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ON THE SET OF POINTS OF DISCONTINUITY FOR FUNCTIONS WITH CLOSED GRAPHS

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For two topological spaces X and Y and any function $f : X \to Y$ the subset $\{(x, f(x)); x \in X\}$ of the space $X \times Y$ (with Tychonoff's topology) is called the graph of f and is denoted by $G(f)$. We denote by $C_f(D_f)$ the set of all such points at which the function f defined on X is continous (discontinuous).

I. Baggs [1] dealt with the set of points of discontinuity of functions with closed graphs. In this paper we shall generalize some results of the paper [1].

1. PRELIMINARIES

First we recall definitions and some basic properties.

Proposition A. *Let a function $f : X \to Y$ have a closed graph. If K is a compact subset of Y then $f^{-1}(K)$ is a closed subset of X.* (See [2; Theorem 3.6].)
This proposition has the following corollary.

Proposition B. *Let a function $f : X \to Y$ have a closed graph. Then $f^{-1}(y)$ is a closed subset of X for each $y \in Y$.* (See [7; Theorem 1].)

Proposition C. *Let $f : X \to Y$ be any function where Y is a locally compact Hausdorff space. If for each compact $K \subset Y, f^{-1}(K)$ is closed, then $G(f)$ is closed.* (See [9; Theorem 6].)

Definition 1. Let X and Y be topological spaces, let $f : X \to Y$ be a function and let $p \in X$. Then f is said to be *c-continuous at p* provided the following holds: if U is an open subset of Y containing $f(p)$ such that $Y - U$ is compact, then there is an open subset V of X containing p such that $f(V) \subset U$. The function f is said to be *c-continuous (on X)* provided f is c-continuous at each point of X. (See [3; Definition 1].)

Proposition D. *Let $f : X \to Y$ be any function where Y is Hausdorff. Then the following statements are equivalent:*

(1) *f is c-continuous, and*

(2) *if K is a compact subset of Y, then $f^{-1}(K)$ is a closed subset of X.* (See [3; Theorem 1].)

Corollary 1. *Let $f : X \to Y$ be any function where Y is a locally compact Hausdorff space. Then f is c-continuous if and only if $G(f)$ is closed.*

Definition 2. A function $f : X \to Y$ is *locally bounded at* $x_0 \in X$ if and only if there exists a compact subset K of Y such that $x_0 \in \operatorname{Int}(f^{-1}(K))$. We denote by B_f the set of all such points at which the function f is locally bounded.

Lemma A. *Let $f : X \to Y$ be given. Then $G(f)$ is closed if and only if for each $x \in X$ and $y \in Y$, where $y \neq f(x)$, there exist open sets U and V containing x and y, respectively, such that $f(U) \cap V = \emptyset$.* (See [8; Lemma].)

Theorem 1. *Let $f : X \to Y$ be given. If $G(f)$ is closed, then*

$$B_f \subset C_f .$$

Proof. We may assume that Y has at least two elements (in the opposite case we evidently have $B_f = C_f$). Let the set $G(f)$ be closed. Let $x_0 \in B_f$. By Definition 2 there exists a compact set K (in Y) such that $x_0 \in \operatorname{Int}(f^{-1}(K))$. Let V be an open neighbourhood of the point $f(x_0)$. Since $K - V$ is compact and $G(f)$ is closed, $f^{-1}(K - V)$ is closed by Proposition A. Put

$$U = \operatorname{Int}(f^{-1}(K)) - f^{-1}(K - V) .$$

Evidently U is an open neighbourhood of the point x_0. We shall prove that $f(U) \subset V$. Let $x \in U$. Since $f(x) \in K$ and $f(x) \notin K - V$, evidently $f(x) \in V$. Hence $x_0 \in C_f$.

Corollary 2. *Let $f : X \to Y$ be any function where Y is a locally compact space. If $G(f)$ is closed, then $B_f = C_f$.*

The converse to Corollary 2 is not necessarily true as the following example shows.

Example 1. Let $X = Y = R$ (where R denotes the set of all real numbers) with the usual topology. Define a function $f : X \to Y$ as follows:

$$f(x) = \begin{cases} \dfrac{1}{x} \sin \dfrac{1}{x} & \text{for} \quad x \neq 0 ; \\ 0 & \text{for} \quad x = 0 . \end{cases}$$

Then $G(f)$ is not closed, but Y is locally compact and $B_f = C_f$.

Lemma 1. *Let the function $f : X \to Y$ have a closed graph. If K is a compact subset of Y then $f^{-1}(K) - B_f$ is nowhere dense in X.*

Proof. Let K be a compact subset of Y. Put

$$A = f^{-1}(K) - B_f .$$

By Definition 2 it is easy to see that B_f is open. By Proposition A the set $f^{-1}(K)$ is closed. Therefore A is closed. Now we shall prove that $\mathrm{Int}(A) = \emptyset$. Let $x \in \mathrm{Int}(A)$. Then A is a neihbourhood of the point x such that $f(A) \subset K$. Since K is compact by Definition 2 we have $x \in B_f$. This leads to a contradiction because $x \in A \subset \subset X - B_f$. Therefore A is nowhere dense in X.

Theorem 2. *Let $f : X \to Y$ be any function where Y is a σ-compact space (i.e. Y is the countable union of compact sets). If $G(f)$ is closed, then $X - B_f$ is closed and of the first category (in X).*

Proof. By the assumption, $Y = \bigcup_{n=1}^{\infty} K_n$, where each K_n is compact. Let $n \in N$ (where N denotes the set of all positive integers). Put

$$A_n = f^{-1}(K_n) - B_f .$$

By Lemma 1 the set A_n is nowhere dense in X. Hence $X - B_f = \bigcup_{n=1}^{\infty} A_n$ is of the first category in X.

2. REAL FUNCTIONS WITH CLOSED GRAPHS

Let X be a topological space. Denote by $U(X)$ the class of all real functions defined on X with closed graphs.

From Corollary 2, Lemma 1, Proposition B and Theorem 2 we obtain the following three theorems.

Theorem 3. *Let $f \in U(X)$. Then the set D_f is closed and of the first category (in X).*

Theorem 4. *Let $f \in U(X)$. Then $D_f \cap f^{-1}(0)$ is closed and nowhere dense (in X).*

Theorem A. *Let X be a T_2 Baire space. If $f : X \to R^n$ (R^n – the Euclidean n-space) has a closed graph, then D_f is closed and nowhere dense in X.* (See [1], and for metric spaces see [5; Theorems 4 and 5].)

Theorem B. *A set $F \subset R$ is closed and nowhere dense if and only if there exists a function $f : R \to R$ such that f has a closed graph and $D_f = F$.* (See [1].)

Theorem 5. *Let F be a closed, G_δ and nowhere dense subset of a normal topological space X. Let $u : X \to \langle 0, 1 \rangle$ be a continuous function such that $u^{-1}(0) = F$. Define a function $g : X \to R$ as follows:*

$$g(x) = \begin{cases} 1/u(x) & \textit{for} \quad x \in X - F\,, \\ 0 & \textit{for} \quad x \in F\,. \end{cases}$$

Then g has a closed graph and $D_g = F$.

Proof. We show that the graph of g is closed. Let $\{(x_\alpha, g(x_\alpha))\}_{\alpha \in A}$ be a convergent net of points of the graph of g, i.e. there exist x_0 and y_0 such that $(x_\alpha, g(x_\alpha)) \to$ $\to (x_0, y_0) \in X \times R$. We distinguish two cases.

a. Let there exist α_0 such that for every $\alpha > \alpha_0$ we have $x_\alpha \in F$. Since $x_\alpha \to x_0$ and F is closed, we obtain $x_0 \in F$. Hence $g(x_0) = 0 = y_0$.

b. For each α let there exist $\beta > \alpha$ such that $x_\beta \notin F$. It follows from the definition of g that $g(x) \geqq 1$ whenever $x \in X - F$. The convergence of the net $\{g(x_\alpha)\}_{\alpha \in A}$ implies that there is α_0 such that for every $\alpha > \alpha_0$ we have $x_\alpha \in X - F$. Since u is continuous at the point x_0 and $g(x_\alpha) \to y_0$, we obtain $x_0 \in X - F$. Since g is continuous on the set $X - F$, it is not difficult to verify that $g(x_\alpha) \to g(x_0)$. Hence $g(x_0) = y_0$.

Finally, we show that $D_g = F$. Evidently g is continuous on the set $X - F$. Let $x \in F$. Because the set F is nowhere dense, we have $\omega_g(x) \geqq 1$ for the oscillation of g in x. Hence $x \in D_g$. The following example shows that there exists a metric space X and a function $f \in U(X)$ such that D_f is not nowhere dense.

Example 2. Let $X = \{x_1, x_2, \ldots\}$ be a countably dense subset of R. Define a function $f : X \to R$ as follows:

$$f(x_n) = n \quad (n = 1, 2, \ldots)\,.$$

Then f has a closed graph, but $D_f = X$ is not nowhere dense in X.

Proposition 1. *Let X be a topological space. Let $f \in U(X)$. Then $|f| \in U(X)$.*

Proof. Let $x_0 \in X$. Let $y \neq |f(x_0)|$. First suppose that $y \geqq 0$. Since $y \neq f(x_0)$, by Lemma A there exist $\delta_1 > 0$ and a neighbourhood U_1 of the point x_0 such that

$$f(U_1) \cap (y - \delta_1,\ y + \delta_1) = \emptyset\,.$$

Since $-y \neq f(x_0)$, by Lemma A there exist $\delta_2 > 0$ and neighbourhood U_2 of the point x_0 such that

$$f(U_2) \cap (-y - \delta_2,\ -y + \delta_2) = \emptyset\,.$$

Put

$$U = U_1 \cap U_2\,,$$

$$\delta = \min(\delta_1, \delta_2)\,,$$

$$V = (y - \delta, \; y + \delta).$$

Let $x \in U$. If $f(x) \geqq 0$, since $f(x) \notin (y - \delta_1, \; y + \delta_1)$, we have $\big| |f(x)| - y \big| = |f(x) - y| \geqq \delta_1 \geqq \delta$. Therefore $|f(x)| \notin V$. If $f(x) < 0$, then $f(x) \notin (-y - \delta_2, -y + \delta_2)$, hence $\big| y - |f(x)| \big| = |y + f(x)| \geqq \delta_2 \geqq \delta$. Therefore $|f(x)| \notin V$. In the case $y < 0$ put $U = X$, $V = (-\infty; 0)$. Then by Lemma A the function f has a closed graph.

Proposition 2. *Let X be a topological space. Let α be a real number. Let $f \in U(X)$. Then $\alpha . f \in U(X)$.*

Proof. It is obvious that for $\alpha = 0$ we have $\alpha . f \in U(X)$. Suppose that $\alpha \neq 0$. Let $x_0 \in X$. Let K be a compact subset of R such that $\alpha . f(x_0) \notin K$. Since K is closed, there exists $\varepsilon > 0$ such that

$$(3) \qquad (\alpha . f(x_0) - \varepsilon, \;\; \alpha . f(x_0) + \varepsilon) \cap K = \emptyset .$$

Let $k > 0$ be a bound of the set K (i.e., $K \subset \langle -k, k \rangle$). Put

$$h = \max(k, k/|\alpha|),$$

$$K_1 = \langle -h, h \rangle - (f(x_0) - \varepsilon/|\alpha|, f(x_0) + \varepsilon/|\alpha|).$$

Since $f(x_0) \notin K_1$ and K_1 is compact, there exists a neighbourhod U of the point x_0 such that

$$(4) \qquad f(U) \cap K_1 = \emptyset .$$

Let $x \in U$. If $\alpha . f(x) \notin \langle -k, k \rangle$, evidently $\alpha . f(x) \notin K$. Let $\alpha . f(x) \in \langle -k, k \rangle$. Then $f(x) \in \langle -h, h \rangle$, therefore by (4) we have $f(x) \in (f(x_0) - \varepsilon/|\alpha|, f(x_0) + \varepsilon/|\alpha|)$. Thus $|\alpha . f(x) - \alpha . f(x_0)| = |\alpha| . |f(x) - f(x_0)| < |\alpha| . \varepsilon/|\alpha| = \varepsilon$, hence by (3) we have $\alpha . f(x) \notin K$. Then Corollary 1 yields $\alpha . f \in U(X)$.

Remark 1. Propositions 1 and 2 are proved in the paper [6] for X a metric space.

It is known that the class $U(X)$ is not closed with respect to addition (see [6; Example 3]). We prove that $U(X)$ is closed with respect to addition of nonnegative functions.

Theorem 6. *Let X be a topological space. Let $f, g \in U(X)$ be nonnegative functions. Then $f + g \in U(X)$.*

Proof. Let $x_0 \in X$. Let K be a compact subset of R such that $f(x_0) + g(x_0) \notin K$. The closedness of the set K implies that there exists $\varepsilon > 0$ such that

$$(5) \qquad (f(x_0) + g(x_0) - \varepsilon, \;\; f(x_0) + g(x_0) + \varepsilon) \cap K = \emptyset .$$

Let $k > 0$ be a bound of the set K (i.e. $K \subset \langle -k, k \rangle$). Put

$$K_1 = \langle 0, k \rangle - (f(x_0) - \varepsilon/2, f(x_0) + \varepsilon/2),$$

$$K_2 = \langle 0, k\rangle - (g(x_0) - \varepsilon/2,\ g(x_0) + \varepsilon/2)\,.$$

Since $f \in U(X)$, by Corollary 1 there exists a neighbourhood U_1 of the point x_0 such that

$$f(U_1) \cap K_1 = \emptyset\,. \tag{6}$$

Since $g \in U(X)$, by Corollary 1 there exists a neighbourhood U_2 of the point x_0 such that

$$g(U_2) \cap K_2 = \emptyset\,. \tag{7}$$

Put

$$U = U_1 \cap U_2\,.$$

Let $x \in U$. If $f(x) + g(x) > k$, evidently $f(x) + g(x) \notin K$. Let $f(x) + g(x) \in \langle 0, k\rangle$. Since by (6) we have $f(x) \in \langle 0, k\rangle - K_1$, by the definition of K_1 we obtain

$$f(x) \in (f(x_0) - \varepsilon/2\ f(x_0) + \varepsilon/2)\,. \tag{8}$$

Since by (7) we have $g(x) \in \langle 0, k\rangle - K_2$, by the definition of K_2 we obtain

$$g(x) \in (g(x_0) - \varepsilon/2,\ g(x_0) + \varepsilon/2)\,. \tag{9}$$

From (8) and (9) it follows that $|(f(x) + g(x)) - (f(x_0) + g(x_0))| \leqq |f(x) - f(x_0)| + |g(x) - g(x_0)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, hence by (5) we have $f(x) + g(x) \notin K$. Therefore $(f + g)(U) \cap K = \emptyset$. By Corollary 1 we obtain $f + g \in U(X)$.

Corollary 3. *Let X be a topological space. Let $f, g \in U(X)$. Then $|f| + |g| \in U(X)$.*

Definition 3. A topological space X is called *perfectly normal* if and only if it is normal and each closed subset of X is G_δ. (See [4], p. 181.)

Theorem 7. *Let X be a perfectly normal topological space. Then $A \subset X$ is closed and of the first category in X if and only if there exists a function $f \in U(X)$ such that $D_f = A$.*

Proof. *Necessity* follows from Theorem 3. *Sufficiency*. Let $A \subset X$ be closed and of the first category in X. Then $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is a closed nowhere dense subset of X, $A_n \subset A_{n+1}$ $(n = 1, 2, \ldots)$. Let $g : X \to \langle 0, 1\rangle$ be a continuous function such that $g^{-1}(0) = A$. Let $g_n : X \to \langle 0, 1\rangle$ $(n = 1, 2, \ldots)$ be continuous functions such that for each $n \in N$

$$g_n^{-1}(0) = A\,, \tag{10}$$

$$g_n(x) \geqq g(x) \quad \text{for each} \quad x \in X\,. \tag{11}$$

The existence of functions g, g_n $(n = 1, 2, \ldots)$ follows from Urysohn's lemma. For each $n \in N$ define a function $f_n : X \to R$ as follows:

$$f_n(x) = \begin{cases} 1/g_n(x) & \text{for} \quad x \in X - A_n , \\ 0 & \text{for} \quad x \in A_n . \end{cases}$$

By Theorem 5 we have $f_n \in U(X)$ $(n = 1, 2, \ldots)$. Consider the series

$$\sum_{n=1}^{\infty} ((1/2^n) \, . \, f_n) \, . \tag{12}$$

We now show that the series (12) is convergent to some function $f : X \to R$. If $x \in A_m$ for some $m \in N$, then

$$0 \leqq \sum_{n=1}^{\infty} ((1/2^n) \, . \, f_n(x)) = \sum_{n=1}^{m} ((1/2^n) \, . \, f_n(x)) < +\infty \, .$$

If $x \in X - A$, then

$$0 \leqq \sum_{n=1}^{\infty} ((1/2^n) \, . \, f_n(x)) = \sum_{n=1}^{\infty} ((1/2^n) \, . \, (1/g_n(x))) \leqq$$

$$\leqq \sum_{n=1}^{\infty} ((1/2^n) \, . \, (1/g(x))) = 1/g(x) < +\infty \, .$$

We now show that $D_f = A$. First we shall prove that $X - A \subset C_f$. Let $b \in X - A$. Since $g(b) > 0$ and g is continuous at the point b, there exists a neighbourhood U of the point b such that

$$\forall x \in U : g(x) > g(b)/2 \, . \tag{13}$$

Evidently $U \subset X - A$. Hence by (13) we have for each $x \in U$

$$f_n(x) = 1/g_n(x) \leqq 1/g(x) < 2/g(b) \quad (n = 1, 2, \ldots) \, .$$

Therefore the series (12) is uniformly convergent on the set U. Since all functions f_n are continuous on U, the function f is continuous at the point b. Now we show that $A \subset D_f$. Let $a \in A$. Then $a \in A_m$ for some $m \in N$. We shall prove that for each neighbourhood V of the point a and for each $n \in N$ there exists a point $y \in V$ such that $f(y) > n$. Let V be a neighbourhood of the point a. Let $n \in N$. Since g_m is continuous at the point a, there exists a neighbourhood W of the point a such that

$$\forall x \in W : g_m(x) < 2^{-m}/n \, . \tag{14}$$

Since A_m is closed and nowhere dense, there exists a point $y \in V \cap W$ such that $y \in$ $\in X - A_m$. Hence by (14) we have

$$f(y) \geqq (1/2^m) \, . \, (1/g_m(y)) > n \, .$$

Therefore $a \in X - B_f = D_f$.

Now we shall prove that $f \in U(X)$. Let K be a compact subset of R. We now show that $X - f^{-1}(K)$ is open. Let $x_1 \in A_m - f^{-1}(K)$ for some $m \in N$. Put

$$h = \sum_{n=1}^{m} ((1/2^n) . f_n) .$$

By Theorem 5, Proposition 2 and Theorem 6 we obtain $h \in U(X)$. Since $f(x_1) \notin K$ and K is closed, there exists $\varepsilon > 0$ such that

$$(f(x_1) - \varepsilon, f(x_1) + \varepsilon) \cap K = \emptyset .$$

Let $k > 0$ be a bound of the set K (i.e. $K \subset \langle -k, k \rangle$). Put

$$K_1 = \langle 0, k \rangle - (f(x_1) - \varepsilon, f(x_1) + \varepsilon) .$$

Since $h(x_1) = f(x_1) \notin K_1$, K_1 is compact and $h \in U(X)$, by Proposition A the set $X - h^{-1}(K_1)$ is an open neighbourhood of the point x_1. Since g_m is continuous at the point x_1, there exists a neighbourhood U_1 of the point x_1 such that $U_1 \subset X - h^{-1}(K_1)$ and for each $x \in U_1$ we have

$$g_m(x) < 2^{-m}/(f(x_1) + \varepsilon) . \tag{15}$$

If $x \in U_1 \cap A_m$, then $f(x) = h(x) \notin K_1$. Therefore $f(x) \notin K$. If $x \in U_1 - A_m$, then by (15) we have

$$h(x) \geqq (1/2^m) . f_m(x) = (1/2^m) . (1/g_m(x)) > f(x_1) + \varepsilon .$$

Since $h(x) \notin K_1$, we obtain $h(x) \notin \langle 0, k \rangle$. Hence $f(x) \geqq h(x) > k$, then $f(x) \notin K$. Therefore the point x_1 has a neighbourhood U_1 such that $U_1 \subset X - f^{-1}(K)$.

Let $x_2 \in (X - A) - f^{-1}(K)$. Since $x_2 \in C_f$, the set

$$U_2 = X - f^{-1}(K) = f^{-1}(R - K)$$

is a neighbourhood of the point x_2.

Therefore the set $X - f^{-1}(K)$ is open. By Proposition C we have $f \in U(X)$.

This theorem has the following corollary.

Theorem C. *Let X be a Baire metric space. Then $F \subset X$ is closed and nowhere dense in X if and only if there exists a function $f \in U(X)$ such that $D_f = F$.*

The following example shows that the assumption "X is perfectly normal" in Theorem 7 cannot be replaced by the assumption "X is normal".

Example 3. Let $X = \{\omega; \omega \leqq \Omega\}$ (where Ω denotes the first uncountable ordinal number) with the order topology. It is well known that X is a normal space, and the set $\{\Omega\}$ is closed and nowhere dense in X but for each $f \in U(X)$ we have $D_f \neq \{\Omega\}$. (See [1].)

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