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NONSTANDARD ANALYSIS AND GENERALIZED RIEMANN INTEGRALS

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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1. INTRODUCTION

It is known that A. Robinson has proposed, under the name of non-standard analysis, an alternative approach to analysis which rehabilitates up to some extent the old, fruitful but heuristic concepts of infinitesimals and infinitely large numbers. The nonstandard treatment of Riemann integration was already mentioned by Robinson in his pioneering paper [25] and he gave detailed versions in his books [27] and [28], initiating also in [28] the nonstandard version of Lebesgue's measure and integral. This theory was extensively developed, specially after Loeb's discovery [13] of a new way of constructing rich standard measure spaces from nonstandard ones. We refer to [2] for a recent survey.

On the other hand, the (classical) Riemann sum approach to integration was completely renewed by Kurzweil's discovery [10] that Perron's integration [21] could be developed from a technically simple but conceptually important modification of Riemann's definition. Such a result was obtained independently by Henstock [4] and then developed in many directions by a number of authors (see the books or survey works [5, 6, 11, 12, 15, 18, 19, 22, 28]).

In this paper, we shall propose a non-standard approach to some aspects of the generalized Riemann integration theory. We shall give nonstandard characterizations of the concepts of S-integral (or Kurzweil-Henstock integral) over a compact interval of $\mathbb{R}^n$ and of one of its generalizations, the M-integral [9], which provides a divergence theorem for mere differentiable vector fields in the same way as Perron's integral [21] allows a fundamental theorem of the calculus for every differentiable function of one variable. We hope that the reader will appreciate the formal transparence of those characterizations as well as the technical simplicity of the nonstandard proofs given as examples, namely that of the Cauchy criterion of M-integrability and that of the divergence theorem. Those are only samples, and we hope to develop the approach in subsequent works.

Let us finally notice that among the various existing approaches to nonstandard analysis, in this paper we have chosen Nelson's Internal Set Theory [20] which seems particularly appropriate to the questions considered here and whose spirit and methodology remain very close to that of the working mathematician.

2. NELSON'S APPROACH TO NONSTANDARD ANALYSIS

Nelson's approach [20] to Robinson's nonstandard analysis is based on internal set theory (IST). This theory starts with the usual ZFC axiomatic set theory (Zermelo-Fraenkel set theory with the axiom of choice, see e.g. [1]) and adjoin to the usual undefined binary predicate $\in$ a new undefined unary predicate *standard* (st). Recall that in ZFC every mathematical object is a set and we write $\mathrm{st}(x)$ for "x is standard". The axioms of IST are the usual ones of ZFC plus three others, which will be stated after some terminology is introduced.

Formulas are written using the usual symbols of formal logic and the predicates, and respecting syntactic rules. A *formula* of IST is called *internal* if it does not involve the new predicate "st" otherwise, is called *external*. When some sets are uniquely defined and extensively used, *constants* are introduced to simplify the language (like $\emptyset$ for the empty set, $\mathbb{N}$ for the natural numbers, $\cup$ for the union of sets ...). They can always be replaced by formulas using only the formal language. Such a constant wil be called *standard* if its definition does not involve the predicate "st" and *non-standard* otherwise. Consequently, all the constants of classical mathematics (ZFC) are standard. An internal formula will be called *standard* if it contains only standard constants, and *nonstandard* otherwise. So the formula "$x \in \mathbb{R}$" is internal standard and the formula "st x" is the simplest external formula. We shall use the following abbreviations:

$$\begin{aligned}
&(\forall^{\mathrm{st}}x)\ F(x) && \text{for} && (\forall x)\ (\mathrm{st}(x) \Rightarrow F(x))\\
&(\exists^{\mathrm{st}}x)\ F(x) && \text{for} && (\exists x)\ (\mathrm{st}(x) \wedge F(x))\\
&(\forall^{\mathrm{fin}}x)\ F(x) && \text{for} && (\forall x)\ (x \text{ finite} \Rightarrow F(x))\\
&(\exists^{\mathrm{fin}}x)\ F(x) && \text{for} && (\exists x)\ (x \text{ finite} \wedge F(x)) .
\end{aligned}$$

We can now state the three axioms which, added to those of ZFC, govern the use of the new predicate "st".

1. **The transfer principle** (T)

For any *standard formula* $A(x, t_1, \ldots, t_n)$ containing no other free variables than $x, t_1, \ldots, t_n$, the following statement is an axiom:

$$(\forall^{\mathrm{st}}t_1)\,(\forall^{\mathrm{st}}t_2)\ldots(\forall^{\mathrm{st}}t_n)\,[(\forall^{\mathrm{st}}x)\ A(x, t_1, \ldots, t_n) \Rightarrow (\forall x)\ A(x, t_1, \ldots, t_n)]$$

(or, equivalently,

$$(\forall^{\mathrm{st}}t_1)\,(\forall^{\mathrm{st}}t_2)\ldots(\forall^{\mathrm{st}}t_n)\ [(\exists x)\ A(x, t_1, \ldots, t_n) \Rightarrow (\exists^{\mathrm{st}}x)\ A(x, t_1, \ldots, t_n)]$$

2. **The principle of idealization** (I)

For any *internal formula* $B(x, y)$ with free variables x, y and possibly other free variables, the following assertion is an axiom:

$$(\forall^{\mathrm{stfin}} z)(\exists x)(\forall y \in z)\, B(x, y) \Leftrightarrow (\exists x)(\forall^{\mathrm{st}} y)\, B(x, y)\,.$$

3. **The principle of standardization** (S)

For any *formula* (internal or external) $C(z)$ with a free variable z and possibly other free variables, the following assertion is an axiom:

$$(\forall^{\mathrm{st}} x)(\exists^{\mathrm{st}} y)(\forall^{\mathrm{st}} z)\,[z \in y \Leftrightarrow (z \in x) \wedge C(z)]\,.$$

One can prove (see [20]) that IST is a conservative extension of ZFC, i.e. that every internal statement which can be proved in IST can be proved in ZFC and IST can be used freely in proving conventional theorems.

An easy consequence of the transfer principle is that if there exists a unique x such that $A(x)$, where the standard formula $A(x)$ contains only one free variable x, then x is standard. Consequently, every specific object of conventional mathematics is a standard set ($\mathbb{N}$, $\mathbb{R}$, $L^2(\mathbb{R})$, ...). Also, standard sets are equal if and only if they have the same standard elements. But a standard set may contain nonstandard elements, as follows from the following consequence of the principle of idealization: every element of a set v is standard if and only if v is standard and finite (see [20]). Consequently, every infinite set contains a nonstandard element. This is the case in particular for $\mathbb{N}$ and the axioms of transfer and idealization easily imply that $m \in \mathbb{N}$ is non-standard if and only if m is *infinitely large*, i.e. such that $(\forall^{\mathrm{st}} n)(n \in \mathbb{N} \Rightarrow$ $\Rightarrow n < m)$. The principle of standardization plays the role of a partial substitute to the fact that we may not use external predicates to define subsets (the replacement axiom holds within ZFC only).

We urge the reader to consult the interesting paper [20] for more information and striking remarks on the use of IST, and the monographs [3] and [14] for more applications and details.

3. MICROGAUGES AND MICROPARTITIONS IN $\mathbb{R}^n$

In IST, we can distinguish in $\mathbb{R}$ the *infinitesimals* x, which are such that

$$(\forall^{\mathrm{st}} \varepsilon)(\varepsilon \in \mathbb{R}_+ \Rightarrow |x| \leqq \varepsilon)\,,$$

the *limited numbers* x, which are such that

$$(\exists^{\mathrm{st}} r)(r \in \mathbb{R}_+ \wedge |x| \leqq r)$$

and the *unlimited* ones x, which are not limited, and hence such that

$$(\forall^{\mathrm{st}} r)(r \in \mathbb{R}_+ \Rightarrow |x| > r)\,.$$

Of course the infinitesimals are limited and using (T) one can show that 0 is the only standard infinitesimal. The existence of infinitesimals and of unlimited elements of $\mathbb{R}$ in IST is a consequence of the idealization principle. We shall say that two real numbers x and y are *infinitely close*, and we shall write $x \simeq y$, if $x - y$ is infinitesimal; this clearly is an equivalence relation. One can prove that every limited real number x is infinitely close to a unique standard real number which is called its *standard part* and denoted by $°x$. Similarly, in $\mathbb{R}^n$, we shall write $x \simeq y$ if $x_i \simeq y_i$ for all $1 \leqq i \leqq n$ and write $°x$ for $(°x_1, \ldots, °x_n)$ when x is *limited*, i.e. each x_i is limited.

Recall that a *gauge* on a set E is, by definition, a positive function on E. Such a gauge will be called standard if its graph is a standard subset of $E \times \mathbb{R}_+^*$ (with $\mathbb{R}_+^* = \{x \in \mathbb{R}: x > 0\}$). Then, necessarily, E is standard. We shall use the following consequence of the idealization principle. We denote as usual by B^A the set of mappings from A into B.

Lemma 1. *Let $E \neq \emptyset$ be a standard set. Then there exists a mapping $\mu: E \to \mathbb{R}_+^*$ such that*

$$(1) \qquad (\forall^{st}\delta)\,(\forall x)\,[(\delta \in \mathbb{R}_+^{*E}) \wedge (x \in E) \Rightarrow \mu(x) \leqq \delta(x)]\,.$$

Proof. Let us introduce in $\mathbb{R}_+^{*E}$ the (internal) order relation $\delta_1 \leqq \delta_2$ by $\delta_1(x) \leqq \delta_2(x)$ for all $x \in E$ and define the internal formula $B(\eta, \delta)$ by

$$(\eta \in \mathbb{R}_+^{*E}) \wedge (\delta \in \mathbb{R}_+^{*E}) \wedge (\eta \leqq \delta)\,.$$

Then

$$(\forall^{\mathrm{stfin}} z)\,(\exists \eta)\,(\forall \delta \in z)\, B(\eta, \delta)$$

because, if $z = \{\delta_1, \ldots, \delta_m\}$ it suffices to define η by

$$\eta(x) = \min(\delta_1(x), \ldots, \delta_m(x))\,, \quad x \in E\,.$$

Then, the principle of idealization implies the existence of μ such that $(\forall^{st}\delta)\, B(\mu, \delta)$, i.e. such that (1) holds, and the proof is complete.

A mapping $\mu: E \to \mathbb{R}_+^*$, with E standard, which satisfies (1) will be called a *microgauge* on E.

Remarks 1. If $\mu: E \to \mathbb{R}_+^*$ is a microgauge, then $\mu(x)$ is an infinitesimal for each $x \in E$. This follows from the fact that, for each standard real $r > 0$, the constant mapping $\delta: E \to \mathbb{R}_+^*$, $x \to r$ is a standard gauge on E.

2. The converse is false, i.e. a gauge η on E with $\eta(x)$ infinitesimal for each $x \in E$ is not necessarily a microgauge. For example, on $E = [0, 1]$, no constant infinitesimal gauge $\eta: x \to \varepsilon$ ($\varepsilon > 0$ infinitesimal) is a microgauge because we have $\eta(\varepsilon/2) = \varepsilon > \delta(\varepsilon)$ for the standard gauge δ on $[0, 1]$ defined by $\delta(x) = x$ if $x \neq 0$ and $\delta(0) = 1$.

3. As a standard mapping takes standard values at standard points, we see that the only restriction to the value of a microgauge at a standard point is to be infinitesimal.

Let now $E \subset \mathbb{R}^n$ be an open standard set, and let $f\colon \mathbb{R}^n \to \mathbb{R}^p$ be a standard function defined on E. The transfer principle immediately implies that f is differentiable on E if and only if f is differentiable at each standard point of E. Moreover, it is a classical exercise in IST to show that if $a \in E$ is standard, then f is *differentiable at* a if and only if there exists a linear mapping $L\colon \mathbb{R}^n \to \mathbb{R}^p$ such that

$$\frac{f(a+h)-f(a)}{|h|} \simeq \frac{L(h)}{|h|}$$

whenever $h \neq 0$ and $h \simeq 0$ (see e.g. [3] or [14]). Such an L, necessarily unique and hence standard, is called the total derivative of f at a and is denoted by f'_a. The concept of microgauge provides a global-like characterization of the differentiability of f on E which we shall use in proving the divergence theorem.

Lemma 2. *Let $E \subset \mathbb{R}^n$ be open, standard and let $f\colon \mathbb{R}^n \to \mathbb{R}^p$ be a standard function defined on E. Then f is differentiable on E if and only if there exists a standard mapping $f'_{\cdot}\colon E \to \mathbf{L}(\mathbb{R}^n, \mathbb{R}^p)$, $x \to f'_x$ such that, for each microgauge μ on E and each $x \in E$, one has*

$$\frac{f(x+h)-f(x)}{|h|} \simeq \frac{f'_x(h)}{|h|}$$

whenever $0 < |h| \leqq \mu(x)$ and $x + h \in E$.

Proof. *Necessity.* By assumption and (T), the total derivative $f'_{\cdot}\colon E \to L(\mathbb{R}^n, \mathbb{R}^p)$ is a standard mapping and, by assumption

$$(\forall\varepsilon)(\exists\delta)\Big[(\varepsilon \in \mathbb{R}^*_+) \wedge (\delta \in \mathbb{R}^{*E}_+) \wedge (x \in E) \wedge (h \in \mathbb{R}^n \setminus \{0\}) \wedge$$

$$\wedge\ (x + h \in E) \wedge (|h| \leqq \delta(x)) \Rightarrow \left|\frac{f(x+h)-f(x)}{|h|} - \frac{f'_x(h)}{|h|}\right| \leqq \varepsilon\Big]. \tag{2}$$

As the formula in [...] is standard and contains only the free variables ε and δ, it follows from (T) applied to (2) restricted to the standard ε that

$$(\forall^{st}\varepsilon)(\exists^{st}\delta)\,[\ldots]\,. \tag{3}$$

Now, if μ is a microgauge on E and $0 < |h| \leqq \mu(x)$, $x + h \in E$, we have $|h| \leqq \delta(x)$ for any standard gauge δ on E and hence, by (3), we get

$$(\forall^{st}\varepsilon)\Big[(\varepsilon \in \mathbb{R}^*_+) \wedge (x \in E) \wedge (h \in \mathbb{R}^n \setminus \{0\}) \wedge (x + h \in E) \wedge (|h| \leqq \mu(x)) \Rightarrow$$

$$\Rightarrow \left|\frac{f(x+h)-f(x)}{|h|} - \frac{f'_x(h)}{|h|}\right| \leqq \varepsilon\Big],$$

and then

$$\frac{f(x+h)-f(x)}{|h|} \simeq \frac{f'_x(h)}{|h|}$$

whenever $x \in E$, $h \in \mathbb{R}^n \setminus \{0\}$, $x + h \in E$ and $|h| \leqq \mu(x)$.

Sufficiency. By assumption,

$$(\forall^{st}\varepsilon)\left\{(\exists\delta)\left[(\varepsilon \in \mathbb{R}^*_+) \wedge (\delta \in \mathbb{R}^{*E}_+) \wedge (x \in E) \wedge (h \in \mathbb{R}^n \setminus \{0\}) \wedge (x + h \in E) \wedge \right.\right.$$
$$\left.\left. \wedge (|h| \leqq \delta(x)) \Rightarrow \left|\frac{f(x+h)-f(x)}{|h|} - \frac{f'_x(h)}{|h|}\right| \leqq \varepsilon\right]\right\}.$$

It suffices to take for δ a microgauge μ on E. As the formula $\{\ldots\}$ is standard and contains only the free variable ε, we can apply (T) to obtain $(\forall\varepsilon)\{\ldots\}$ which expresses that f is differentiable on E and completes the proof.

Now, let $I =]a_1, b_1] \times \ldots \times]a_n, b_n]$ be a right-closed interval in $\mathbb{R}^n$ and $\bar{I}$ its closure. Recall that a *P-partition* of I (see e.g. [12], [15]) is a finite family $\{(x^1, I^1), \ldots, (x^q, I^q)\}$ where $I^j \subset I$ are right-closed intervals in $\mathbb{R}^n$ which partition I (i.e. $I = \bigcup_{j=1}^{q} I^j$ and $I^j \cap I^k = \emptyset$ if $j \neq k$) and x^j are elements of $\mathbb{R}^n$ such that $x^j \in \bar{I}^j$, $1 \leqq j \leqq q$. If δ is a gauge on $\bar{I}$, such a P-partition will be called *δ-fine* if $I^j \subset B[x^j; \delta(x^j)]$, $1 \leqq j \leqq q$ where $B[y; r]$ is the closed ball of center y and radius r in $\mathbb{R}^n$ with the norm $|x| = \max_{1 \leqq j \leqq n} |x_i|$. When I is standard, a P-partition of I which is μ-fine for some microgauge μ on $\bar{I}$ will be called a *micropartition*. Notice that in such a micropartition, q is necessarily unlimited.

As each theorem of conventional mathematics remains valid in IST, so is *Cousin's lemma*, which insures, for each left-open interval $I \subset \mathbb{R}^n$ and each gauge δ on $\bar{I}$, the existence of a regular δ-fine P-partition of I (see e.g. [12], [15–17], [9]). Recall that a *regular* P-partition of I means that each I^j is similar to I $(1 \leqq j \leqq q)$. When I is standard, if we take for δ a microgauge μ on $\bar{I}$, we obtain immediately the existence of a regular micropartition of I.

Let us finally observe that the nonstandard approach to Riemann integral [26, 27] implies the use of *infinitesimal P-partitions of I*, i.e. P-partitions $\{(x^1, I^1), \ldots, (x^q, I^q)\}$ such that each I^j has an infinitesimal diameter $(1 \leqq j \leqq q)$. They correspond of course to δ-fine P-partitions for a constant infinitesimal gauge δ on $\bar{I}$. Clearly, every micropartition of I is an infinitesimal P-partition of I but the converse is not true (see Remarks 1 and 2).

4. NONSTANDARD CHARACTERIZATIONS OF GENERALIZED RIEMANN INTEGRALS

Let $I \subset \mathbb{R}^n$ be a right-closed interval and f a function of $\mathbb{R}^n$ into $\mathbb{R}^p$ defined on $\bar{I}$.

The following rather unconspicuous modification of the concept of Riemann integral was introduced independently by Kurzweil [10] and Henstock [4] and shown equivalent, for $n = 1$, to Perron's integral [10, 21]. It is based upon the concept of *Riemann sum* associated to f and to a P-partition $\Pi = \{(x^1, I^1), \ldots, (x^q, I^q)\}$ of I, defined as usual by

$$S(I, f, \Pi) = \sum_{j=1}^{q} m_n(I^j) f(x^j)$$

where $m_n(I^j) = \prod_{k=1}^{n} (b_k^j - a_k^j)$ is the n-measure of the right-closed interval $I^j =]a_1^j, b_1^j] \times \ldots \times]a_n^j, b_n^j]$, $(1 \leqq j \leqq q)$.

Definition 1. *f is S-integrable over $\bar{I}$ if there exists $J \in \mathbb{R}^p$ such that for each $\varepsilon > 0$, we can find a gauge δ over $\bar{I}$ such that, for each δ-fine P-partition Π of I, one has*

$$|S(I, f, \Pi) - J| \leqq \varepsilon .$$

It is easy to show that such a J is necessarily unique. It is called the *S-integral* of f over $\bar{I}$ and denoted by $\int_I f$ or $\int_I f(x) \, dm_n(x)$. The letter S indicates that the S-integral is defined by means of Riemann sums; because of its equivalence when $n = 1$ to the Perron integral, it is also called the *P-integral* and Henstock's original terminology was the *Riemann-complete integral.*

The S-integral has a lot of interesting properties (see e.g. [5, 6, 11, 12, 15, 18, 19]) and, in particular, when $n = 1$, it integrates every derivative f' over an interval $[a, b]$, with the value $f(b) - f(a)$ for the integral. This suggested that the n-dimensional S-integral should integrate the divergence of each differentiable vector field over a closed interval $\bar{I}$ and provide a Stokes-type theorem. That does not seem to be the case however and the author introduced in [17] a generalized Riemann integral which reduces to the S-integral when $n = 1$ and achieves this program. This was done by inserting in Definition 1 a non-uniformity with respect to a geometric property of a partition of I called its irregularity. However, the new integral lacked some useful properties (e.g. getting integrability on $\bar{I} = \bar{I}^1 \cup \bar{I}^2$ from that on $\bar{I}^1$ and $\bar{I}^2$) and this led Jarník, Kurzweil and Schwabik [9] to injecting in the author's definition another notion of irregularity which eliminates the mentioned flaws without losing the Stokes theorem. They obtained in this way the following concept, which they called the M-integral, and which depends upon the following notion of irregularity of a P-partition.

Definition 2. *If $\Pi = \{(x^1, I^1), \ldots, (x^q, I^q)\}$ is a P-partition of the right-closed interval $I \subset \mathbb{R}^n$, the irregularity $\Sigma(\Pi)$ of Π is the real number*

$$\Sigma(\Pi) = \sum_{j=1}^{q} \int_{\partial I^j} |x - x^j| \, dm_{n-1}(x)$$

where ∂I^j is the boundary of I^j and the integral is just the sum of $(n-1)$-dimensional (Riemann) integrals over the $(n-1)$-dimensional intervals which constitute ∂I^j.

Definition 3. *f is M-integrable over $\bar{I}$ if there exists $J \in \mathbb{R}^p$ such that, for each $\varepsilon > 0$ and each $C > 0$, one can find a gauge δ over $\bar{I}$ such that, for each δ-fine P-partition Π of I with $\Sigma(\Pi) \leqq C$, one has*

$$|S(I, f, \Pi) - J| \leqq \varepsilon .$$

Again such a J is necessarily unique and it exists and coincides with $\int_I f$ if f is S-integrable. It will therefore also be noted $\int_I f$ or $\int_I f(x)\,\mathrm{d}m_n(x)$, and called the *M-integral* of f over $\bar{I}$.

Remark 3. If δ is a gauge on $\bar{I}$, the regular δ-fine P-partition $\Pi = \{(x^1, I^1), \ldots$ $\ldots, (x^q, I^q)\}$ whose existence follows from Cousin's lemma has the following property. If, for a right-closed interval $K =]c_1, d_1] \times \ldots \times]c_n, d_n]$, we denote by

$$l(K) = \max_{1 \leqq k \leqq n} (d_k - c_k)$$

its longest edge and by $m_{n-1}(\partial K)$ the $(n-1)$-dimensional measure of its boundary, then the regularity of Π implies the existence of $c_j \in]0, 1[$ such that $\sum_{j=1}^{q} c_j^n = 1$,

$$m(I^j)/m(I) = c_j^n , \quad l(I^j)/l(I) = c_j , \quad m_{n-1}(\partial I^j)/m_{n-1}(\partial I) = c_j^{n-1} \quad (1 \leqq j \leqq q)$$

and hence

$$\Sigma(\Pi) \leqq \sum_{j=1}^{q} l(I^j)\, m_{n-1}(\partial I^j) = \left(\sum_{j=1}^{q} c_j^n\right) l(I)\, m_{n-1}(\partial I) = l(I)\, m_{n-1}(\partial I) .$$

Thus Definition 3 is meaningful.

We shall now state, prove and compare nonstandard characterizations of the S- and M-integrals when I and f are standard.

Let $I \subset \mathbb{R}^n$ be a standard right-closed interval and $f\colon \mathbb{R}^n \to \mathbb{R}^p$ a standard function defined on $\bar{I}$. Denote by $\mathbf{P}(I, \delta)$ the set of δ-fine P-partitions of I.

Theorem 1. *f is S-integrable over $\bar{I}$ if and only if there exists a standard $J \in \mathbb{R}^p$ such that*

$$S(I, f, \Pi) \simeq J$$

for each micropartition Π of I.

Proof. *Necessity.* By Definition 2, the uniqueness of J and (T), there exists a standard $J \in \mathbb{R}^p$ such that

$$(4) \qquad (\forall \varepsilon)\,(\exists \delta)\,\{(\forall \Pi)\,[(\varepsilon > 0) \wedge (\delta \in \mathbb{R}_+^{*I}) \wedge (\Pi \in \mathbf{P}(I, \delta)) \Rightarrow \Rightarrow |S(I, f, \Pi) - J| \leqq \varepsilon]\} .$$

Now, $\{\dots\}$ is standard and depends only on the free variables ε and δ so that, by (T) applied to

$$(\forall^{\mathrm{st}}\varepsilon)\,(\exists\delta)\,\{\dots\}\,,$$

we obtain

$$(\forall^{\mathrm{st}}\varepsilon)\,(\exists^{\mathrm{st}}\delta)\,\{\dots\}\,. \tag{5}$$

Now, if Π is a micropartition of I, then Π is δ-fine for each standard $\delta \in \mathbb{R}_+^{*\bar{I}}$ and hence, by (5),

$$(\forall^{\mathrm{st}}\varepsilon)\,[(\varepsilon > 0) \Rightarrow |S(I,f,\Pi) - J| \leqq \varepsilon]$$

i.e. $S(I,f,\Pi) \simeq J$.

Sufficiency. By assumption there exists a standard $J \in \mathbb{R}^p$ such that

$$(\forall^{\mathrm{st}}\varepsilon)\,\{(\exists\delta)\,(\forall\Pi)\,[(\varepsilon > 0) \wedge (\delta \in \mathbb{R}_+^{*\bar{I}}) \wedge (\Pi \in \mathbf{P}(I,\delta)) \Rightarrow \Rightarrow |S(I,f,\Pi) - J| \leqq \varepsilon]\}\,. \tag{6}$$

It suffices to take for δ a microgauge on $\bar{I}$. Again, we can apply (T) to (6) to obtain (4) and the proof is complete.

Let us now denote by $\mathbf{P}(I,\delta,C)$ the set of δ-fine P-partitions Π of I such that $\Sigma(\Pi) \leqq C$, and let us call a micropartition Π of I such that $\Sigma(\Pi)$ is limited a *regular micropartion* of I. Remark 3 and the fact that I is standard implies the existence of regular micropartitions of I.

Theorem 2. *f is M-integrable over $\bar{I}$ if and only if there exists a standard $J \in \mathbb{R}^p$ such that*

$$S(I,f,\Pi) \simeq J$$

for each regular micropartition Π of I.

Proof. *Necessity*. By Definition 3, the uniqueness of J and (T), there exists a standard $J \in \mathbb{R}^p$ such that

$$(\forall\varepsilon)\,(\forall C)\,(\exists\delta)\,\{(\forall\Pi)\,[(\varepsilon > 0) \wedge (C > 0) \wedge (\delta \in \mathbb{R}_+^{*\bar{I}}) \wedge (\Pi \in \mathbf{P}(I,\delta,C)) \Rightarrow \Rightarrow |S(I,f,\Pi) - J| \leqq \varepsilon]\}\,. \tag{7}$$

By (T) applied to

$$(\forall^{\mathrm{st}}\varepsilon)\,(\forall^{\mathrm{st}}C)\,(\exists\delta)\,\{\dots\}\,,$$

we obtain

$$(\forall^{\mathrm{st}}\varepsilon)\,(\forall^{\mathrm{st}}C)\,(\exists^{\mathrm{st}}\delta)\,\{\dots\}\,. \tag{8}$$

Hence, if Π is a regular micropartition of I, then $\Sigma(\Pi) \leqq C$ for some standard $C > 0$ and Π is δ-fine for each standard $\delta \in \mathbb{R}_+^{*\bar{I}}$, and we deduce from (8) that

$$(\forall^{\mathrm{st}}\varepsilon)\,[(\varepsilon > 0) \Rightarrow |S(I,f,\Pi) - J| \leqq \varepsilon]$$

i.e. $S(I,f,\Pi) \simeq J$.

Sufficiency. By assumption, there exists a standard $J \in \mathbb{R}^p$ such that

$$(9) \qquad (\forall^{st}\varepsilon)(\forall^{st}C)\{(\exists\delta)(\forall\Pi)[(\varepsilon > 0) \wedge (\delta \in \mathbb{R}_+^{*I}) \wedge (\Pi \in P(I, \delta, C)) \Rightarrow \Rightarrow |S(I, f, \Pi) - J| \leqq \varepsilon]\}.$$

It sufffices to take for δ a microgauge on $\bar{I}$, so that the corresponding Π are regular micropartitions. Two consecutive applications of (T) to (9) imply (7) and complete the proof.

Remark 4. It is immediate that the characterizations of the S-integrability and M-integrability given respectively by Theorem 1 and Theorem 2 can also be expressed by *f is S-integrable over $\bar{I}$ if and only if there exists a standard $J \in \mathbb{R}^p$ such that for each $C > 0$ and each micropartition Π of I with $\Sigma(\Pi) \leqq C$, one has*

$$(10) \qquad S(I, f, \Pi) \simeq J$$

and *f is M-integrable over $\bar{I}$ if and only if there exist a standard $J \in \mathbb{R}^p$ such that for each standard $C > 0$ and each micropartition Π of I with $\Sigma(\Pi) \leqq C$, one has (10). This formulation makes more apparent the usual nonstandard distinction between uniformity and non-uniformity.*

Remark 5. The IST-characterization of R-integrability (i.e. Riemann integrability) of f over $\bar{I}$ is the existence of a standard $J \in \mathbb{R}^p$ such that $S(I, f, \Pi) \simeq J$ for all infinitesimal P-partitions Π of I (as defined at the end of Section 3). This fact and Theorems 1 and 2 make quite transparent how the increasing generality from the R-integral to the S-integral and from the S-integral to the M-integral is obtained by successively restricting the choice of the allowed P-partitions.

5. NONSTANDARD PROOFS OF SOME PROPERTIES OF THE GENERALIZED RIEMANN INTEGRALS

In this section, we want to illustrate the use of nonstandard techniques to prove some (known) properties of the generalized Riemann integrals. For the sake of brevity, we shall only select a few characteristic ones and consider only the M-integral.

We first consider the IST-proof of the nonstandard formulation of the *Cauchy condition for M-integrability.*

Theorem 3. *Let $I \subset \mathbb{R}^n$ be a standard right-closed interval and f a standard function of $\mathbb{R}^n$ into $\mathbb{R}^p$ defined on $\bar{I}$. Then f M-integrable over $\bar{I}$ if and only if*

$$(11) \qquad S(I, f, \Pi) \simeq S(I, f, \tilde{\Pi})$$

for all regular micropartitions Π and $\tilde{\Pi}$ of I.

Proof. Necessity. Let $J = \int_I f$ and let Π and $\tilde{\Pi}$ be regular micropartitions of I. Then,

$$S(I, f, \Pi) \simeq J \simeq S(I, f, \tilde{\Pi})$$

and the result follows from the transitivity of $\simeq$.

Sufficiency. If there exists a regular micropartition $\tilde{\Pi}$ such that $S(I, f, \tilde{\Pi})$ is limited (i.e. each component is a limited real number), then, by taking $J = {}^\circ(S(I, f, \tilde{\Pi}))$, (11) and Theorem 2 imply that f is M-integrable over $\bar{I}$. It remains therefore to show the existence of such a $\tilde{\Pi}$.

By assumption,

$$(12) \qquad (\forall^{st} C)(\exists \delta)(\forall \Pi)(\forall \tilde{\Pi})\left[(C > 0) \wedge (\delta \in \mathbb{R}_+{}^I) \wedge (\Pi \in \mathbf{P}(I, \delta, C)) \wedge \wedge (\tilde{\Pi} \in \mathbf{P}(I, \delta, C)) \Rightarrow |S(I, f, \Pi) - S(I, f, \tilde{\Pi})| \leqq 1\right],$$

as follows immediately by taking for δ a microgauge on $\bar{I}$. By (T) we obtain the existence of a standard δ_s such that (12) holds and for this standard gauge δ_s, Cousin's lemma implies that

$$(13) \qquad (\forall^{st} C)(\exists \Pi)\left[C \geqq l(I)\, m_{n-1}(\partial I) \Rightarrow \Pi \in \mathbf{P}(I, \delta_s, C)\right].$$

By (T) we deduce that there exists a standard Π_s such that (13) holds and hence the corresponding Riemann sum $S(I, f, \Pi_s)$ is a standard element of $\mathbb{R}^p$. Taking $C \geqq l(I)\, m_{n-1}(\partial I)$, $\Pi = \Pi_s$ and an arbitrary regular micropartition $\tilde{\Pi}$ in (12) with the standard gauge δ_s (so that $\tilde{\Pi} \in \mathbf{P}(I, \delta_s, C)$) we obtain

$$|S(I, f, \Pi_s) - S(I, f, \tilde{\Pi})| \leqq 1,$$

i.e.

$$|S(I, f, \tilde{\Pi})| \leqq 1 + |S(I, f, \Pi_s)|,$$

which shows that $S(I, f, \tilde{\Pi})$ is limited and completes the proof.

Let now $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $I \subset \mathbb{R}^n$ a right-closed interval and f a function of $\mathbb{R}^n$ into $\mathbb{K}^n$ which is differentiable on an open domain Ω containing $\bar{I}$. Let us denote by ω_f the $(n-1)$-form on Ω defined by

$$\omega_f = \sum_{i=1}^{n} (-1)^{i-1} f_i \, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n .$$

As ω_f is continuous, the integral $\int_{\partial I} \omega_f$ over the (oriented) boundary ∂I of I is defined classically through Riemann integrals (see e.g. [17]), and, as ω_f is differentiable, the divergence $\operatorname{div} f = \sum_{i=1}^{n} (\partial f_i / \partial x_i)$ exists, but is not necessarily continuous. We have, however, the following *divergence theorem* (see e.g. [17, 9]).

Theorem 4. *Let f be a function of $\mathbb{R}^n$ into $\mathbb{K}^n$ which is differentiable on an open domain Ω. Then, for each right-closed interval $I \subset \mathbb{R}^n$ such that $\bar{I} \subset \Omega$, $\operatorname{div} f$ is M-integrable and*

$$\int_I \operatorname{div} f = \int_{\partial I} \omega_f .$$

Proof. By (T), it suffices to prove the theorem for f, Ω and I standard. Let $\Pi =$

$= \{(x^1, I^1), \ldots, (x^q, I^q)\}$ be a regular micropartition of I. Then, by a well-known result, we have

$$\int_{\partial I} \omega_f = \sum_{j=1}^{q} \int_{\partial I^j} \omega_f$$

and hence

$$\int_{\partial I} \omega_f - S(I, \operatorname{div} f, \Pi) = \sum_{j=1}^{q} \left[\int_{\partial I^j} \omega_f - \operatorname{div} f(x^j)\, m(I^j) \right]. \tag{14}$$

Now, if we define g^j and h^j by

$$g^j(y) = f(x^j) + f'_{x^j}(y - x^j),$$
$$h^j(y) = f(y) - g^j(y) \quad (1 \leqq j \leqq q),$$

and observe that by the divergence theorem for smooth mappings or by direct calculation we have

$$\int_{\partial I^j} \omega_{g^j} = \int_{I^j} \mathrm{d}\omega_{g^j} = \operatorname{div} f(x^j)\, m(I^j),$$

we deduce from (14) that

$$\int_{\partial I} \omega_f - S(I, \operatorname{div} f, \Pi) = \sum_{j=1}^{q} \int_{\partial I^j} \omega_{h^j}. \tag{15}$$

Now, as f is differentiable on Ω, Lemma 2 and the fact that Π is a micropartition imply that

$$\frac{h^j(y)}{|y - x^j|} \simeq 0$$

for all $y \in \bar{I}^j$ $(1 \leqq j \leqq q)$ with $y \neq x^j$. Thus, if $\varepsilon > 0$ is any standard real number, we have

$$|h^j(y)| \leqq \varepsilon |y - x^j|$$

for all $y \in \bar{I}^j$ $(1 \leqq j \leqq q)$. Consequently,

$$\left| \sum_{j=1}^{q} \int_{\partial I^j} \omega_{h^j} \right| \leqq \sum_{j=1}^{q} \left| \int_{\partial I^j} \omega_{h^j} \right| \leqq \sum_{j=1}^{q} \varepsilon \int_{\partial I^j} |y - x^j|\, \mathrm{d}m_{n-1}(y) = \varepsilon \sum(\Pi).$$

As Π is regular, $\sum(\Pi)$ is limited, and hence

$$\left| \sum_{j=1}^{q} \int_{\partial I^j} \omega_{h^j} \right| \leqq \varepsilon' \tag{16}$$

for every standard $\varepsilon' > 0$. It then follows from (15) and (16) that $\int_{\partial I} \omega_f \simeq$ $\simeq S(I, \operatorname{div} f, \Pi)$ and the proof is complete.

Remark 6. The theory of the M-integral has been developed and generalized in very interesting directions by Jarník and Kurzweil in [7] and [8]. Also, an alternative approach to divergence theorems through generalized Riemann integrals has been initiated by Pfeffer [23, 24]. We shall study those questions from a nonstandard viewpoint in subsequent papers.

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