$$\hat W f(\zeta) = -\iota \int_G \operatorname{grad} \varphi \operatorname{grad} h_\zeta \, d\mathscr{H}_2 .$$

If further $\mu \in \mathscr{C}'(K)$, $\hat{\mu} \in \mathscr{C}'(\hat{K})$, $\mu \approx \hat{\mu}$ then (see also (2.44))

$$\langle \mu, \hat{W}f \rangle = -\iota \int_K \int_G \operatorname{grad} \varphi(x) \operatorname{grad}_x h_\zeta(x) \, d\mathscr{H}_2(x) \, d\mu(\zeta) =$$

$$= -\iota \int_G \operatorname{grad} \varphi(x) \operatorname{grad} \int_K h_\zeta(x) \, d\mu(\zeta) \, d\mathscr{H}_2(x) =$$

$$= -\iota \int_G \operatorname{grad} \varphi \operatorname{grad} U_{\hat{\mu}} \, d\mathscr{H}_2 = -\iota \langle \varphi, N_G U_{\hat{\mu}} \rangle = -\iota \langle \varphi, {}^\wedge N_G U \mu \rangle .$$

We have thus proved that (2.47) is valid for any $\mu \in \mathscr{C}'(K)$ and every $f \in \mathscr{C}(K)$ of the form $f = \varphi|_K$, where $\varphi \in \mathscr{D}$, $\operatorname{spt} \varphi \cap \hat{K} = \emptyset$. Now let $\mu \in \mathscr{C}'(K)$ be fixed and $f \in \mathscr{C}(K)$ arbitrary. Then there are $\varphi_n \in \mathscr{D}$ $(n = 1, 2, \ldots)$ such that $\operatorname{spt} \varphi_n \cap \hat{K} = \emptyset$ and $f_n = \varphi_n|_K \to f$ uniformly on K. As $\operatorname{grad} U_{\hat{\mu}}$ is bounded on K we obtain from (2.45) that

$$\lim_{n \to +\infty} \langle f_n, N_G U_{\hat{\mu}} \rangle = \langle f, N_G U_{\hat{\mu}} \rangle . \tag{2.48}$$

For $\zeta \in K$ we further have

$$|\hat{W}f(\zeta) - \hat{W}f_n(\zeta)| = |W(\bar{g}(\zeta), f - f_n)| = \left| \frac{1}{\pi} \int_{\langle a,b \rangle} (f - f_n)(\psi(t)) \, d\vartheta_{\bar{g}(\zeta)}(t) \right| \leqq$$

$$\leqq \frac{1}{\pi} \|f - f_n\| \, v^\psi(\bar{g}(\zeta)) .$$

Since $\operatorname{var}[\psi; \langle a, b \rangle] < \infty$ (and $\operatorname{dist}(K, \hat{K}) > 0$) then surely

$$\sup_{z \in \hat{K}} v^\psi(z) < \infty$$

from which it is seen that

$$\lim_{n \to +\infty} \langle \mu, \hat{W}f_n \rangle = \langle \mu, \hat{W}f \rangle . \tag{2.49}$$

Consequently,

$$\langle \mu, \hat{W}f_n \rangle = \langle f_n, -\iota {}^\wedge N_G U \mu \rangle = -\iota \langle f_n, N_G U_{\hat{\mu}} \rangle$$

and now it follows immediately from (2.48), (2.49) that (2.47) holds for each $f \in \mathscr{C}(K)$ (and each $\mu \in \mathscr{C}'(K)$).

By (2.27) we have

$$(I - \iota\bar{H}) = (I - \iota(\bar{W} - \hat{W})) = (I - \iota\bar{W}) + \iota\hat{W} .$$

The following assertion follows immediately from Proposition 2.10 and Lemma 2.12 (note that (2.36) implies $\operatorname{var}[\psi; \langle a, b \rangle] < \infty$; see for example [7]).

2.13. Theorem. *Suppose that the condition (2.36) is fulfilled. Then the operators $(I - \iota\bar{H})$ and $(N_G U - {}^\wedge N_G U)$ are adjoint to each other.*

2.14. Remark. If $\varphi \in \mathscr{D}$, $\mu \in \mathscr{C}'(K)$, $\hat{\mu} \in \mathscr{C}'(\hat{K})$, $\mu \approx \hat{\mu}$ then

$$(2.50)\qquad \langle\varphi, (N_G U - {}^\wedge N_G U)\,\mu\rangle = \langle\varphi, N_G U_\mu\rangle - \langle\varphi, {}^\wedge N_G U\mu\rangle =$$
$$= \langle\varphi, N_G U_\mu\rangle - \langle\varphi, N_G U_{\hat\mu}\rangle = \langle\varphi, N_G(U_\mu - U_{\hat\mu})\rangle\,.$$

The functional $(N_G U - {}^\wedge N_G U)\,\mu$ can be thus considered as a generalized normal derivative (with respect to G) of the difference of the potentials U_μ and $U_{\hat\mu}$.

2.15. From now on we shall suppose that L with the parametrization Φ is positively oriented. In the same way as we have defined the double layer potential $W(\cdot, f)$ for the curve K and $f \in \mathscr{C}(K)$ we can define the double layer potential for the curve L and $f \in \mathscr{C}(L)$ which will be denoted by $W_L(\cdot, f)$. Recall once more that we have denoted $S^+ = \operatorname{Ext} K \cap \operatorname{Int} L$. Since L is "sufficiently smooth" we have for each $f \in \mathscr{C}(L)$, $\zeta \in L$

$$W_L^i f(\zeta) = \lim_{\substack{z \to \zeta \\ z \in S^+}} W_L(z, f) = W_L(\zeta, f) + f(\zeta)\,,$$

$$W_L^e f(\zeta) = \lim_{\substack{z \to \zeta \\ z \in \operatorname{Ext} L}} W_L(z, f) = W_L(\zeta, f) - f(\zeta)\,.$$

For $f \in \mathscr{C}(L)$, $z \in S^+$ denote

$$(2.51)\qquad H_L f(z) = W_L(z, f) - W(\bar g(z), f)\,.$$

Since $\bar g(z) \in \operatorname{Ext} L$ for $z \in S^+$ and $\bar g(\zeta) = \zeta$ for $\zeta \in L$ (and $\bar g$ is continuous) we have for $f \in \mathscr{C}(L)$, $\zeta \in L$

$$(2.52)\qquad \lim_{\substack{z \to \zeta \\ z \in S^+}} H_L f(z) = 2 f(\zeta)\,.$$

We are going to solve the Dirichlet problem on S^+. Let $u_K \in \mathscr{C}(K)$, $u_L \in \mathscr{C}(L)$. The problem is to find a function h harmonic on S^+ and such that

$$(2.53)\qquad \lim_{\substack{z \to \zeta \\ z \in S^+}} h(z) = u_K(\zeta)$$

for $\zeta \in K$ and

$$(2.54)\qquad \lim_{\substack{z \to \zeta \\ z \in S^+}} h(z) = u_L(\zeta)$$

for $\zeta \in L$. We shall look for the solution in the form

$$(2.55)\qquad h(z) = H f_K(z) + H_L f_L(z) + a\, h_{z_0}(z)\,,$$

where $f_K \in \mathscr{C}(K)$, $f_L \in \mathscr{C}(L)$, $a \in R^1$ is a constant, $z_0 \in G$ is fixed.

It follows from (2.22), (2.52) that

$$(2.56)\qquad \lim_{\substack{z \to \zeta \\ z \in S^+}} h(z) = 2 f_L(\zeta) + a\, h_{z_0}(\zeta)$$

for $\zeta \in L$. Thus (2.54) is fulfilled if and only if for $\zeta \in L$ we have

$$(2.57)\qquad f_L(\zeta) = \tfrac{1}{2}(u_L(\zeta) - a\, h_{z_0}(\zeta))\,.$$

In this way f_L can be found. However, we still do not know the constant a and the function f_K.

From now on we shall always suppose that

$$(2.58) \qquad (\omega\bar{H} =) \frac{1}{\pi} \lim_{r\to 0+} \sup_{\zeta\in K} v_r^{\psi}(\zeta) < 1 ;$$

recall that $\omega\bar{H}$ is the reciprocal value of the Fredholm radius of $\bar{H}$ (cf. Lemma 2.4) and that under the condition (2.58) all the solutions of the homogeneous equation $(I - \iota\bar{H}) f = 0$ are just all the constant functions on K (Corollary 2.7), in particular the null-space of $(I - \iota\bar{H})$ is of dimension one. If (2.58) is fulfilled then also the null-space of the adjoint operator to $(I - \iota\bar{H})$ (which is equal to $(N_G U - {}^{\wedge}N_G U)$) is of dimension one and Fredholm's alternative is valid (see for example [14], or also [12], appendix IV). Thus the following assertion holds.

2.16. Theorem. *Suppose that the condition* (2.58) *is fulfilled and let* $\mu_0 \in \mathscr{C}'(K)$ *be a non-trivial solution of the homogeneous equation*

$$(2.59) \qquad (N_G U - {}^{\wedge}N_G U)\, \mu = 0 .$$

If $u \in \mathscr{C}(K)$ *then the equation*

$$(2.60) \qquad (I - \iota\bar{H}) f = u$$

has a solution $f \in \mathscr{C}(K)$ *if and only if*

$$(2.61) \qquad \langle \mu_0, u \rangle = \int_K u \, \mathrm{d}\mu_0 = 0 .$$

If the condition (2.61) *is fulfilled then the solution of* (2.60) *is determined uniquely up to an additive constant.*

2.17. In what follows let μ_0 be a non-trivial solution of (2.59). It is seen from (2.23), (2.25) (with regard to the fact that the equations (2.24), (2.28) are equivalent to each other) that the function h of the form (2.55) satisfies (2.53) if and only if

$$u_K(\zeta) = -\iota(I - \iota\bar{H}) f_K(\zeta) + H_L f_L(\zeta) + a\, h_{z_0}(\zeta)$$

for $\zeta \in K$, that is, f_K is a solution of the equation

$$(2.62) \qquad (I - \iota\bar{H}) f_K = -\iota(u_K - H_L f_L|_K - a h_{z_0}|_K) .$$

However, in accordance with Theorem 2.16 the equation (2.62) admits a solution if and only if

$$\langle \mu_0, u_K - H_L f_L - a h_{z_0} \rangle = 0 .$$

Since f_L is of the form (2.57) the last condition can be written in the form

$$(2.63) \qquad a\langle \mu_0, \tfrac{1}{2} H_L h_{z_0} - h_{z_0} \rangle = \langle \mu_0, \tfrac{1}{2} H_L u_L - u_K \rangle$$

which determines the constant a provided

$$\langle \mu_0, \tfrac{1}{2} H_L h_{z_0} - h_{z_0} \rangle \neq 0 . \tag{2.64}$$

2.18. Lemma. *Let $\mu_0 \in \mathscr{C}'(K)$ be a non-trivial solution of the homogeneous equation* (2.59) *and let $z_0 \in G$. Then* (2.64) *is valid.*

Proof. Suppose that

$$\langle \mu_0, \tfrac{1}{2} H_L h_{z_0} - h_{z_0} \rangle = 0 .$$

Then the equation

$$(I - \iota \bar{H}) f = (\tfrac{1}{2} H_L h_{z_0} - h_{z_0})|_K$$

has a solution in $\mathscr{C}(K)$ owing to Theorem 2.16. Let f be a solution of that equation. Then Hf is harmonic on S^+ and for $\zeta \in L$ we have

$$\lim_{\substack{z \to \zeta \\ z \in S^+}} H f(z) = 0$$

and at the same time

$$\lim_{\substack{z \to \zeta \\ z \in S^+}} H f(z) = -\tfrac{1}{2} H_L \, h_{z_0}(\zeta) + h_{z_0}(\zeta)$$

for $\zeta \in K$. Since we have

$$\lim_{\substack{z \to \zeta \\ z \in S^+}} (\tfrac{1}{2} H_L \, h_{z_0}(z) - h_{z_0}(z)) = h_{z_0}(\zeta) - h_{z_0}(\zeta) = 0$$

for $\zeta \in L$ it is seen that the function

$$\tfrac{1}{2} H_L h_{z_0} - h_{z_0} + Hf$$

has zero limits on $\partial S^+ = K \cup L$ (with respect to S^+), that is, this function vanishes on S^+ (being harmonic there) and thus

$$h_{z_0}(z) = \tfrac{1}{2} H_L \, h_{z_0}(z) + H f(z) \tag{2.65}$$

for $z \in S^+$.

For $z \notin K$ denote

$$F_1(z) = \frac{1}{\pi} \int_{\psi} \frac{f(\zeta)}{\zeta - z} \, d\zeta .$$

Then by (2.5) (for $z \notin K$)

$$W(z, f) = \operatorname{Im} F_1(z) .$$

Further, we have

$$W(\bar{g}(z), f) = \operatorname{Im} F_1(\bar{g}(z)) = -\operatorname{Im} \bar{F}_1(\bar{g}(z))$$

and thus

$$H f(z) = W(z, f) - W(\bar{g}(z), f) = \operatorname{Im} (F_1(z) + \bar{F}_1(\bar{g}(z)))$$

for $z \in S^+$. For $z \notin L$ denote

$$F_2(z) = \frac{1}{\pi} \int_{\Phi} \frac{h_{z_0}(\zeta)}{\zeta - z} \, d\zeta .$$

Similarly we obtain for $z \in S^+$

$$H_L h_{z_0}(z) = \operatorname{Im} (F_2(z) + \bar{F}_2(\bar{g}(z))) .$$

Now it follows from (2.65) that for $z \in S^+$,

$$\frac{1}{\pi} \ln \frac{1}{|z - z_0|} = h_{z_0}(z) = \operatorname{Im} [F_1(z) + \bar{F}_1(\bar{g}(z)) + \tfrac{1}{2}(F_2(z) + \bar{F}_2(\bar{g}(z)))] .$$

F_1, F_2 are holomorphic on S^+ and also $\bar{F}_1(\bar{g}(z)), \bar{F}_2(\bar{g}(z))$ are holomorphic there, that is, h_{z_0} is on S^+ the imaginary part of a holomorphic function. But $z_0 \in G$, which is a contradiction.

Now the following assertion can be established.

2.19. Theorem. *Suppose that the condition (2.58) is fulfilled, let μ_0 be a non-trivial solution of (2.59) and let $z_0 \in G$. Then for each boundary conditions $u_K \in \mathscr{C}(K)$, $u_L \in \mathscr{C}(L)$ there exist $a \in R^1$, $f_K \in \mathscr{C}(K)$, $f_L \in \mathscr{C}(L)$ such that the function h of the form (2.55) is the solution of the Dirichlet problem on S^+ with the boundary conditions u_K, u_L (that is, the conditions (2.53), (2.54) are fulfilled). The constant a is then determined by (2.63), f_L is of the form (2.57) and f_K is a solution of (2.62); f_K is determined uniquely up to an additive constant.*

2.20. The erm h_{z_0} plays a role in the solution of the form (2.55). It was just this form of the solution that J. M. Sloss employed in [16] (see also [17]). In order to make the best of the advantages of the reflection function it seems natural to replace h_{z_0} by the function

$$v(z) = h_{z_0}(z) - h_{z_0}(\bar{g}(z)) \tag{2.66}$$

and find the solution of the Dirichlet problem in the form

$$h(z) = H f_K(z) + H_L f(z) + a\, v(z) . \tag{2.67}$$

This form of the solution is in a sense more elegant since $v(\zeta) = 0$ for $\zeta \in L$ and we can thus take simply $f_L = \frac{1}{2} u_L$. The constant a has to be chosen in such a way that the equation

$$(I - \iota \bar{H}) f_K = -\iota(u_K - \tfrac{1}{2} H_L u_L|_K - a v|_K) \tag{2.68}$$

admits a solution, that is (Theorem 2.16), a is detemined by the condition

$$a \langle \mu_0, v \rangle = \langle \mu_0, u_K - \tfrac{1}{2} H_L u_L \rangle . \tag{2.69}$$

Here it is necessary, of course, to know that $\langle \mu_0, v \rangle \neq 0$.

2.21. Lemma. *Let $\mu_0 \in \mathscr{C}'(K)$ be a non-trivial solution of (2.59), $z_0 \in G$ and let v be defined by (2.66). Then*

$$\langle \mu_0, v \rangle \neq 0 \,. \tag{2.70}$$

Proof. Suppose that $\langle \mu_0, v \rangle = 0$. Then in accordance with Theorem 2.16 there is an $f \in \mathscr{C}(K)$ such that

$$(I - \iota \bar{H}) f = v|_K = (h_{z_0} - h_{z_0} * \bar{g})|_K \,,$$

that is, for $\zeta \in K$ we have

$$H^e f(\zeta) = -h_{z_0}(\zeta) + h_{z_0}(\bar{g}(\zeta)) \,.$$

It means that the function

$$Hf + h_{z_0} - h_{z_0} * \bar{g}$$

(which is harmonic on S^+) has zero limit on $\partial S^+ = K \cup L$ (with respect to S^+) and, consequently, vanishes on S^+. Since this function is harmonic even on $S = S^+ \cup \cup L \cup S^-$ it vanishes there (S^- is defined by (2.20)). Thus

$$W(z, f) + h_{z_0}(z) = W(\bar{g}(z), f) + h_{z_0}(\bar{g}(z))$$

for $z \in S$.

If we put

$$h(z) = W(z, f) + h_{z_0}(z)$$

for $z \in S$ the last equality can be written in the form

$$h(z) = h(\bar{g}(z))$$

($z \in S$). If $\boldsymbol{n}$ stands for the normal to L then according to Remark 1.3

$$\frac{\partial h}{\partial \boldsymbol{n}}(\zeta) = 0$$

for each $\zeta \in L$, that is

$$\frac{\partial}{\partial \boldsymbol{n}} W(\zeta, f) = -\frac{\partial}{\partial \boldsymbol{n}} h_{z_0}(\zeta) \,.$$

But this is impossible since $W(\cdot, f)$ is the imaginary part of a holomorphic function and thus

$$\int_L \frac{\partial}{\partial \boldsymbol{n}} W(\cdot, f) \, \mathrm{d}\mathscr{H}_1 = 0 \,,$$

while

$$\int_L \frac{\partial}{\partial \boldsymbol{n}} h_{z_0} \, \mathrm{d}\mathscr{H}_1 \neq 0$$

as $z_0 \in G$.

Finally, we obtain the following assertion.

2.22. Theorem. *Suppose that the condition* (2.58) *is fulfilled, let* μ_0 *be a nontrivial solution of* (2.59), *let* $z_0 \in G$ *and let* v *be the function defined by* (2.66). *Then for each boundary conditions* $u_K \in \mathscr{C}(K)$, $u_L \in \mathscr{C}(L)$ *there are* $a \in R^1$ *and* $f_K \in \mathscr{C}(K)$ *such that the function* h *of the form*

$$h(z) = H f_K(z) + \tfrac{1}{2} H_L u_L + a\, v(z)$$

is the solution of the Dirichlet problem on S^+ *with the boundary conditions* u_K, u_L. *The constant* a *is then determined by* (2.69) *and* f_K *is a solution of* (2.68); f_K *is determined uniquely up to an additive constant.*

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Souhrn

REFLEXE A DIRICHLETOVA ÚLOHA NA DVOJNÁSOBNĚ SOUVISLÝCH OBLASTECH

EVA DONTOVÁ

V článku je řešena Dirichletova úloha pro dvojnásobně souvislou rovinnou oblast omezenou dvěma Jordanovými křivkami v případě, že „vnější" křivka je analytická a má reflexní funkci (vnitřní křivka může být nehladká). V tomto případě lze příslušnou soustavu dvou integrálních rovnic redukovat na jednu integrální rovnici uvažovanou na „vnitřní" křivce.

Резюме

РЕФЛЕКСИЯ И ЗАДАЧА ДИРИХЛЕ ДЛЯ ДВУСВЯЗНЫХ ОБЛАСТЕЙ

EVA DONTOVÁ

В статье решается задача Дирихле для двусвязной области ограниченной двумя кривыми Жордана, из которых „внешняя" кривая аналитическая и обладает рефлексной функцией („внутренная" кривая, вообще говоря, не гладкая). В этом случае соответствующая система интегральных уравнений превращается в одно интегральное уравнение на внутренней кривой.

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