Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 110 (1985), No. 3, 237--240

Persistent URL: `http://dml.cz/dmlcz/118230`

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SOME REMARKS ON A RECENT THIRD ORDER NONLINEAR OSCILLATION RESULT

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(Received October 11, 1983)

In a recent paper [2], Mehri considers the problem of oscillation of the solutions of the nonlinear third order differential equation

$$x''' + f(t, x) = 0, \tag{1}$$

where $f(t, x)$ is a continuous function of the variables $t \geqq t_0$ and $|x| < \infty$, satisfying the following sign and monotonicity conditions:

$$x f(t, x) > 0, \quad x \neq 0 \quad \text{for all} \quad t \geqq t_0, \tag{2}$$

$$|f(t, x_1)| \leqq |f(t, x_2)| \quad \text{if} \quad |x_1| \leqq |x_2|, \quad x_1 x_2 \geqq 0. \tag{3}$$

Mehri shows ([2], Theorem 1) that for equation (1) to be oscillatory (i.e., all non-trivial solutions of (1) to be oscillatory) it is necessary that conditions

$$\int^{\infty} t^2 |f(t, C)| \, dt = \infty, \quad \int^{\infty} |f(t, Ct^2)| \, dt = \infty \tag{4}$$

be satisfied for any number $C \neq 0$.

Then he gives the following theorem ([2], Theorem 2) about the sufficiency condition for equation (1) to be oscillatory.

Theorem M. *If the condition*

$$\int^{\infty} |f(t, C)| \, dt = \infty \tag{5}$$

is satisfied for every constant $C \neq 0$, then (1) is oscillatory.

Mehri also gives the following corollary for the special case of (1), namely

$$x''' + a(t) f(x) = 0. \tag{6}$$

Corollary M. *Let $a(t) \geqq 0$, $f(x)$ be continuous functions satisfying the conditions*

$$x f(x) > 0, \quad x \neq 0 \tag{7}$$

$$|f(x_1)| \leqq |f(x_2)| \quad \textit{when} \quad |x_1| \leqq |x_2|\,, \quad x_1 x_2 \geqq 0\,,$$

and

(8) $$\sup |f(x)| < \infty\,.$$

Then (6) *is oscillatory if and only if*

(9) $$\int^{\infty} a(t)\,\mathrm{d}t = \infty\,.$$

The argument given in the second part of the proof of Theorem M is incorrect and the falsity of Theorem M and Corollary M can be shown by the following examples.

Example 1. $f(t, x) = x^3 e^{2t}$, $t \geqq 0$; $|x| < \infty$.

Example 2. Let $$f_1(x) = \begin{cases} x^3\,, & |x| \leqq 1 \\ 2 - \dfrac{1}{x}\,, & x > 1 \\ -2 - \dfrac{1}{x}\,, & x < -1\,, \end{cases}$$

$$a(t) = e^{2t} \quad \text{for} \quad t \geqq 0 \quad \text{and} \quad f(t, x) = a(t) f_1(x)\,.$$

Remarks.

(i) $f(t, x)$ in Example 1 (or Example 2) satisfies the conditions (2)–(5), and $x(t) = e^{-t}$ is a bounded nonoscillatory solution of (1). Hence Theorems 2 and 3 of [2] are false*); and condition (4) (though a necessary condition), is not a sufficient condition.

(ii) $f(t, x)$ in Example 2 satisfies (7)–(9), and $x(t) = e^{-t}$ is a bounded nonoscillatory solution of (6). Hence the sufficiency part of Corollary *M* is false and the necessary part follows from Theorem 1 of [2] which does not require (8).

(iii) For $f(t, x)$ in Example 2 we have:

(a) for each $\delta > 0$,

(10) $$\left| \int^{\infty} \inf_{\delta \leqq |x| < \infty} f(t, x)\,\mathrm{d}t \right| = \infty\,;$$

(b) $f(t, x)$ is strongly lower semi-continuous from the left for $x > 0$, upper semi-continuous from the right for $x < 0$, smooth at infinity and also

*) **Theorem 3** of [2]. *If for any nonzero constant C we can find constants $\lambda \neq 0$ and $M > 0$, depending on C, such that the inequality $|f(t, C)| \geqq M|f(t, \lambda t^2)|$ is satisfied for t sufficiently large, then for every solution of equation* (1) *to be oscillary condition* (5) *is necessary and sufficient.*

(11) $$\left|\int^{\infty} f(t, x)\, dt\right| = \infty \quad \text{for each} \quad x \neq 0\,,$$

and

(12) $$|f(t, x)| \geqq 1 \quad \text{for all } t \text{ and all} \quad |x| \geqq 1\,;$$

(c)

(13) $$\liminf_{|x| \to \infty} |f_1(x)| > 0\,,$$

(d)

(14) $$\lim_{|x| \to \infty} \int_0^x f_1(u)\, du = \infty\,.$$

Therefore various sufficiency conditions similar to those given in Theorem 1 and Corollaries 2 and 3 of [3] for second order nonlinear equations will not be adequate for (1) or (6).

(iv) The following result in the frame-work of [2] is possible.

Under conditions (2), (3) *and* (5) *every solution x of* (1) *is oscillatory or such that*

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} x'(t) = \lim_{t \to \infty} x''(t) = 0 \quad \textit{monotonically}\,.$$

Proof. Assume the contrary and let $x(t)$ be a nonoscillatory solution which may be assumed to be positive for $t \geqq t_0$. Then $x'''(t) < 0$; hence $x''(t)$ is non-increasing and $x'(t)$ is concave and consequently $x''(t) > 0$ for $t \geqq t_0 > 0$. Now, if $x'(t) > 0$ for $t \geqq t_0$, then $x(t)$ is non-decreasing and

$$x''(t) = x''(t_0) - \int_{t_0}^t f(s, x(s))\, ds \leqq x''(t_0) - \int_{t_0}^t f(s, x(t_0))\, ds\,.$$

This implies $\lim_{t \to \infty} x''(t) = -\infty$ which is a contradiction. If $x'(t) < 0$ for $t \geqq t_0$, then $x(t)$ is non-increasing. Here we consider two cases:

Case 1. $\lim_{t \to \infty} x(t) = \alpha > 0$. Then $x(t) \geqq \alpha$ for $t \geqq t_1$. From the identity

$$t_1\, x''(t_1) - x'(t_1) = t\, x''(t) - x'(t) + \int_{t_1}^t s f(s, x(s))\, ds\,,$$

it follows that

$$A = \frac{t_1 x''(t_1) - x'(t_1)}{t_1} \geqq \int_{t_1}^t f\,(s, \alpha)\, ds\,,$$

which is a contradiction.

Case 2. $\lim_{t \to \infty} x(t) = 0$. From the fact that $x(t) > 0$, $x''(t) > 0$ for $t \geqq t_0$ it follows that $x'(t)$ is non-decreasing and $\lim_{t\to\infty} x'(t) = \beta$ where $-\infty < \beta \leqq 0$. This implies that $x'(t) \leqq \beta$ or f all $t \geqq t_0$, and hence $x(t_0) \geqq x(t) - \beta(t - t_0)$ which is impossible for $\beta < 0$. Therefore $\lim_{t\to\infty} x'(t) = 0$. Now $x'(t) < 0$, $x'''(t) < 0$ for $t \geqq t_0$ imply that $x''(t)$ is non-increasing and $\lim_{t\to\infty} x''(t) = \gamma$ where $0 \leqq \gamma < \infty$. This implies that

$$x'(t_0) \leqq x'(t) - \gamma(t - t_0) \quad \text{for} \quad t \geqq t_0$$

which again is impossible for $\gamma > 0$, and hence $\gamma = 0$.

(v) Conclusions of (iv) hold for equation (6) under (7) and (9).

Acknowledgement. The author is very grateful to Professor V. Ďurikovič for his valuable comments and suggestions.

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