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Dalibor Klucký; Libuše Marková

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A CONTRIBUTION TO THE THEORY OF TACNODAL QUARTICS

DALIBOR KLUCKÝ, LIBUŠE MARKOVÁ, Olomouc
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In [4]–[6] some types of quartics are investigated whose divisors of inflection points have an order divisible by 4 so that it is possible that such a divisor may be determined by another curve of a suitable degree. As it will be shown below, the tacnodal quartic (the plain quartic with just one point-singularity which is tacnodal) has just 12 inflection points, more exactly: its divisor of inflection points has the order 12. There is a natural question under what conditions the inflection points divisor is determined by a cubic. It was the original aim of this article to find such conditions as well as their geometrical meaning. Unfortunately, we did not succeed in expressing these conditions in an elegant form as in [4]–[6].

For next we introduce the following notations: $\mathbf{S}_2$ – a given complex projective plane, (F) – a curve determined by the form F, in particular (K) – the considered tacnodal quartic, A – the tacnodal point, a – the tangent of (K) at A, P, P^* – the different places of (K) with the centre A.

1. Now, let B be an arbitrary regular point of (K) and let b be the tangent of (K) at B. Let us choose the coordinate projective frame (A_0, A_1, A_2, E) in $\mathbf{S}_2$ so that $A_0 = A$, $A_1 = a \cap b$, $A_2 = B$ (Fig. 1).

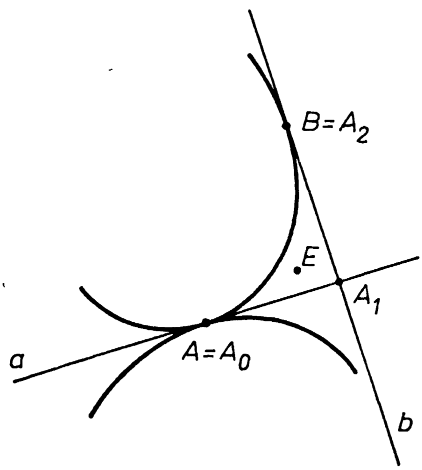

Fig. 1.

Then the curve (K) has the following equation

$$(1) \qquad (K)\colon ax_0^2x_2^2 + bx_0x_1^2x_2 + cx_0x_1x_2^2 + dx_0x_2^3 + ex_1^4 + fx_1^3x_2 + gx_1^2x_2^2 = 0\,;$$

$$a \neq 0\,, \quad d \neq 0\,, \quad e \neq 0\,.$$

Let us denote as usual

$$(2) \qquad K_i = \frac{\partial K}{\partial x_i}\,, \quad K_{ij} = \frac{\partial^2 K}{\partial x_i\,\partial x_j} \quad (i, j = 0, 1, 2)\,.$$

Therefore, the polars of the 1st order of A, B with respect to (K) have the equations

$$(3) \qquad K_0 = 0\,, \quad K_2 = 0\,.$$

For the Hessian $H = \det [K_{ij}]$ of K we can find

$$\begin{aligned}(4) \qquad H = {} & -18d^2gx_2^6 + [(6c^2d - 24adg - 18b^2d)\,x_0 - 54d^2fx_1]\,x_2^5 + \\ & + [(6ac^2 - 24a^2g - 24abd)\,x_0^2 + (-24acg + 6c^3 - 12bcd - \\ & - 72adf)\,x_0x_1 + (-24ag^2 + 6c^2g - 108d^2c + 36bdg - 54cdf)\,x_1^2]\,x_2^4 + \\ & + [-24a^2bx_0^3 - 72a^2fx_0^2x_1 + (-144ade + 24abg + 12bc^2 - 12b^2d - \\ & - 72acf)\,x_0x_1^2 + (-24afg + 24bcg - 12c^2f - 144cde)\,x_1^3x_2^3 + \\ & + [-144a^2ex_0^2x_1^2 + (-144ace + 12b^2c - 24abf)\,x_0x_1^3 + \\ & + (48aeg + 6b^2g + 6bcf - 72bde - 18af^2 - 48c^2e)\,x_1^4]\,x_2^2 + \\ & + [(-96abe + 6b^3)\,x_0x_1^4 + (6b^2f - 48bce)\,x_1^5]\,x_2 - 12b^2ex_1^6\,.\end{aligned}$$

The places P, P^* with the common centre $A = A_0$ and the common tangent $a = A_0A_1$ have the following parametrizations:

(5a) (5b) P: $\bar{x}_0 = 1$, $\bar{x}_1 = t$, $\bar{x}_2 = mt^2 +$ members of degreee $\geqq 3$, $m \neq 0$

P^*: $\bar{x}_0^* = 1$, $\bar{x}_1^* = t$, $\bar{x}_2^* = m^*t^2 +$ members of degree $\geqq 3$, $m^* \neq 0$

As $K(\bar{x}_0, \bar{x}_1, \bar{x}_2) = 0$ and, in the same way, $K(\bar{x}_0^*, \bar{x}_1^*, \bar{x}_2^*) = 0$, we get from (1), (5a) (5b):

$$\begin{aligned}(6) \qquad & (am^2 + bm + e)\,t^4 + (\text{members of degree} \geqq 5) = 0\,, \\ & (am^{*2} + bm^* + e)\,t^4 + (\text{members of degree} \geqq 5) = 0\,.\end{aligned}$$

This means that m and m^* are the roots of the equation

$$(7) \qquad a\xi^2 + b\xi + e = 0\,.$$

Since any root of (7) leads to a certain place of (K) with the centre A, the assumption

$$(8) \qquad b^2 - 4ae \neq 0$$

gives a sufficient condition for the existence of two different places of (K) with the centre A. Under the condition (8) we get

$$(9) \qquad m \neq m^* .$$

In what follows we will suppose that (8) is fulfilled.

2. We determine the class τ of (K) by the method described in [2], pp. 116–117. Let us choose a fixed point $C \in \mathbf{S}_2$ and for any place W of (K), put

$$\varepsilon_C(W) \begin{cases} = 0, \text{ if the tangent of } W \text{ does not contain } C; \\ = \text{the degree of } W, \text{ if the tangent of } W \text{ contains } C, \\ \quad \text{but the centre of } W \text{ is different from } C; \\ = \text{the sum of the degree and of the class of } W, \\ \quad \text{if } C \text{ is the centre of } W. \end{cases}$$

Then the divisor $T_C = \sum_W \varepsilon_C(W) \, . \, W$, where W runs over all places of (K), is well defined. Its order $\sum_W \varepsilon_C(W)$ does not depend on the point C and is equal to τ. We will call the divisor T_C the *tangential divisor with respect to the point* C.

Let $C = (c_0, c_1, c_2)$ and let $F = \sum_{0 \leqq i \leqq 2} c_i K_i$ be the first polar form of C with respect to K. If we denote, as usual, for any form $\phi \in \mathbf{C}[x_0, x_1, x_2]$ by $O_W(\phi)$ the order of W on the form ϕ, we may write

$$O_W(F) = \delta_C(W) + \varepsilon_C(W) ,$$

where $\delta_C(W) \geqq 0$. It is well-known that for any place W whose centre is a regular point of (K) the identity $O_W(F) = \varepsilon_C(W)$ ($\Rightarrow \delta_C(W) = 0$) holds.

Now let us put $C = B$. As $B = A_2$ does not lie on the common tangent a of the places P, P^*, we have

$$O_P(K_2) = \delta_B(P) , \quad O_{P^*}(K_2) = \delta_B(P^*) \ ^{*)}$$

By substituting (5a) or (5b) into the left hand side of (3), we get, respectively,

$$K_2(\bar{x}_0, \bar{x}_1, \bar{x}_2) = (2am + b)\, t^2 + \text{members of degree} \geqq 3 ,$$

$$K_2(\bar{x}_0^*, \bar{x}_1^*, \bar{x}_2^*) = (2am^* + b)\, t^2 + \text{members of degree} \geqq 3 .$$

With respect to (8) we have $2am + b \neq 0$, $2am^* + b \neq 0$, so that $O_P(K_2) = O_{P^*}(K_2) = 2$. Hence

$$\tau + 2 + 2 = \sum_W \varepsilon_B(W) + O_P(K_2) + O_{P^*}(K_2) =$$

$$= \sum_{W \neq P, P^*} \varepsilon_B(W) + O_P(K_2) + O_{P^*}(K_2) = \sum_{W \neq P, P^*} O_W(K_2) + O_P(K_2) + O_{P^*}(K_2) =$$

*) If $C = A_2$, then, of course, $F = K_2$.

$$= \sum_W O_W(K_2) = 4.3 = 12\,, \quad \text{hence} \quad \tau = 8\,.$$

Proposition 1. *The tangential divisor* T_A *is determined on* (K) *by a conic.*

Proof. Let us consider the conic Q having the equation

(10) $$(Q)\colon\ 2ax_0x_2 + bx_1^2 + cx_1x_2 + dx_2^2 = 0\,.$$

If we take the first polar (K_0) of the point A as a divisor in the plane $\mathbf{S}_2$, then

(11) $$(K_0) = a + (Q)$$

Let W be a place of (K) whose centre U is a regular point of (K). Then

$$\varepsilon_A(W) = O_W(K_0) = O_W(Q)\,.$$

Hence $O_W(Q) > 0$ if and only if the tangent of W contains A; in this case

$$O_W(Q) = \varepsilon_A(W) = \text{the class of } W\,.$$

Now let $W = P, P^*$. Then we can evaluate $Q(\bar{x}_0, \bar{x}_1, \bar{x}_2) = (2am + b)\, t^2 +$ members of degree $\geqq 3$. As $2am + b \neq 0$, we get $O_P(Q) = 2$. In the same way we obtain $O_{P^*}(Q) = 2$. On the other hand, $\varepsilon_A(P)$ as well as $\varepsilon_A(P^*)$ equal the sum of the degree and the class of the place P, P^*, respectively. Thus

$$O_P(Q) = 2 = \varepsilon_A(P)\,; \quad O_{P^*}(Q) = 2 = \varepsilon_A(P^*)\,.$$

Finally

$$\mathsf{T}_A = \sum_W \varepsilon_A(W)\, W = \sum_W O_W(Q)\, W\,.$$

3. For any place W of (K) let us denote by τ_W the class of W. Then the divisor

$$\mathsf{I} = \sum_{W \neq P, P^*} (\tau_W - 1)\, W$$

is the divisor of inflection points. We will determine its order $i = \sum_{W \neq P, P^*} (\tau_W - 1)$.

It is well-known that for any place W of (K), $W \neq P$, P^*, the relation

$$\tau_W - 1 = O_W(H)$$

is true, therefore $i = \sum_{W \neq P, P^*} O_W(H)$.

Hence

$$i + O_P(H) + O_{P^*}(H) = \sum_{W \neq P, P^*} O_W(H) + O_P(H) + O_{P^*}(H) = \sum_W O_W(H) = 24\,.$$

Substituting (5a) into (4) we have

$$H(\bar{x}_0, \bar{x}_1, \bar{x}_2) = (-24a^2bm^3 - 144a^2em^2 - (96abe + 6b^3)\, m -$$
$$- 12b^2e)\, t^6 + \text{members of degree} > 6$$

and in the same way

$$H(\bar{x}_0^*, \bar{x}_1^*, \bar{x}_2^*) = (-24a^2bm^{*3} - 144a^2em^{*2} - (96abe + 6b^3)\, m^* - \\ - 12b^2e)\, t^6 + \text{members of degree} > 6\,.$$

Let us suppose that neither m nor m^* is a solution of the equation

$$\text{(12a)} \qquad -24a^2b\xi^3 - 144a^2e\xi^2 - (96abe + 6b^3)\,\xi - 12b^2e = 0\,.$$

(If we use the resultant of the equation (7) and (12), we obtain, that our assumption is fulfilled if and only if

$$\text{(12b)} \qquad 6\,.\,24^2a^4e^3 - 108\,.\,24a^3b^2e + 27\,.\,24a^2b^4e - 25ab^6 \neq 0$$

is true.)

With respect to this assumption we establish that $O_P(H) = O_{P^*}(H) = 6$, consequently

$$i = 12\,.$$

4. Now we will determine the quadratic form

$$Q = Bx_1 + Cx_2 + Dx_0x_2 + Ex_1x_2$$

so that the form $H - QK$ may be written as

$$\text{(13)} \qquad H - QK = x_2^2\bar{K}\,,$$

where $\bar{K}$ is a suitable form of degree 4.

Using (4) we get that (13) is true if and only if

$$\text{(14)} \qquad B = -12b^2\,; \quad D = [-6b(16ae - 3b^2)]/e\,; \\ E = [-6b(8ce - 3bf)]/e\,;$$

moreover,

$$\text{(15)} \quad \bar{K} = \{-18d^2gx_2^4 + [(6c^2d - 24adg - 18bd^2)\,x_0 - 54d^2fx_1]\,x_2^3 + \\ + [(6ac^2 - 24a^2g - 24abd - Dd)\,x_0^2 + (-24acg + 6c^3 - 12bcd - \\ - 72adf - Ed)\,x_0x_1 + (-24ag^2 + 6c^2g - 108d^2c + 36bdg - \\ - 54cdf)\,x_1^2]\,x_2^2 + (-24a^2b - Da)\,x_0^3 + (-72a^2f - Dc - Ea)\,x_0^2x_1 + \\ + (-144ade + 24adg + 12bc^2 - 12b^2d - 72acf - Dg - \\ - Ec - Bd)\,x_0x_1^2 + (-24afg + 24bcg - 12c^2f - 144cde - \\ - Eg)\,x_1^3]\,x_2 + [(-144a^2e - Db - Ba)\,x_0^2x_1^2 + (-144ace + \\ + 12b^2e - 24abf - Df - Eb - Bc)\,x_0x_1^3 + (48aeg + 6b^2g + \\ + 6bcf - 72bde - 18af^2 - 48c^2e - Ef - Bg)\,x_1]\} - C\,. \\ .\,\{ax_0^2x_2^2 + bx_0x_1^2x_2 + cx_0x_1x_2^2 + dx_0x_2^3 + ex_1^4 + fx_1^3x_2 + gx_1^2x_2^2\}\,.$$

Substituting (5a) as well as (5b) into (15), we obtain, respectively,

$$\bar{K}(\bar{x}_0, \bar{x}_1, \bar{x}_2) = [(-24a^2b + 6ab(16ae - 3b^2)/e)\, m + (-144a^2e + \\ + 6b^2(16ae - 3b^2)/e + 12ab^2]t^2 + \text{members of degree} \geqq 3\,,$$

$$\bar{K}(\bar{x}_0^*, \bar{x}_1^*, \bar{x}_2^*) = [(-24a^2b + 6ab(16ae - 3b^2)/e)\, m^* + (-144a^2e + \\ + 6b^2(16ae - 3b^2)/e + 12ab^2]t^2 + \text{members of degree} \geqq 3\,.$$

This implies $O_P(\bar{K}) \geqq 2$; $O_{P^*}(\bar{K}) = 2$. Evidently, any place W of the curve (K) having as its centre an inflection point of (K) has the same order on the Hessian H and on the form $\bar{K}$. This means that the curve $(\bar{K})$ cuts the (K) in the divisor

$$\text{(16)} \qquad \mathsf{I} + 2P + 2P^*\,.$$

As (16) has order 16 and (K), $(\bar{K})$ are obviously different, then $(\bar{K})$ intersects (K) in the divisor (16) – the divisor (16) is determined on (K) by $(\bar{K})$.

Let us denote by K_0 the form in the first exterior brackets of (15). Then

$$\text{(17)} \qquad \bar{K} = K_0 - CK\,,$$

which means that the curves (15) together with (K) form a pencil of curves of fourth degree.

5. Let us suppose that a cubic curve $(\mathscr{C})$ without multiple components determines on (K) the divisor I. Let S denote the set of all places of $(\mathscr{C})$ whose centres are inflection points of (K).

Obviously the divisor

$$\sum_{W \in S} O_W(K)\, \overline{W}$$

has order 12. Now, we will investigate the number $\sum_{W \in S} O_{\overline{W}}(\bar{K})$. Let us choose any place $\overline{W} \in S$ and let us denote by U its centre ($\Rightarrow U$ is an inflection point of (K) and it lies on $(\bar{K})$). Let W be the place of (K) with the centre U. We shall distinguish the following cases:

(a) U is a regular point of $(\mathscr{C})$. Then $O_W(\mathscr{C}) = O_{\overline{W}}(K) = \tau_W - 1 =$ the intersection multiplicity of $(\mathscr{C})$ and (K) at the point U. Now, for $\tau_W = 2$ we have: $O_{\overline{W}}(K) = 1 \leqq \leqq O_{\overline{W}}(\bar{K})$. Let $\tau_W = 3$. Then $O_{\overline{W}}(K) = 2$ and (K) and $(\mathscr{C})$ have a common tangent u at the point U. As $(\bar{K})$ determines the divisor (16) on (K), we get that either U is regular point of $(\bar{K})$ and $(\bar{K})$ has the tangent u at U or U is a multiple point of $(\bar{K})$. In both cases $O_W(\bar{K}) \geqq 2 \Rightarrow O_{\overline{W}}(K) \leqq O_{\overline{W}}(\bar{K})$.

(b) U is a singular (double) point of $(\mathscr{C})$.

Let U be the centre of two (linear) places $\overline{W}, \overline{W}'$ of $(\mathscr{C})$. As $O_W(\mathscr{C}) = O_{\overline{W}}(K) + O_{\overline{W}'}(K)$ and $O_W(\mathscr{C}) = 2$, hence $O_{\overline{W}}(K) = O_{\overline{W}'}(K) = 1 \Rightarrow O_{\overline{W}}(K) \leqq O_{\overline{W}}(\bar{K})$, $O_{\overline{W}'}(K) \leqq O_{\overline{W}'}(\bar{K})$.

Let U be the centre of one (quadratic) place $\overline{W}$. Then $2 \leqq O_{\overline{W}}(K) = O_{\overline{W}}(\mathscr{C}) \leqq$ $\leqq 2 \Rightarrow O_{\overline{W}}(K) = 2$. Obviously $O_{\overline{W}}(\overline{K}) \geqq 2 \Rightarrow O_{\overline{W}}(K) \leqq O_{\overline{W}}(\overline{K})$.

We have proved the following lemma:

Lemma. *Let the divisor* I *on* (K) *be determined by a cubic curve* $(\mathscr{C})$ *(without multiple components). Let* S *be the set of all places of* $(\mathscr{C})$ *whose centre is an inflection point of* (K). *Then for any quartic* $(\overline{K})$ *from the relation* (13) *the divisor* $\sum_{\overline{W} \in S} O_{\overline{W}}(\overline{K})\, \overline{W}$ *has an order at least* 12. *Moreover for any place* $\overline{W} \in S$ *the inequality* $O_{\overline{W}}(K) \leqq O_{\overline{W}}(\overline{K})$ *holds.*

6. Now we are able to prove:

Proposition. *The divisor* I *on* (K) *is determined by a cubic curve* $(\mathscr{C})$ *if and only if there exists* C *so that the form* $\overline{K} = K_0 - \mathsf{C}K$ *is divisible by* x_2. *In this case* $(\overline{K}) = (\mathscr{C}) + a$.

Proof. 1. Let us suppose that I is determined by a suitable cubic $(\mathscr{C})$. Let us choose a place $\overline{W}_0$ which does not belong to the set S. Then there exists a uniquely determined parameter C such that the place $\overline{W}_0$ has a positive order on the form $\overline{K} = K_0 - \mathsf{C}K$. Hence the divisor

$$\sum_{\overline{W} \in S} O_{\overline{W}}(\overline{K})\, \overline{W} + O_{\overline{W}}(\overline{K})\, \overline{W}_0$$

has an order $\geqq 13$ and for any place $\overline{W} \in S$ we have

$$O_{\overline{W}}(K) \leqq O_{\overline{W}}(\overline{K}).$$

If $(\mathscr{C})$ is irreducible, then it is obviously a component of $(\overline{K})$, the other component being the tangent $a = (x_2) \Rightarrow (\overline{K}) = (\mathscr{C}) + (x_2)$.

Let $(\mathscr{C})$ be reducible e.g. $(\mathscr{C}) = (l_1) + (l_2) + (l_3)$, where (l_1). $(l_2), (l_3)$ are distinct lines. The place $\overline{W}_0$ has its centre, say, on (l_1) therefore $(\overline{K}) = (\mathscr{C}_1) + (l_1)$, where $(\mathscr{C}_1)$ is a cubic curve. As (l_2) and (l_3) determine on $(\mathscr{C}_1)$ divisor of order $\geqq 4$, hence $(\mathscr{C}_1) =$ $= (l_2) + (l_3) + (x_2)$ consequently $(\overline{K}) = (l_1) + (l_2) + (l_3) + (x_2) = (\mathscr{C}) + (x_2)$.

Now, let $(\mathscr{C})$ have a double linear component (l), i.e. $(\mathscr{C}) = 2(l) + (l')$, where (l') is a line different from (l). Let us choose a point $D \in (l')$, $D \notin (K)$. Then there exists again a unique parameter C such that the point D lies on $(\overline{K}) = (K_0 - \mathsf{C}K)$, thus (l') is a component of $(\overline{K})$, so that $(\overline{K}) = (\mathscr{C}_1) + (l')$, $(\mathscr{C}_1)$ is a cubic curve. Let J and J' be the divisors determined on (K) by (l') and (l), respectively. Their order is equal to 4 and clearly $\mathsf{I} = 2\mathsf{J} + \mathsf{J}'$. Hence $(\mathscr{C}_1)$ determines on (l) a divisor of order 4. Then we have $(\mathscr{C}_1) = (Q_1) + (l)$, where (Q_1) is a conic, hence $(\overline{K}) = (Q_1) + (l) + (l')$. As (l) determines on (K) the divisor J of order 4, then (Q_1) must determine the divisor $\mathsf{J} + 2P + 2P^* \Rightarrow (Q_1) = (l) + (x_2) \Rightarrow (\overline{K}) = (l) + (l) + (l') + (x_2) = 2(l) + (l') +$ $+ (x_2) = (\mathscr{C}_1) + (x_2)$.

Finally, let $(\mathscr{C}) = 3(l)$, where (l) is a line. Then any inflection point has order 3 and the tangent at it has 5-point intersection with (K) – a contradiction.
2. If for some parameter C the form $\bar{K} = K_0 - \mathrm{C}K$ is divisible by x_2, then $\bar{K} = x_2 \, . \, \mathscr{C}$, where $\mathscr{C}$ is a cubic form. We get $(\bar{K}) = (\mathscr{C}) + (x_2) = (\mathscr{C}) + a$ and I is determined by $(\mathscr{C})$.

If we use the just proved proposition together with the relations (14) and (15), we get:

Theorem. *The divisor* I *of inflection points of a tacnotal quartic* (K) *is "in general" not determined by a cubic curve. More exactly: If the coordinate system is chosen so that* (K) *has the equation* (1), *and* (K) *fulfils the assumptions* (8) *and* (12b), *then the divisor* I *is determined by a cubic curve if and only if the condition*

$$(18) \qquad 6ab^2e - 8a^2e^2 - b^4 = 0$$

$$8ace^2 - 4abef - b^3f = 0$$

are fulfilled.

7. It is well known that any tacnodal quartic (K) may be transformed by a quadratic transformation into a quartic (K_1) possessing two ordinary double points (a binodal quartic). Thus, the genus of (K) = the genus of $(K_1) = 1$. Hence the curve (K) is elliptic.

8. As an example, let us consider a quartic (K) having with respect to a choosen projective coordinate system $\langle A_0, A_1, A_2, E\rangle$ the equation

$$(19) \qquad (K)\colon\ ax_0^2x_2^2 + x_0x_1^2x_2 + x_0x_2^3 + x_1^4 = 0\,,$$

where the coefficient a is an arbitrary solution of the equation

$$8a^2 + 6a - 1 = 0 \quad (\Rightarrow a = (1/8(-3 \pm \sqrt{}17))\,.$$

By direct verification we may establish that the coefficients of (19) fulfil the conditions (8), (12b) as well as both conditions (18). Moreover, the quartic (19) obviously has a tacnodal point A_0 with the tangent $x_2 = 0$ and has no more singular points.

Finally, we will prove that (K) is irreducible. If the curve (K) has a linear component (l), then (l) must be equal to a which is obviously impossible. Therefore, if (K) is reducible, then

$$(K) = (Q_1) + (Q_2)$$

and $(Q_1), (Q_2)$ have a 4-point intersection at the point A_0, hence

$$Q_1(x) = \beta x_1^2 + \gamma x_2^2 + \delta x_0x_2 + \varepsilon x_1x_2\,,$$

$$Q_2(x) = \beta' x_1^2 + \varrho\gamma x_2^2 + \varrho\delta x_0x_2 + \varrho\varepsilon x_1x_2\,, \quad \varrho \neq 0$$

[see [1] pp. 211–222]. It is clear that $Q_1 \, . \, Q_2$ is not proportional to K.

If we replace in (19) the coefficient a by 1 we obtain a quartic (K')

$$(K'): \; x_0^2x_2^2 + x_0x_1^2x_2 + x_0x_2^3 + x_1^4 = 0 .$$

By a similar argument as above we establish that (K') is an irreducible quartic, whose divisor I of inflection points is not determined by any cubic curve.

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Authors' address: 771 46 Olomouc, Leninova 26 (Přírodovědecká fakulta UP).