Časopis pro pěstování matematiky

Vachtang Michailovič Kokilashvili; Jiří Rákosník

Weighted inequalities for anisotropic maximal functions

Časopis pro pěstování matematiky, Vol. 110 (1985), No. 4, 384–393

Persistent URL: `http://dml.cz/dmlcz/118255`

Časopis pro pěstování matematiky, roč. 110 (1985), Praha

WEIGHTED INEQUALITIES FOR ANISOTROPIC MAXIMAL FUNCTIONS

VACHTANG MICHAILOVIČ KOKILASHVILI, Tbilisi, JIŘÍ RÁKOSNÍK, Praha

(Received April 4, 1984)

1. INTRODUCTION

1.1. Let $\boldsymbol{R}^n$ be the n-dimensional Euclidean space of points $x = (x_1, \ldots, x_n)$. By a weight function (shortly a weight) we shall mean a measurable function which is non-negative and finite a.e. in $\boldsymbol{R}^n$.

1.2. If $1 < p < \infty$ and w is a weight function, we denote by $L^p_w(\boldsymbol{R}^n)$ the weighted Lebesgue space of all measurable functions f with the norm

$$\|f\|_{p,w} = \left(\int_{\boldsymbol{R}^n} |f(x)|^p \, w(x) \, \mathrm{d}x\right)^{1/p} < \infty .$$

Similarly, the norm in $L^\infty_w(\boldsymbol{R}^n)$ is defined by

$$\|f\|_{\infty,w} = \operatorname{ess\,sup} |f(x)| ,$$

where the essential supremum is taken with respect to the measure μ_w:

$$\mu_w e = \int_e w(x) \, \mathrm{d}x , \quad e \subset \boldsymbol{R}^n \text{ measurable} . \tag{1.1}$$

The Lebesgue measure of e will be denoted by $|e|$. The number p' is always defined by $1/p + 1/p' = 1$.

1.3. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a fixed vector from $\boldsymbol{R}^n$ with $\alpha_i > 0$, $i = 1, \ldots, n$. For $x \in \boldsymbol{R}^n$ and $t > 0$ we define the one-parametric parallelepiped

$$E(x, t) = \{y \in \boldsymbol{R}^n;\ |y_i - x_i| \leqq \tfrac{1}{2} t^{\alpha_i},\ i = 1, \ldots, n\}$$

and by $\mathbf{E} = \mathbf{E}(\alpha)$ we denote the set of all $E(x, t)$ with $x \in \boldsymbol{R}^n$, $t > 0$.

1.4. Let $f \in L_{\mathrm{loc}}(\boldsymbol{R}^n)$. The anisotropic maximal function Mf is defined by

$$Mf(x) = \sup_{t>0} |E(x, t)|^{-1} \int_{E(x,t)} |f(y)| \, \mathrm{d}y . \tag{1.2}$$

If $\alpha_1 = \ldots = \alpha_n$ then $E(x, t)$ is a cube and Mf becomes the usual Hardy-Littlewood maximal function. B. Muckenhoupt [8] gave the complete characterization of the weighted spaces L^p_w, $1 < p < \infty$, for which such an operator $M: L^p_w \to L^p_w$ is continuous. In 1978 B. Muckenhoupt stated the following problems [9]: When, for a given integral operator T and a weight w, is there a weight v such that the operator $T: L^p_w \to L^p_v$ is bounded? And, conversely, when for a given weight v can such a weight w be found that $T: L^p_w \to L^p_v$ is bounded?

In papers of P. Koosis [6], L. Carleson and P. Jones [1], J. L. Rubio de Francia [10], W. S. Young [12], E. T. Sawyer [11] and A. E. Gatto and C. E. Gutiérrez [3] these problems were solved for the Hardy-Littlewood maximal operator and for singular integral operators.

In the present paper we give answers to these questions in the case of anisotropic maximal functions (1.2).

2. THE CHARACTERIZATION OF THE WEIGHT v

2.1. In Theorem 2.4 we shall characterize weights v for which there exists such a weight w that the inequality

$$(2.1) \qquad \int_{R^n} [Mf(x)]^p v(x) \, dx \leqq c \int_{R^n} |f(x)|^p w(x) \, dx$$

holds for all $f \in L^p_w(R^n)$ with a constant c independent of f. The method of proof comes from [3].

First of all we shall prove an analogue of the lemma by C. Fefferman and E. M. Stein [2] for the following modified maximal functions (cf. [8]):

$$(2.2) \qquad f^*(x) = \sup_{t < \tau(x)} |E(x, t)|^{-1} \int_{E(x,t)} |f(y)| \, dy,$$

$$(2.3) \qquad f_*(x) = \sup_{t < 2\tau(x)} |E(z, t)|^{-1} \int_{E(z,t)} |f(y)| \, dy,$$

where the supremum is taken over all $E(z, t) \in x$, and

$$(2.4) \qquad \tau(x) = \tfrac{1}{2}[1 + \max_i (2|x_i|)^{1/\alpha_i}], \quad x \in R^n.$$

Let us note that we can suppose

$$(2.5) \qquad \alpha_i \geqq 1, \quad i = 1, \ldots, n,$$

since $E(x, t) = \tilde{E}(x, t^\gamma)$, where $\gamma = \min_i \alpha_i$ and $\tilde{E}(x, t) = \{y \in R^n;\ |y_i - x_i| \leqq \leqq \frac{1}{2} t^{\alpha_i/\gamma}\}$, and, consequently, $\mathbf{E}(\alpha) = \mathbf{E}(\alpha/\gamma)$.

2.2. Lemma. *Let $1 < p < \infty$ and let f, g be measurable functions, g finite and positive a.e. in $\boldsymbol{R}^n$. Then the inequality*

$$\int_{\boldsymbol{R}^n} [f^*(x)]^p g(x) \, dx \leqq c \int_{\boldsymbol{R}^n} |f(x)|^p g_*(x) \, dx \tag{2.6}$$

holds with a constant $c > 0$ independent of f and g.

Proof. We shall first prove that the operator $f \mapsto f^*$ is of the weak type $(1, 1)$ with respect to the measures μ_{g_*} and μ_g (see (1.1)).

Let $s > 0$ be given. We denote

$$H_s = \{x \in \boldsymbol{R}^n;\ f^*(x) > s\}, \quad \text{and} \quad H_s^m = H_s \cap \{x \in \boldsymbol{R}^n;\ |x| \leqq m\}, \quad m \in \boldsymbol{N}.$$

By (2.2), for each $x \in H_s^m$ there exists $t < \tau(x)$ such that

$$|E(x, t)|^{-1} \int_{E(x,t)} |f(y)| \, dy > s. \tag{2.7}$$

Applying de Guzmán's covering lemma ([4]) we select sequences $x^{(j)} \in H_s^m$ and $t_j > 0$, $j \in \boldsymbol{N}$, so that

$$t_j < \tau(x^{(j)}), \tag{2.8}$$

$$\bigcup_j E(x^{(j)}, t_j) \supset H_s^m, \quad \sum_j \chi_j(x) \leqq \vartheta_n, \quad x \in H_s^m, \tag{2.9}$$

where χ_j stays for the characteristic function of the set $E_j = E(x^{(j)}, t_j)$ and ϑ_n depends only on the dimension n. By (1.1) and (2.8) we obtain

$$\mu_g(H_s^m) \leqq \sum_j \int_{E_j} g(x) \, dx \leqq s^{-1} \sum_j |E_j|^{-1} \int_{E_j} g(x) \, dx \int_{E_j} |f(y)| \, dy. \tag{2.10}$$

However, for $y \in E_j$ we have $(2|y_i - x_i^{(j)}|)^{1/\alpha_i} \leqq t_j$, $i = 1, \ldots, n$, and, according to (2.5),

$$|x_i^{(j)}|^{1/\alpha_i} \leqq |y_i - x_i^{(j)}|^{1/\alpha_i} + |y_i|^{1/\alpha_i}, \quad i = 1, \ldots, n.$$

Hence, by (2.4) and (2.8),

$$\begin{aligned} t_j < \tau(x^{(j)}) &< 2\tau(x^{(j)}) - t_j \leqq \\ &\leqq 1 + \max_i (2|x_i^{(j)}|)^{1/\alpha_i} - \max_i (2|y_i - x_i^{(j)}|)^{1/\alpha_i} \leqq \\ &\leqq 1 + \max_i (2|y_i|)^{1/\alpha_i} = \tau(y). \end{aligned}$$

Consequently,

$$|E_j|^{-1} \int_{E_j} g(x) \, dx \leqq g_*(y), \qquad y \in E_j,$$

and from (2.10) and (2.9) we obtain

$$\mu_g(H_s^m) \leqq s^{-1} \sum_j \int_{E_j} |f(y)| \, g_*(y) \, dy \leqq \vartheta_n s^{-1} \int_{\boldsymbol{R}^n} |f(y)| \, g_*(y) \, dy. \tag{2.11}$$

Passing to the limit for $m \to \infty$ and assuming that ϑ_n depends only on n we can write (2.11) with H_s instead of H_s^m which is the weak type (1,1) inequality for the operator $f \mapsto f^*$ with respect to the measures μ_g and μ_{g_*}.

On the other hand, since $g(x) > 0$ for a.a. $x \in \boldsymbol{R}^n$ and so $g_*(x) > 0$ as well, it can be easily seen, that the operator $f \mapsto f^*$ is continuous from $L^\infty_{g_*}(\boldsymbol{R}^n)$ into $L^\infty_g(\boldsymbol{R}^n)$ and, all the more, of the weak type (∞, ∞) with respect to the measures μ_{g_*} and μ_g.

The assertion of the lemma now follows from the Marcinkiewicz interpolation theorem (see e.g. [13]).

2.3. Remarks. (i) Let

$$\tilde{M} f(x) = \sup |E|^{-1} \int_E |f(y)| \,\mathrm{d}y\,, \tag{2.12}$$

where the supremum is taken over all $E \in \mathbf{E}$ which contain the point x. It can be seen (cf. [5], Lemma 2.3) that

$$M f(x) \leqq \tilde{M} f(x) \leqq 2^{|\alpha|/\gamma} M f(x)\,, \quad x \in \boldsymbol{R}^n\,, \tag{2.13}$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\gamma = \min_i \alpha_i$.

(ii) Let us define the "anisotropic norm" ϱ by

$$\varrho(x) = \Big(\sum_{i=1}^n |x_i|^{2/\alpha_i}\Big)^{|\alpha|/2n}\,, \quad x \in \boldsymbol{R}^n\,. \tag{2.14}$$

One can easily verify that $[1 + \varrho^n(x)]^s \in L^1(\boldsymbol{R}^n)$ if and only if $s < -1$.

2.4. Theorem. *Let v be a weight on $\boldsymbol{R}^n$ and $1 < p < \infty$. The following conditions are equivalent:*

(i) *There exists a weight w positive a.e. in $\boldsymbol{R}^n$ and such that the inequality* (2.1) *holds for all $f \in L^p_w(\boldsymbol{R}^n)$ with a constant independent of f.*

(ii) *Let ϱ be defined by* (2.14). *Then*

$$\int_{\boldsymbol{R}^n} \frac{v(x)}{[1 + \varrho^n(x)]^p} \,\mathrm{d}x < \infty\,. \tag{2.15}$$

If the condition (ii) *is satisfied, the weight w in* (i) *can be taken in the form*

$$w(x) = v_*(x) + [1 + \varrho^n(x)]^\beta\,, \quad \beta > p - 1\,. \tag{2.16}$$

Proof. Suppose first that the condition (i) is fulfilled. Let the function $f > 0$ and the set $E \in \mathbf{E}$ be such that

$$\int_{\boldsymbol{R}^n} f^p(x)\, w(x) \,\mathrm{d}x < \infty \quad \text{and} \quad 0 < \int_E f(x) \,\mathrm{d}x < \infty\,.$$

There exists $t > 0$ such that $E \subset E(0, t)$. Then for all $y \in E$ we have $(2|y_i|)^{1/\alpha_i} \leqq t$ and for $x \in \boldsymbol{R}^n$ (by use of (2.5))

$$(2|y_i - x_i|)^{1/\alpha_i} \leqq 2^{1/\alpha_i}(|y_i|^{1/\alpha_i} + |x_i|^{1/\alpha_i}) \leqq t + \max_i (2|x_i|)^{1/\alpha_i} .$$

Thus, for all $x \in \boldsymbol{R}^n$,

$$E \subset E(x, t + \max_i (2|x_i|)^{1/\alpha_i}) ,$$

and so

$$M f(x) \geqq |E(x, t + \max_i (2|x_i|)^{1/\alpha_i})|^{-1} \int_E f(y) \, \mathrm{d}y . \tag{2.17}$$

By simple estimates we get

$$|E(x, t + \max_i (2|x_i|)^{1/\alpha_i})| = [t + \max_i (2|x_i|)^{1/\alpha_i}]^{|\alpha|} \leqq c_1[1 + \varrho^n(x)] \tag{2.18}$$

with $c_1 > 0$ independent of $x \in \boldsymbol{R}^n$. Hence, from (2.17) and (2.18) we conclude

$$\int_{\boldsymbol{R}^n} \frac{v(x)}{[1 + \varrho^n(x)]^p} \, \mathrm{d}x \leqq c_2 \int_{\boldsymbol{R}^n} [M f(x)]^p \, v(x) \, \mathrm{d}x \leqq$$

$$\leqq c c_2 \int_{\boldsymbol{R}^n} f^p(x) \, w(x) \, \mathrm{d}x < \infty ,$$

which is (2.15).

Conversely, suppose that the condition (ii) is fulfilled. Since $p > 1$, by Remark 2.3 (ii), $\int_{\boldsymbol{R}^n} [1 + \varrho^n(x)]^{-p} \, \mathrm{d}x < \infty$. Hence, the function $v + 1$ satisfies the condition (ii) as well, and so we can suppose that v is positive.

We can write

$$M f(x) \leqq f^*(x) + f^\#(x) , \tag{2.19}$$

where f^* is given by (2.2) and

$$f^\#(x) = \sup_{t \geqq \tau(x)} |E(x, t)|^{-1} \int_{E(x,t)} |f(y)| \, \mathrm{d}y .$$

According to Lemma 2.2 there is a constant $c_3 > 0$ such that

$$\int_{\boldsymbol{R}^n} [f^*(x)]^p \, v(x) \, \mathrm{d}x \leqq c_3 \int_{\boldsymbol{R}^n} |f(x)|^p \, v_*(x) \, \mathrm{d}x . \tag{2.20}$$

Similarly as in (2.18) we obtain for $t \geqq \tau(x)$ the estimate

$$|E(x, t)| \geqq c_4[1 + \varrho^n(x)] .$$

By means of Hölder's inequality, for $\beta \in \boldsymbol{R}^1$ we get

$$f^\#(x) \leqq c_4^{-1}[1 + \varrho^n(x)]^{-1} \int_{\boldsymbol{R}^n} |f(y)| \, \mathrm{d}y \leqq$$

$$\leqq c_4^{-1}[1 + \varrho^n(x)]^{-1} \left(\int_{\boldsymbol{R}^n} [1 + \varrho^n(z)]^{-\beta p'/p} \, \mathrm{d}z \right)^{1/p'} \times$$

$$\times \left(\int_{R^n} |f(y)|^p \, [1 + \varrho^n(y)]^\beta \, \mathrm{d}y \right)^{1/p}$$

and, following the Remark 2.3 (ii), for $\beta > p - 1$

$$\int_{R^n} [f^\#(x)]^p \, v(x) \, \mathrm{d}x \leqq \tag{2.21}$$

$$\leqq c_5 \left(\int_{R^n} \frac{v(x)}{[1 + \varrho^n(x)]^p} \, \mathrm{d}x \right) \int_{R^n} |f(y)|^p \, [1 + \varrho^n(y)]^\beta \, \mathrm{d}y \, .$$

According to (2.15) the first integral on the right hand side of (2.21) is finite.

Since $v_*(x)$ is finite for a.a. $x \in R^n$, we conclude from (2.19), (2.20) and (2.21) that the inequality (2.1) holds with the weight w defined by (2.16).

3. THE INVERSE PROBLEM

3.1. Now we turn our attention to the question for which weights w there exists a weight v such that the operator M defined by (1.2) is bounded from L^p_w into L^p_v. The characterization of such weights and the idea of the proof is due to J. L. Rubio de Francia [10].

Theorem. *Let w be a weight positive a.e. in R^n. Let $1 < p < \infty$. The following conditions are equivalent:*

(i) *There exists a weight v positive a.e. in R^n and such that the inequality* (2.1) *holds for all $f \in L^p_w(R^n)$ with a constant independent of f.*

(ii) $w^{-p'/p} \in L_{\mathrm{loc}}(R^n)$ *and*

$$\limsup_{t \to \infty} |E(0, t)|^{-p'} \int_{E(0,t)} w^{-p'/p}(x) \, \mathrm{d}x < \infty \, .$$

Let us recall several assertions which we shall employ in the proof of the theorem:

3.2. Proposition. (B. Maurey [7], Corollary 5 of Theorem 2). *Let $E \subset R^n$ be a measurable set, $0 < q \leqq p \leqq \infty$, $1/q = 1/p + 1/r$, and let I be a set of indices. Let $\{f_i; \, i \in I\}$ be such a set of functions from $L^q(E)$ that*

$$\int_E \left(\sum_{i \in I} |\alpha_i f_i|^p \right)^{q/p} \mathrm{d}x < \infty$$

for each system $\{\alpha_i \in R^1; \, i \in I\}$ with

$$\sum_{i \in I} |\alpha_i|^p < \infty \, .$$

Then there exists a function $g \in L^r(E)$ such that

$$\int_E |f_i(x) \, g^{-1}(x)|^p \, \mathrm{d}x \leqq 1 \quad \textit{for all} \quad i \in I \, .$$

3.3. Let (Y, S, v) be a σ-finite measure space, T a σ-algebra of Lebesgue measurable sets in R^n. On the σ-algebra $T \times S$ we define the measure λ as the product of the Lebesgue measure and of v. For a λ-measurable function $f: R^n \times Y \to R^1$ we define the vector-valued anisotropic maximal function

$$M_{(1)}f(x, y) = \sup_{t>0} |E(x, t)|^{-1} \int_{E(x,t)} |f(z, y)| \, dz .$$

In [5], Lemma 3.1 an assertion is proved a special case of which we state here:

Proposition. *Let* $1 < \vartheta < \infty$. *Let a weight* w *in* R^n *satisfy the condition* $A_1(\alpha)$, *i.e.*

$$Mw(x) \leqq c_1 \, w(x) \quad \textit{for a.a.} \quad x \in R^n .$$

Then there exists a constant $c_2 > 0$ *such that for all* $s > 0$ *and for all* λ-*measurable functions* $f: R^n \times Y \to R^1$,

$$\mu_w \left\{ x \in R^n; \left(\int_Y [M_{(1)}f(x, y)]^\vartheta \, dv \right)^{1/\vartheta} > s \right\} \leqq$$

$$\leqq c_2 s^{-1} \int_{R^n} \left(\int_Y |f(x, y)|^\vartheta \, dv \right)^{1/\vartheta} w(x) \, dx .$$

3.4. The following analogue of Kolmogorov's inequality can be derived in the usual way from Proposition 3.3:

Proposition. *Let* $0 < p < 1 \leqq \vartheta < \infty$. *If the weight* w *satisfies the condition* $A_1(\alpha)$, *then there exists a constant* $c > 0$ *such that the inequality*

$$\int_e \left(\int_Y [M_{(1)}f(x, y)]^\vartheta \, dv \right)^{p/\vartheta} w(x) \, dx \leqq$$

$$\leqq \frac{c}{1 - p} (\mu_w e)^{1-p} \left(\int_{R^n} \left(\int_Y |f(x, y)|^\vartheta \, dv \right)^{1/\vartheta} w(x) \, dx \right)^p$$

holds for all $e \subset R^n$, $\mu_w e < \infty$ *and for all* λ-*measurable functions* $f: R^n \times Y \to R^1$.

3.5. Proof of Theorem 3.1. Suppose that the condition (i) of the theorem is satisfied. Since $v > 0$ a.e. in R^n, it can be deduced in the usual way that $w^{-p'/p} \in L_{\text{loc}}(R^n)$. Denoting $E = E(0, t)$ and $f(z) = w^{-p'/p}(x) \chi_E(x)$, where χ_E is the characteristic function of the set E, we have

$$M f(x) \geqq c_1 \tilde{M} f(x) \geqq c_1 |E|^{-1} \int_E w^{-p'/p}(y) \, dy , \quad x \in E ,$$

(cf. Remark 2.3 (i)) and

$$\int_{R^n} f^p(x) \, w(x) \, dx = \int_E w^{-p'/p}(x) \, dx .$$

Hence by (2.1),

$$\int_E v(x)\,\mathrm{d}x\left(|E|^{-p'}\int_E w^{-p'/p}(x)\,\mathrm{d}x\right)^{p-1} \leqq c_2\,,$$

and the second condition of (ii) follows since

$$\limsup_{t\to\infty}\int_{E(0,t)} v(x)\,\mathrm{d}x > 0\,.$$

On the contrary, let us suppose that the weight w satisfies the condition (ii) of Theorem 3.1. We cover $\boldsymbol{R}^n$ by a sequence of non-overlapping parallelepipeds $E_j \in \mathbf{E}$ and for each j we shall prove that there exists a weight v_{E_j} positive on E_j and such that

$$\int_{E_j}[Mf(x)]^p\, v_{E_j}(x)\,\mathrm{d}x \leqq \int_{\boldsymbol{R}^n}|f(x)|^p\, w(x)\,\mathrm{d}x\,. \tag{3.1}$$

Then the inequality (2.1) holds with $v(x) = \sum_{j=1}^{\infty} 2^{-j}\, v_{E_j}(x)\,\chi_{E_j}(x)$.

So, let $E \in \mathbf{E}$ be given. There exists $T > 0$ such that

$$E \subset E(0, T)\,, \tag{3.2}$$

$$|E(0,t)|^{-p'}\int_{E(0,t)} w^{-p'/p}(x)\,\mathrm{d}x \leqq K < \infty \quad \text{for} \quad t \geqq T\,. \tag{3.3}$$

Given a number $t > 0$ we set $\tilde{t} = 2^{1/\gamma}t$, $\gamma = \min_i \alpha_i$. For $f \in L^p_w(\boldsymbol{R}^n)$ we denote $f''(x) = f(x)\,\chi_{E(0,\tilde{T})}(x)$ and $f'(x) = f(x) - f''(x)$. If $y \in E(0, T)$ and $t > 0$ then for $z \in E(y, t)$ we have

$$|z_i| \leqq |y_i| + |y_i - z_i| \leqq \tfrac{1}{2}T^{\alpha_i} + \tfrac{1}{2}t^{\alpha_i}\,, \quad i = 1, \ldots, n\,,$$

i.e.

$$|z_i| \leqq \begin{cases} \tfrac{1}{2}\tilde{T}^{\alpha_i} & \text{for} \quad t \leqq T, \\ \tfrac{1}{2}\tilde{t}^{\alpha_i} & \text{for} \quad t > T. \end{cases}$$

So we get

$$E(y, t) \subset E(0, \tilde{T}) \quad \text{for} \quad t \leqq T, \tag{3.4}$$

$$E(y, t) \subset E(0, \tilde{t}) \quad \text{for} \quad t > T, \tag{3.5}$$

and, moreover,

$$|E(0, \tilde{t})| = 2^{|\alpha|/\gamma}|E(0, t)|\,. \tag{3.6}$$

It follows from (3.2)–(3.6) that for $x \in E$,

$$Mf'(x) \leqq \sup_{t > T}|E(0,t)|^{-1}\int_{E(0,\tilde{t})}|f'(y)|\,\mathrm{d}y \leqq$$

$$\leqq 2^{|\alpha|/\gamma} \sup_{t>T} |E(0, \tilde{t})|^{-1} \left(\int_{E(0,\tilde{t})} w^{-p'/p}(y)\, dy \right)^{1/p'} \times$$

$$\times \left(\int_{R^n} |f'(y)|^p\, w(y)\, dy \right)^{1/p} \leqq 2^{|\alpha|/\gamma} K^{p'} \|f'\|_{p,w} .$$

Integrating this inequality over E we obtain

$$\int_E [M f'(x)]^p\, |E|^{-1}\, c_1\, dx \leqq \int_{R^n} |f'(x)|^p\, w(x)\, dx , \tag{3.7}$$

where $c_1 = 2^{-|\alpha|p/\gamma} K^{-pp'}$.

Now, we shall seek the weight for estimating Mf'' by means of Maurey's factorization theorem (Proposition 3.2). Let $H = \{h_i;\ i \in I\}$ be the set of all functions $h \in L^p_w(R^n)$ with supp $h \subset E(0, \tilde{T})$ and such that

$$\int_{R^n} |h(x)|^p\, w(x)\, dx \leqq 1 . \tag{3.8}$$

Let $\{\alpha_i \in R^1;\ i \in I\}$ be such that $\sum_{i\in I} |\alpha_i|^p < \infty$ and let $0 < q < 1$. By Proposition 3.4 there exists $c_2 > 0$ such that

$$\int_E (\sum_{i\in I} |\alpha_i M h_i(x)|^p)^{q/p}\, dx \leqq \tag{3.9}$$

$$\leqq \frac{c_2}{1-q} |E|^{1-q} \left(\int_{R^n} (\sum_{i\in I} |\alpha_i h_i(x)|^p)^{1/p}\, dx \right)^q .$$

Using the Hölder inequality and the Fubini theorem we obtain

$$\int_{R^n} (\sum_{i\in I} \alpha_i\, h_i(x)|^p)^{1/p}\, dx \leqq \tag{3.10}$$

$$\leqq \left(\int_{E(0,\tilde{T})} \sum_{i\in I} |\alpha_i\, h_i(x)|^p\, w(x)\, dx \right)^{1/p} \left(\int_{E(0,\tilde{T})} w^{-p'/p}(x)\, dx \right)^{1/p'} .$$

From (3.3), (3.8), (3.9) and (3.10) conclude that

$$\int_E (\sum_{i\in I} |\alpha_i M h_i(x)|^p)^{q/p}\, dx \leqq c_3 (\sum_{i\in I} |\alpha_i|^p)^{q/p} < \infty ,$$

where c_3 depends on c_2, p, q, w and T. Since the last estimate verifies that the set $\{Mh;\ h \in H\}$ satisfies the assumptions of Proposition 3.2, there exists a function $g \in L^r(E)$, $1/r = 1/q - 1/p$, such that

$$\int_E [Mh(x)]^p\, |g(x)|^{-p}\, dx \leqq 1 \quad \text{for all} \quad h \in H .$$

In particular, if we take $h = f''\|f''\|_{p,w}^{-1}$, we obtain

$$\int_E [Mf''(x)]^p |g(x)|^{-p} \, dx \leqq \int_{R^n} |f''(x)|^p w(x) \, dx . \tag{3.11}$$

If we put $v_E(x) = 2^{1-p} \min \left(|g(x)|^{-p}, c_1|E|^{-1}\right)$, $x \in E$, the estimate (3.1) follows from (3.7) and (3.11).

References

[1] *L. Carleson, P. Jones:* Weighted norm inequalities and a theorem of Koosis. Mittag-Leffler Inst., Rep. No. 2, 1981.

[2] *C. Fefferman, E. M. Stein:* Some maximal inequalities. Amer. J. Math. Soc. *93* (1971), 107—115.

[3] *A. E. Gatto, C. E. Gutiérrez:* On weighted norm inequalities for the maximal function. Studia Math. *81* (1983), 59—62.

[4] *M. de Guzmán:* A covering lemma with applications to differentiability of measures and singular integral operators. Studia Math. *34* (1970), 290—317.

[5] *V. M. Kokilashvili, J. Rákosník:* Weighted inequalities for vector-valued anisotropic maximal functions. (To appear in Z. Anal. Anwendungen (1985).)

[6] *P. Koosis:* Moyennes quadratiques pondérées de fonctions périodiques et de leur conjuguées harmoniques. C. R. Acad. Sci. Paris, Ser. A 291 (1980), 255—257.

[7] *B. Maurey:* Théorèmes de factorisation pour les operateur linéaires à valeur dans les espaces L^p. Astérisque, No. 11, Soc. math. de France, 1974.

[8] *B. Muckenhoupt:* Weighted norm inequalities for the Hardy maximal function. Trans. Amer. Math. Soc. *165* (1972), 207—226.

[9] *B. Muckenhoupt:* Weighted norm inequalities for classical operators. Harm. Anal. Euclidean Spaces. Proc. Symp. Pure Math., vol. 35, part 1, Amer. Math. Soc., Providence, R. I. 1979, 51—60.

[10] *J. L. Rubio de Francia:* Boundedness of maximal functions and singular integrals in weighted L^p spaces. Proc. Amer. Math. Soc. 83 (1981), 673—679.

[11] *E. T. Sawyer:* Two weighted norm inequalities for certain maximal and integral operators. Proc. Conference Harmonic Analysis. Lecture Notes Math., vol. 908, Springer-Verlag Berlin—Heidelberg—New York, 1982, 102—107.

[12] *W. S. Young:* Weighted norm inequalities for the Hardy-Littlewood maximal functions. Proc. Amer. Math. Soc. *85* (1982), 24—26.

[13] *A. Zygmund:* Trigonometric series, vol. II, Second Ed. Cambridge Univ. Press, 1959.

Authors' addresses: V. M. Kokilashvili, Mathematical Institute of Georgian Academy of Sciences, ul. Ruchadze 1, 380093 Tbilisi, USSR; J. Rákosník, 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).